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GRAVIMETRIC DEFLECTIONS OF THE VERTICAL BY DIGITAL COMPUTER

Dezsö Nagy

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Gravimetric Deflections of the Vertical by Digital Computer

DEZSŐ NAGY

ABSTRACT—A digital computer program has been developed to calculate plumb-line deflections from gravity data. A region of 1,200 × 1,200 km for which free air anomalies were available was subdivided into units of 50 × 50 km. With n denoting the number of points per unit, each unit was represented by one gravity anomaly calculated as an average in cases where $0 \leqslant n < 6$ and where $n > 50$. For $6 \leqslant n \leqslant 50$ the integral mean, obtained from a fitted surface of second order in two variables, was used.

Weighting functions were derived for and calculations were done in the rectangular plane coordinate system. For units with sufficient points for surface fitting, the contributions to the deflection components at the centre of the unit from within the unit itself were computed first, and then the effect of the outer region was added. The contributions from the outer region were obtained as the sum of the products of gravity anomalies and weighting coefficients over all units. The computations were repeated with three different origins in order to analyse the effect of the change in the number of points and the point distribution within the 50 × 50 km units. This analysis shows that non-uniform point distribution may seriously distort the fitted surface, giving erroneous values for the horizontal gradients and hence for the contribution to the deflection components at the centre from within the unit element.

The program solves for the gravimetric deflections *relative* to the origin. To make the two sets of deflections comparable it was necessary to transform the astro-geodetic deflections from Clarke's spheroid to the International Ellipsoid and add a constant term to all gravimetric deflections. This constant term, representing the effect from beyond the region of integration, is the difference between the astro-geodetic and gravimetric deflections at the origin. A visual comparison of the plotted deflections shows generally good agreement both in direction and in magnitude, indicating that the choice of weighting function, grid distance, and order of fitted surface was suitable. The accuracy of the astro-geodetic and gravimetric deflections is estimated at ± 1 and ± 2 seconds of arc respectively.

RÉSUMÉ—On a mis au point un programme pour calculatrice numérique afin de calculer les déviations du fil à plomb à l'aide de données gravimétriques. Une région de 1,200 km sur 1,200 pour laquelle on connaissait les anomalies à l'air libre a été divisée en unités de 50 km sur 50. La lettre "n" désignant le nombre de points par unité, chaque unité a été représentée par une anomalie de la gravité prise comme moyenne dans les cas où $0 \leqslant n < 6$ et $n > 50$. Dans le cas de $6 \leqslant n \leqslant 50$, on a utilisé la moyenne intégrale calculée à partir d'une surface agencée de second ordre à deux variables.

On a dérivé les fonctions de pondération et procédé à des calculs dans le système de coordonnées rectangulaires planes. Dans le cas des unités qui comportent suffisamment de points pour l'agencement en surface, on a tout d'abord calculé les contributions aux composantes de déviation au centre de l'unité à partir de données propres à l'unité elle-même, puis on a ajouté l'effet de la région extérieure. Les contributions de la région extérieure ont été obtenues comme la somme des produits des anomalies de la gravité et des coefficients de pondération pour toutes les unités. Les calculs ont été repris à partir de trois origines différentes afin d'analyser l'effet du changement dans le nombre de points et de la répartition des points dans les unités de 50 km sur 50. L'analyse en question démontre que la répartition non uniforme des points peut entraîner une distortion sérieuse de la surface agencée, indiquant des valeurs erronées pour les gradients horizontaux et, partant, pour la contribution aux composantes de déviation au centre qui peut être attribuable à l'élément unitaire lui-même.

Le programme en question vaut pour les déviations gravimétriques relativement à l'origine. Afin de pouvoir comparer les deux séries de déviation, il a fallu transformer les déviations astro-géodésiques de la sphéroïde de Clarke en celles de l'Ellipsoïde international, puis ajouter un terme constant à toutes les déviations gravimétriques. La constante en question, qui représente l'effet de la région en dehors des limites de l'intégration, constitue la différence entre les déviations astro-géodésiques et gravimétriques à l'origine. Une comparaison visuelle des déviations tracées montre qu'il y a d'ordinaire un agencement relativement précis tant pour la direction que pour l'intensité, ce qui indique que le choix de la fonction de pondération, de la distance de quadrillage et de la disposition de la surface agencée était convenable. La précision des déviations astro-géodésiques et gravimétriques a été respectivement évaluée à ± 1 et ± 2 secondes d'arc.

PART I

Introduction

The fundamental problem of geodesy is to determine the size and shape of the earth. By the size, one usually means the size of a reference surface, which is given by a closed mathematical form and best fits the observations. (By 'best' the least-squares fit is meant, i.e. when the sum of the squares of the deviations is minimized). The reference surface which is accepted for world-wide use might be called the *mathematical* surface of the earth. To determine the shape, it is necessary to find the deviations between the reference surface and the actual *physical* surface. The calculations are carried out on the mathematical surface, the measurements are made on the physical surface, but these measurements refer to the local *niveau* (*level*) surface, which is perpendicular to the gravity. The particular niveau surface, which passes through the mean sea level is called the *geoid*. The reference surface was chosen to approximate the geoid. It is an ellipsoid of revolution, the parameters of which have been determined by Hayford (1910) and for reference, they are given below:

$$a = 6\ 378\ 388 \pm 18 \text{ metres}$$
$$\alpha = \frac{a-b}{a} = \frac{1}{297 \pm .5} \tag{1}$$

where a is the equatorial radius of the earth, and
α is the flattening.

These parameters were determined from 765 astronomic observations in the United States using isostatic reductions, with the depth of compensation of 122.2 km and were accepted as the parameters of the International Ellipsoid by the Madrid Assembly of the International Union of Geodesy and Geophysics (Bulletin Géodesique 1924, No. 4, p. 258). This reference surface satisfies the present-day requirements for all geodetic work.

For gravimetric work the International Gravity Formula is used which gives the value of the theoretical (normal) gravity on the surface of the international ellipsoid. By definition, this surface encloses the total mass of the earth, its centre of gravity and rotational axis coincide with that of the earth, and its density distribution is such that the international ellipsoid is an equipotential surface for all practical purposes. The formula is given below:

$$\gamma = 978.049\,(1 + .005\ 2884 \sin^2 \varphi - .000\ 0059 \sin^2 2\,\varphi) \tag{2}$$

where γ is in gals, and
φ is the latitude.

The coefficients in the formula were calculated by Cassinis (1930) and adopted by the Stockholm Assembly of the IUGG in 1930.

According to recent determinations of the corrections to a and α it can be said that the first part of the problem is solved, i.e. the size of the mathematical surface of the earth has been determined. The remaining part of the problem is to find the shape of the physical surface of the earth with reference to the international ellipsoid. This problem is one of height determination: the determination of the height H from the reference surface to point P on the physical surface, measured along a well-defined line. Choosing this line as the direction of normal gravity (normal plumb-line), then the problem can be stated as follows: find H measured along the normal plumb-line of point P. In Figure 1 the relative positions of the above mentioned surfaces are shown.

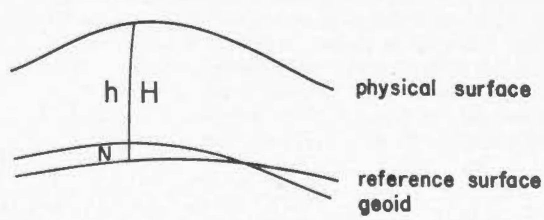

Figure 1

From Figure 1 can be written:

$$H = h + N \tag{3}$$

where H is the height of point P, measured along the normal plumb-line,
h is the elevation of point P, known from spirit levelling and referring to the mean sea level (geoid),
N is the geoidal height i.e. the separation between the geoid and the reference surface. (It is of the order of $\pm$ 50 metres.)

Since in (3), h is available from levelling, the problem of finding the shape of the earth is reduced to finding N for every point of the physical surface.

As is well known, N can be calculated by Stokes' formula:

$$N = \frac{R}{4\pi G} \int S(\psi)\, \Delta g\, d\sigma \tag{4}$$

where R is the mean radius of the earth
G is the mean gravity
$S(\psi)$ is Stokes' function
Δg is the gravity anomaly which corresponds to the surface element
$d\sigma$ is the surface element of the unit sphere.

At present this is the only method which can furnish the value of N with sufficient accuracy. The relative value, i.e. the difference between the geoidal heights of two points ($N_1 - N_0$) not far from each other, can be calculated from plumb-line deflections. From the differences the geoidal heights can be obtained if, at least for one point, N is known.

A variety of methods can be developed to solve the problem. Before deciding what method to use it is necessary to consider such points as: available material; the rate of incoming material per year; the economy and flexibility of securing new material; etc.

There are only a few hundred astro-geodetic deflections available for Canada, and the rate of incoming information is only a few tens per year. Since deflection determinations are tied to the triangulation net, it is only possible to obtain deflection values at points of the net. On the average, it takes about one week of additional field work per station in order to be able to calculate the deflection value.

As for gravity data, there are about 35,000 points where gravity is known and most of them are on punched cards. The yearly increase is about 4,000 to 5,000 new points, usually processed on punched cards within about 6 months from the time of the measurement. The distribution of points can be made to meet the demand.

It was therefore decided to base this study on the gravimetric method for the solution of the problem and to utilize the astro-geodetic deflections, after the proper transformation, for the control of calculation.

To solve the problem, i.e. to calculate N, the work involves numerical integration. The accepted method is to use templates: radial lines divide concentric circles into compartments. The radii of circles are such that for each compartment the same factor is used by which the estimated gravity anomaly for the compartment has to be multiplied, and the sum of these products gives the solution. The points where gravity anomalies are known must be plotted on a map of suitable scale and projection, and then the anomaly map prepared by the usual contouring technique. The main disadvantage of this method is that the estimated mean values of gravity anomalies can be used only once. For another computational point new estimates must be obtained.

In order to avoid this disadvantage the calculation was "digitized", and a grid system of 50-km grid-distance was introduced, which divides the local area for which the calculation was being carried out into "unit" areas. To each unit area, if the number of points was sufficient, a low-order surface was fitted by the method of least squares. Then the gravity anomaly which represents the unit area was estimated from the fitted surface. These estimated values can be used again if the computational point moves parallel to the grid lines by the grid distance or a multiple of it. For each unit area there is a corresponding "weighting" coefficient which is multiplied by the estimated gravity anomaly, and the sum of these products over all unit areas gives the solution. The number of computational points is equal to the number of unit areas. The calculation is adapted for a digital computer (IBM 650, index registers, floating point device, 60-word core memory, 3 tape units). The input is on the format of the Gravity Division, Dominion Observatory, Ottawa.

It is well known that the weighting function is inversely proportional to the distance if N is to be calculated, and inversely proportional to the square of the distance if it is desired to determine deflection values. In other words, N depends on the regional gravity field, while the principal part of the plumb-line deflection can be obtained from a local survey. Besides some other advantages, this was the main reason why it was proposed to solve the problem "indirectly", that is, first the plumb-line deflections are calculated, then, by numerical integration, the geoidal height or height anomaly map is prepared.

It is emphasized here that the object of this study was to find a procedure for the calculation of the deflection of the vertical only, and adapt that for a digital computer. The selection of the parameters, such as the grid distance, the order of fitted surface, etc., was quite arbitrary, and was based on general practice rather than mathematical grounds. To demonstrate the validity of the assumptions made during the development of the procedure, sample calculations were carried out.

Historical Development

In this section some of the important studies consulted in connection with the problem are briefly discussed. This selection, which makes no claim to completeness, includes authors whose approach to the problem seems most pertinent in the opinion of the writer.

Clairaut (1743), starting from the theory of a rotating fluid in relative equilibrium with uniform angular velocity, and assuming that this rotating body is built up from concentric and coaxial shells whose density may vary from shell to shell in any manner, found that the gravity—on an equipotential surface of a rotational ellipsoid which encloses all the attracting mass—can be given approximately by the formula

$$\gamma = \gamma_e (1 + \beta \sin^2 \varphi) \tag{5}$$

This equation is known as *Clairaut's formula.* Substituting $\varphi = 0$, it is seen that γ_e is the gravity at the equator. The coefficient β is related to the flattening α by the equation:

$$\alpha = \frac{5}{2} m - \beta \tag{6}$$

This is well known as *Clairaut's theorem,* accurate to the first order of the flattening. In this equation

$m = \dfrac{\omega^2 a}{\gamma_e}$ i.e. the ratio of the centrifugal force to the gravity at the equator,

$\beta = \dfrac{\gamma_p - \gamma_e}{\gamma_e}$

where γ_p is the gravity at the pole.

The coefficient β might be called the "gravitational" flattening.

About one hundred years later Stokes (1849a) showed that the assumptions made by Clairaut for the derivation of the variation of gravity on an equipotential surface are not necessary: if it is merely assumed that the surface of the earth is a surface of equilibrium, then, using only the theory of gravitation "there exists a necessary connexion between the *form of the surface** and the *variation of gravity** along it, so that the one being given the other follows". When the surface is an ellipsoid of revolution of small eccentricity, then the variation of gravity on the surface is given (approximately) by Clairaut's formula.

In another paper Stokes (1849b) actually "inverted" Clairaut's formula: from the variation of gravity on an unknown surface of equilibrium he obtained a closed expression which enables one to determine the surface. The method is independent of the density distribution of the earth, it requires only gravity anomalies on the mean sea-level surface and assumes a sphere as reference surface. The method can be applied when the surface is nearly spherical. Stokes' function is given below:

$$S(\psi) = \operatorname{cosec} \tfrac{1}{2}\psi + 1 - 6 \sin \tfrac{1}{2}\psi - 5 \cos \psi \tag{7}$$
$$- 3 \cos \psi \, ln \, (\sin \tfrac{1}{2}\psi + \sin^2 \tfrac{1}{2}\psi)$$

To avoid ambiguity in connection with the figure of the earth Listing (1873) suggested the name of *geoid*† for that particular niveau (level) surface which passes through the mean sea-level. This surface is understood to be the figure of the earth, the determination of which is the main objective of geodesy.

Bruns (1878) derived a relation for the distance between two equipotential surfaces of the same mass with different density distribution. The name *Bruns' term* is associated with this approximation and because of its important role in the problem it is discussed in some detail.

Let W be the potential of the actual earth, and U that of an idealized earth (say, of the reference ellipsoid); then the disturbing potential T, at any point, is defined as:

$$T = W - U \tag{8}$$

Furthermore if γ is the theoretical gravity corresponding to U, $\mathfrak{g}$ is the actual gravity and ε is the angle between γ and $\mathfrak{g}$, then:

$$\gamma \cdot \mathfrak{g} = \gamma \, g \cos \varepsilon = \frac{\partial U}{\partial x}\frac{\partial W}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial W}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial W}{\partial z} \tag{9}$$

which gives ε, i.e. the plumb-line deflection.

To get Bruns' term, let point P (given by its coordinates x, y and z) be a point of the surface $W = U_0$, Q another point (along the normal at P) of the surface $U = U_0$ (W and U are equipotential surfaces corresponding to the same mass with different density distributions) and h the distance between the two points P and Q. Then the coordinates of Q are $x + \Delta x$, $y + \Delta y$ and $z + \Delta z$

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*Emphasized by the writer.

† "Wir werden die vorhin definirte mathematische Oberfläche der Erde, von welcher die Oberfläche des Oceans einen Theil bildet, die 'geoidische' Fläche der Erde oder das Geoid nennen, ... "

$$\text{where } \Delta x = h \cos \alpha = -\frac{h}{g}\frac{\partial W}{\partial x}$$

$$\Delta y = h \cos \beta = -\frac{h}{g}\frac{\partial W}{\partial y}$$

$$\Delta z = h \cos \gamma = -\frac{h}{g}\frac{\partial W}{\partial z}$$

The potential U at P is given as:

$$U_P = U_0 - \frac{h}{g}\left(\frac{\partial U}{\partial x}\frac{\partial W}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial W}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial W}{\partial z}\right)$$

But the potential at P from (8) is:

$$U_P = W_P - T_P = U_0 - T_P$$

and equating the right hand side of the last two equations the following form is obtained for any point P on the physical surface

$$T = \frac{h}{g}\left(\frac{\partial U}{\partial x}\frac{\partial W}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial W}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial W}{\partial z}\right)$$

The term in brackets from (9) is equal to $\gamma g \cos \varepsilon$ and after substitution we get for the disturbing potential

$$T = h\,\gamma \cos \varepsilon$$

Since the value of ε is small (less than a minute of arc) we can write

$$h = \frac{T}{\gamma} \tag{10}$$

which is known as *Bruns' term.*

Bruns also obtained the differential equation of physical geodesy and gives it in the following form:

$$g - \gamma' = -\frac{\partial T}{\partial a} + \frac{2}{a}\, h\,\gamma \tag{11}$$

$$\text{where } \gamma' = \gamma + h\,\frac{\partial \gamma}{\partial h} \text{ and}$$

a is the radius of the sphere representing the earth.

All the formulae and theorems just mentioned were treated by Helmert (1884) and in some cases were extended to give higher order accuracy (for example Clairaut's formula was derived including the terms which involve the square of the flattening.) Helmert also introduced the use of a surface coating. His treatment of the problem was, in every respect, definitely the most complete in his time.

Since, for the calculation of geoidal heights, the gravity anomaly on the geoid was required, the problem of the reduction of the gravity observations had special importance and various ways of reduction were investigated. In the next three to four decades, scientists in the field of geodesy were mostly concerned with the problem of reduction, and with criticism and extension of Stokes' and Clairaut's formulae.

Poincaré (1901) has studied the problem of determining the geoid and has paid special attention to finding the local shape of the geoid. He deduces Stokes' solution from his own investigations without referring to Stokes (in the text, reference was made to Helmert).

Cassinis (1930), basing his work on that of Pizzetti and Somigliana, calculated the coefficients in the "extended" Clairaut's formula, which was accepted as the international gravity formula (*see* formula (2)).

Brillouin (1925, 1927) in two short papers obtained an expression for the density, and to avoid the difficulties encountered in connection with the reduction of gravity, he proposed that use be made of a reference surface which is well above the physical surface (100 to 200 km).

Vening Meinesz (1928) derived the relationship between gravity and the plumb-line deflection. The expression for the north-south component of the deflection ξ, has the form:

$$\xi'' = -\frac{\rho''}{4\pi\gamma} \int \Delta g \frac{\partial S(\psi)}{\partial \psi} \cos\alpha \, d\sigma \tag{12}$$

where $$\frac{\partial S(\psi)}{\partial \psi} \sin\psi = \cos^2 \tfrac{1}{2}\psi \left[\operatorname{cosec} \tfrac{1}{2}\psi + 12 \sin \tfrac{1}{2}\psi - 32 \sin^2 \tfrac{1}{2}\psi + \frac{3}{1 + \sin \tfrac{1}{2}\psi} - 12 \sin^2 \tfrac{1}{2}\psi \, ln \left(\sin \tfrac{1}{2}\psi + \sin^2 \tfrac{1}{2}\psi \right) \right]$$

The component of the deflection in the east-west directions η'', can be obtained from (12) by replacing $\cos\alpha$ by $\sin\alpha$. Vening Meinesz also gives the generalized Stokes' function. The disturbing potential T outside the geoid at a distance ρ is given in terms of gravity anomaly Δg on the geoid, which for purposes of integration is simplified to a sphere of radius R. Then

$$T = \frac{R}{4\pi} \int \sum_{n=2}^{\infty} \left(\frac{2n+1}{n-1} \right) P_n (\cos\vartheta) \left(\frac{R}{\rho} \right)^{n+1} \Delta g \, d\sigma \tag{13}$$

where $$\sum_{n=2}^{\infty} \left(\frac{2n+1}{n-1} \right) P_n (\cos\vartheta) \left(\frac{R}{\rho} \right)^{n+1} = 2 \frac{R}{r} + \frac{R}{\rho} - 5 \left(\frac{R}{\rho} \right)^2 \cos\psi - 3 \frac{Rr}{\rho^2} - - 3 \left(\frac{R}{\rho} \right)^2 \cos\psi \, ln \frac{\rho - R\cos\psi + r}{2\rho}$$

This "generalized" Stokes' function reduces to the well known Stokes' function (see formula (7)) by the substitution $\rho = R$ which gives $r = 2 R \sin \tfrac{1}{2}\psi$.

As already mentioned in connection with the calculation of geoidal heights, a number of scientists have considered the problem of gravity reduction from the measured point to the geoid level. Below are a few of the well known methods of gravity reduction:

- Free air
- Bouguer
- Helmert condensation
- Rudzki inversion
- Isostatic based on the hypotheses of:
  - Pratt-Hayford
  - Airy-Heiskanen
  - Vening Meinesz.

In this work only the free air method of reduction is used; for information on other methods refer to Heiskanen (1958).

To summarize, for geodetic work the international ellipsoid was accepted as the reference surface. The connection between geodetic and gravimetric work was established when the international gravity formula was derived, which gives the value of gravity on the surface of the international ellipsoid. Stokes' formula was generalized to give the disturbing potential outside the geoid, and an expression for the relation between gravity and the plumb-line deflection was thus found.

Idelson and Malkin (1931) obtained the fundamental formula of Stokes as the solution of a boundary value problem of potential theory.

Jeffreys (1931) made a very important advance when, starting from Green's theorem, he derived, in a different way, Stokes' formula for the geoidal height and pointed out that the free air gravity anomaly is the appropriate one to use for the calculation.

Malkin (1933) derived formulae to calculate the gravity anomaly Δg and the geoidal height N from the plumb-line deflections and gave the expression to determine the curvature of the geoid from gravity anomalies.

Hirvonen (1934) discussed the history of the theory (Stokes, Helmert, Pizetti, Hopfner) and in the second part of his work he made practical application of Stokes' formula. He estimated the continental undulation of the geoid and in agreement with Helmert it was found to be of the order of ± 50 metres. He introduced a fixed "square" system to carry out the summation required to obtain the geoidal height. The accuracy of the solution was also estimated.

From about this time on, mainly two different treatments of the problem can be observed:

(a) Starting from Stokes' and Vening Meinesz's formulae, the main problem is to reduce the gravity anomaly to the geoid. The calculated values, geoidal height and plumb-line deflection, are obtained at geoid level. To compare astro-geodetic and gravimetric deflections, it is necessary to reduce the former from the physical surface to geoid level. Heiskanen and his co-workers followed this procedure. For further detail along this line reference is made to Heiskanen (1958) and to the reports and publications of the Institute of Geodesy, Photogrammetry and Cartography, Ohio State University, Columbus, Ohio. Of particular interest are two papers:

Tanni (1948) using the available gravity material applied Stokes' formula to obtain the continental undulation of the geoid. Details of calculations are described and the result is given in the form of geoidal contours.

Rice (1952) calculated the plumb-line deflections for a few selected stations where the astro-geodetic values were known. From the comparison, errors in the astro-geodetic values were discovered. This shows the high accuracy of the gravimetrically calculated deflections.

(b) Starting from Jeffreys (1931) a solution of the problem was developed in which only free air anomalies are required. The values obtained from calculation are at the physical surface, not at geoid level. Since it is intended to mainly follow this treatment, some details are given about the work of others along this line.

Idelson (1933) put the disturbing potential T in the form of an integral equation, which he solved. The solution is given in the form of a Neumann-series and the first approximation for T is given as:

$$T = \frac{1}{2\pi} \int \frac{g - \gamma_0}{r}\, d\sigma \equiv \int \frac{\nu}{r}\, d\sigma \tag{14}$$

where $\nu = \dfrac{g - \gamma_0}{2\pi}$ is the surface density of the simple layer placed on a sphere.

Moisseiev (1934) determined the disturbing potential outside and inside the geoid and found the solution of the integral equation for the sum of the two potentials.

Malkin (1935) assumed the disturbing potential T in the form of surface coating, then he calculated the separation between two equipotential surfaces of different density distribution (of the same mass) and as a special case Stokes' solution is discussed. The solution for a triaxial ellipsoid is in the form of a series of Lame functions with variable coefficients.

Kasansky (1935) considered the practical determination of the deflection from gravimetric data. From Vening Meinesz's formula he obtained an approximate expression for small values of ψ and applied it to gravity observations for a region near Moscow.

Tsuboi and Fuchida (1937) initiated a new approach to the problem: assuming that surface densities $\rho(y)$ responsible for the gravity values $g(x)$ are placed along a line y parallel to x at depth d (*see* Figure 2), and furthermore if

$$\rho = \rho_n \cos ny$$

then $$g(x) = 2k^2\rho_n d \int_{-\infty}^{\infty} \frac{\cos ny}{(y - x)^2 + d^2}\, dy$$

i.e. $$g(x) = 2\pi\, k^2\rho_n\, e^{-nd} \cos nx$$

where k^2 is the gravitational constant.

The general formula for $g(x)$ is in the form

$$g(x) = 2\pi\, k^2 \left[\sum A_n e^{-nd} \cos nx + \sum B_n e^{-nd} \sin nx\right] \tag{15}$$

Tsuboi (1937) also gives an expression for the deflection of the vertical $\Delta\theta$. The n-th term contributes:

$$\Delta\theta = \frac{1}{g}\frac{\partial V}{\partial x} = -\frac{2\pi\, k^2\, \rho_n}{g}\, e^{-nd} \sin nx \tag{16}$$

and for the geoidal height h:

$$h = \int \Delta\,\theta\, dx = \frac{2\pi\, k^2\, \rho_n}{g}\, e^{-nd} \cos nx - h_0 \tag{17}$$

In a series of papers—Tsuboi and Fuchida (1937, 1938); Tsuboi (1937, 1938, 1939, 1948, 1952, 1954a, 1954b, 1959a, 1959b, 1959c, 1959d, 1961a, 1961b); Tsuboi, Kaneko, Miyamura, and Yabasi (1939); Tsuboi and Kato (1952); Tsuboi, and Hayatu (1955); Tomoda and Aki (1955); Tsuboi and Tomoda (1958); Tsuboi, Oldham and Waithman (1958); Shimazu (1962)—the method was extended, applied to practical cases and numerical tables were given to facilitate the calculations.

Mihal (1939) critized the wrongful interpretation of the free air gravity anomaly and then interpreted it in the following manner:

"Normal gravity at point A, lying on the physical surface of the Earth, is equal to:

$$\gamma_s = \frac{2\,\gamma_0}{R}\,(H + N)$$

"Consequently, the difference between the gravity of the Earth and normal gravity at point A is determined by the formula

$$G + \frac{2\,\gamma_0}{R}\,(H + N) - \gamma_s$$

"If we discard correction $2\gamma_0 N/R$, we shall obtain Faye's anomaly, i.e. obtain the difference between the gravity of the Earth at point A and the normal gravity at a point lying at distance N above or below point A, according to the sign of N."

This is the first correct interpretation of the free air gravity anomaly known to the writer.

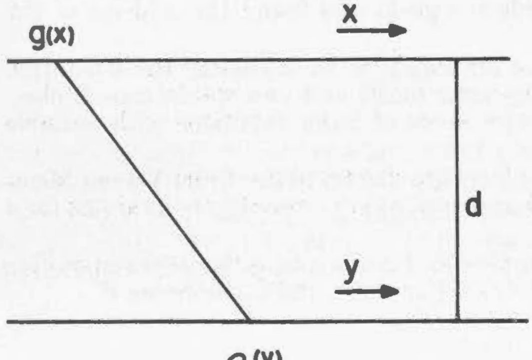

Figure 2

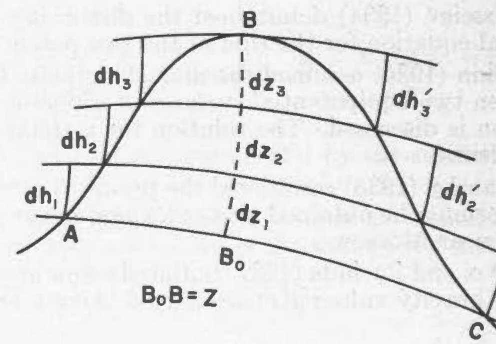

Figure 3

Molodensky (1945), in his fundamental work, summarized the results of the study of the figure of the earth and derived a general solution. From this he obtained Malkin's and Moisseiev's solutions as special cases. Starting from Green's theorem he obtained an integral equation in a curvilinear coordinate system. The integral equation is a linear integral equation of Fredholm type of the second kind with inhomogeneous term. The uniqueness of the solution was also investigated. He considered the joint use of the astro-geodetic and gravimetric data and discussed the error of the reduction and interpolation of the gravity anomalies.

Molodensky (1948) introduced the *quasi-geoid* which hardly deviates more than 2 metres from the geoid. The quasi-geoid height ζ can be obtained from the disturbing potential T *at the physical surface* by dividing it through by γ, i.e.:

$$\zeta = \frac{T}{\gamma}$$

The disturbing potential satisfies the following boundary condition:

$$-\,(g - \gamma) = \left(\frac{\partial T}{\partial \nu} - \frac{T}{\gamma}\;\frac{\partial \gamma}{\partial \nu}\right)_H$$

where ν is the direction of the normal,
H is the height above the reference surface.

The left hand side of the above equation is the free air anomaly at the physical surface. The disturbing potential is assumed to be in the form:

$$T = \int \frac{\varphi}{r} \, dS$$

where φ is the surface density and it is obtained by solving the following integral equation:

$$2\,\pi\,\varphi \cos\alpha = (g - \gamma) + \frac{3}{2\,\rho_0} \int \frac{\varphi}{r} \, dS + \frac{1}{2\,\rho_0} \int \frac{\rho^2 - \rho_0^2}{r^3} \varphi \, dS \tag{18}$$

where ρ_0 is the radius vector to the fixed point P,
ρ is the radius vector to the moving point Q,
r is the distance from P to the point where ρ intersects the reference sphere.

Molodensky also introduced the orthogonal coordinate system of latitude B^*, longitude L^*, and height H above the reference surface measured along the normal plumb-line. If the approximate values of the coordinates of a point on the physical surface are B, L and H respectively, then the differences

$$\Delta B = B^* - B$$

$$\Delta L = L^* - L$$

$$\zeta = H^* - H$$

are so small, that the second powers or the products of these quantities can be neglected. Then from the Taylor series expansion he obtained the following formula for the disturbing potential at any point of the physical surface:

$$T\,(B^*, L^*, H^*) = W\,(B^*, L^*, H^*) - U\,(B^*, H^*) = \gamma\,(B, H)\,\zeta + W_0 - U_0 \tag{19}$$

Thus Molodensky obtained a solution to determine the shape of the earth. This solution is free of assumptions and no reduction problems arise: the free air gravity anomaly at the physical surface must be used and the values obtained from the calculations belong also to the physical surface and not to the geoid.

Eremeev (1950) applying the method of models compared the solution of Molodensky with that of Stokes and Vening Meinesz. He concludes that the result of the calculation by the Vening Meinesz formula also gives the deflection at the physical surface and not at the geoid as previously interpreted. He also showed that even the first approximation of Molodensky's formula gives a better result than can be obtained from Vening Meinesz's formula.

In the past few years considerable research has been carried out on this problem and a partial list of the contributors follows:

Almquist (1959); Arnold (1956a, 1956b, 1959a, 1959b, 1960, 1961); Bjerhammar (1959, 1960a, 1960b, 1960c, 1961); Bragard (1958); de Graaff-Hunter (1957, 1958); Hirvonen (1960); Levallois (1958); Molodensky, Eremeev and Yurkina (1962a, 1962b).

Discussion of Levelling Corrections

Since the result of levelling enters into the calculations it is worthwhile to consider how it is obtained and its relation to gravity.

As is well known the horizontal axis of the levelling instrument is parallel to the local niveau surface. Therefore, the measured height differences give the distances between local equipotential surfaces. The sum of these differences from A to B gives the height difference between the two points. If A is on the geoid then this difference gives the elevation of point B above mean sea-level. It is clear that the elevation depends on the path of levelling because of the non-parallelism of equipotential surfaces of the earth. If both A and C are on the geoid and the elevation of point B is to be determined by spirit levelling starting from both points, then the two results usually differ, (see Figure 3).

If the topography were as indicated by the dotted line (B_0, B,) then the levelling from B_0 to B would give the *orthometric height* of B. The orthometric height is defined as the distance B_0B measured along the plumb-line in natural units, say, in metres.

Because of the flattening of equipotential surfaces of the earth, the distance between two equipotential surfaces at the equator is larger than at the pole. But the work required to move a unit mass from one equipotential surface to another is the same at both the equator and at the pole. Therefore, if the work is used as the measure of height, no ambiguity arises; the points which have the same height are on the same equipotential surface. The *geopotential value C* is defined as the work required to move a unit mass from the geoid to the point in question. This is independent of the path and it is the potential difference between the point in question and the geoid. If A and B are two points not on the geoid then by definition

$$C_A = -\int_0^A gdh$$
$$C_B = -\int_0^B gdh \tag{20}$$

and $$C_{AB} = C_B - C_A = -\int_A^B gdh \tag{21}$$

Therefore the difference in the geopotential values of the two points A and B is also independent of the path.

Another measure of height already mentioned is the orthometric height which is obtained from spirit levelling plus orthometric correction, the orthometric correction being necessary because of the non-parallelism of equipotential surfaces. If L and M are two equipotential surfaces (*see* Figure 4) then taking elementary small distances the work that is performed in moving from L to M is the same along every path, i.e.

$$g_1 \Delta z_1 = g_2 \Delta z_2 \tag{22}$$

Thus neither Δz_1 nor Δz_2 can be used as the orthometric height.

Figure 3 is now used to derive the connection between the orthometric height and the geopotential number. Let A be a point on the geoid and B a point on the physical surface. Furthermore let Z be the orthometric height of B, and G the average gravity along Z. Then the geopotential value of point B is

$$C_B = -\int_0^B gdh \tag{23}$$

To obtain C, the measured height differences and the gravity values *at the physical surface* along the levelling path are required.

On the other hand the work required to move the unit mass from A to B is

$$W_{AB} = -\int_A^{B_0} gdh - \int_{B_0}^B gdz \tag{24}$$

The first term is zero since A and B_0 are on the same equipotential surface. Therefore

$$W_{AB} = -\int_{B_0}^B gdz \tag{25}$$

To obtain this quantity, g is needed inside of the mass along Z. This work is also equal to

$$-W_{AB} = Z_B G \tag{26}$$

Therefore $$Z_B = \frac{1}{G} \int gdz \tag{27}$$

Comparison of (23) and (26) gives

$$Z_B = \frac{C_B}{G} \tag{28}$$

As can be seen, only the geopotential number C can be obtained without assumption. To obtain the orthometric height Z, assumptions must be made.

It is emphasized that in this work the geopotential value has its natural place because of its relationship to the surface value of gravity. On the other hand since for this study only orthometric heights are available, and because the difference between the two heights is only of the order of centimetres, *h* in the remaining part of the study is used to denote elevation above the geoid in metre units.

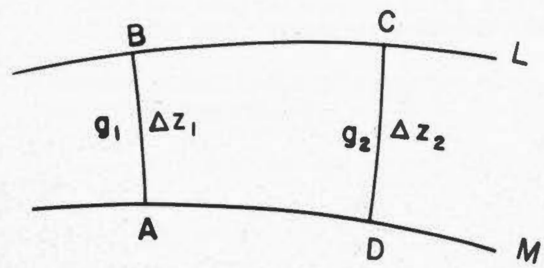

Figure 4

PART II

Statement of the Problem

The potential of the earth W is equal to the sum of the normal potential U and the disturbing potential T:

$$W = U + T \tag{29}$$

Taking the normal derivative of (29) one formally obtains:

$$\frac{\partial W}{\partial n} = \frac{\partial U}{\partial n} + \frac{\partial T}{\partial n} \tag{30}$$

But by definition

$$\frac{\partial W}{\partial n} = -g \cos (g, n) \tag{31}$$

i.e. the left hand side of (31) is the measured value of gravity on the physical surface. If n is taken along the direction of the theoretical gravity γ then (g, γ), i.e. the angle between the actual and theoretical gravity, is of the order of 30″. Therefore replacing $g \cos 30''$ by g, an error of the order of .01 milligal is introduced. Thus

$$\frac{\partial W}{\partial n} = -g \tag{32}$$

In the following pages we evaluate equation (30) at a point of the physical surface say at P. The distance h, which is the elevation of point P above mean sea-level, is measured up from the reference ellipsoid along the normal plumb-line of P. This value is obtained from levelling. Thus

$$h = -\int_0^P g dh \equiv -\int_{P_0}^{P'} g dz \tag{33}$$

i.e. h is a measure of the work required to move a unit mass from the reference surface to point P' (see Figure 5).

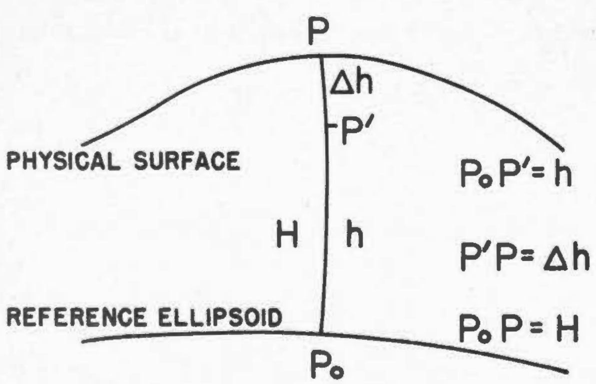

Figure 5

The distance between P' and P is Δh and is called the *height correction* by Bjerhammer. This height correction should not be confused with any correction applied to levelling, as it is an entirely different concept. The height correction is of the same order as the geoidal height, the difference between the height correction and the geoidal height being always less than 2 metres. Then if H is the distance from P_0 to P, from Figure 5

$$H = h + \Delta h \tag{34}$$

Going back to equation (30) it is necessary to evaluate the term $\partial U/\partial n$ at P. From Taylor series expansion one finds:

$$\left.\frac{\partial U}{\partial n}\right|_P = \left.\frac{\partial U}{\partial n}\right|_{P_0} + \left.\frac{\partial^2 U}{\partial n^2}\right|_{P_0} \overline{P_0P} + \ldots \tag{35}$$

In this case P_0 is on the reference surface and $P_0P \equiv H$. The first term of equation (35) is given by the international gravity formula:

$$\left.\frac{\partial U}{\partial n}\right|_{P_0} = -\gamma \tag{36}$$

where n is the direction of the normal to the international ellipsoid.

Therefore (35) takes the form:

$$\left.\frac{\partial U}{\partial n}\right|_P = -\gamma - \frac{\partial \gamma}{\partial n} H + \ldots . \tag{37}$$

For a local area the normal potential can be approximated as:

$$U = f\ \frac{M}{R} \tag{38}$$

where f is the gravitational constant,
M is the total mass of the earth, and
R is the mean radius of the earth.

If n is the normal to the international ellipsoid and R is the radius vector of the point, then the difference between the geodetic and geocentric latitudes φ and ϕ is approximately

$$\varphi - \phi = \tfrac{1}{2}\, e^2 \sin 2\, \varphi$$

$$\text{where } e^2 = \frac{a^2 - b^2}{a^2}$$

This expression reaches the maximum value at $\varphi = 45°$ which, using the parameters of the international ellipsoid is equal to:

$$(\varphi - \phi)''_{\text{max.}} = \tfrac{1}{2}\, \rho''\, e^2 = (.5)\ (206\,265)\ (.006\,722) = 12'$$

This is the approximation if n is taken along R.

$$\text{Then} \qquad \frac{\partial U}{\partial n} = \frac{\partial U}{\partial R} = -\, f\ \frac{M}{R^2} = -\, \gamma \tag{39}$$

$$\text{and} \qquad \frac{\partial^2 U}{\partial R^2} = 2\, f\ \frac{M}{R^3} = -\, 2\ \frac{\gamma}{R} \tag{40}$$

Substituting (40) into (37) we obtain:

$$\left.\frac{\partial U}{\partial n}\right|_P = -\, \gamma + 2\ \frac{\gamma h}{R} + 2\ \frac{\gamma \Delta h}{R} \tag{41}$$

Making use of Bruns' term given by (10) we have

$$\left.\frac{\partial U}{\partial n}\right|_P = -\, \gamma + \frac{2}{R}\ \gamma h + \frac{2}{R}\ T + \ldots . \tag{42}$$

The last term of (30) is $\partial T / \partial n$.

Taking the normal derivative along R and substituting (32) and (42) into (30) we obtain

$$-\, g = -\, \gamma + \frac{2}{R}\ \gamma h + \frac{2}{R}\ T + \ldots . + \frac{\partial T}{\partial R} \tag{43}$$

The first two terms on the right hand side give the theoretical gravity at point P' in Figure 5. The next term gives the correction (to the theoretical gravity) required because of the unknown distance Δh. Therefore, it can be seen that the problem can be solved only by successive approximations: first, the term $2\ T/R$ is neglected, then finding the disturbing potential T, the normal derivative of which satisfies the given boundary condition, and substituting it into (43) a better approximation can be obtained. This kind of solution requires a great deal of calculation and handling of large amounts of data and, because of other larger uncertainties in the calculation due to insufficient knowledge of gravity anomalies, this procedure has not yet been employed. Equation (43) can be given in the well known form:

$$-\, \Delta g = \frac{2}{R}\ T + \frac{\partial T}{\partial R} \tag{44}$$

where Δg is the free air gravity anomaly at the physical surface.

Thus the problem to be solved can be stated as follows: ***find a potential which satisfies the boundary condition specified by*** (44).

To solve the problem there are at least two main lines of attack:

(a) to use the result of the theory of integral equations,

(b) to use spherical harmonics.

Some work was done along the first line. The problem was transferred into a linear integral equation of Fredholm type of the second kind with inhomogenous term. Some approximations were introduced and after defining and using linear integral operators, a formal solution was obtained. It was shown that this solution is given in terms of the resolvent but is based on the assumption that the kernel can be replaced by a degenerate one. The decomposition of the specific kernel of the problem presented such a difficult mathematical problem, that it was necessary to follow the more classical approach to it. This is dealt with in the next section.

Calculation of the Deflection

Let U_1 and W_1 be two equipotential surfaces and T the potential difference between them. Then Bruns' term gives an approximation for the distance between U_1 and W_1:

$$\Delta h = \frac{T}{\gamma \cos \varepsilon} \doteq \frac{T}{g} \doteq \frac{T}{\gamma}$$

Taking the partial derivative of Δh along an aribtrary direction ψ we obtain the deflection of vertical θ along that direction:

$$\theta = \frac{\partial}{\partial \psi} (\Delta h) = \frac{\partial \dfrac{T}{\gamma}}{\partial \psi} = \frac{1}{\gamma} \frac{\partial T}{\partial \psi} \tag{45}$$

If α is the azimuth of this direction measured from the X axis as shown in Figure 6, then the components of deflection along X and Y are:

$$\xi = \theta \cos \alpha = \frac{1}{\gamma} \frac{\partial T}{\partial \psi} \cos \alpha \tag{46}$$

$$\eta = \theta \sin \alpha = \frac{1}{\gamma} \frac{\partial T}{\partial \psi} \sin \alpha \tag{47}$$

To find ξ and η, first it is necessary to find an expression for T as a function of the gravity anomaly. As is known, the solution of Laplace's equation in a spherical system having aximuthal isotropy consists in functions of the kind

$$V = A_n P_n r_n + B_n P_n r^{-n-1}$$

where A_n and B_n are constants and
P_n is the Legendre polynominal of degree n and of argument $\cos \vartheta$.
where ϑ is the angle between the fixed and moving points.

If the solution is required outside or on the boundary of the attracting mass and furthermore to be harmonic and regular at infinity, then $A = 0$ and V can be given as:

$$V = \sum_{n=0}^{\infty} \frac{f_n}{\rho^{n+1}} \tag{48}$$

where f_n is a spherical harmonic of degree n, and
ρ is the radius vector.

Equation (48) can be applied for the disturbing potential, noting that the summation starts from $n=2$ because $T = W - U$ and the expansions for W and U differ only from this term onwards.* Therefore we can write:

$$T = \sum_{n=2}^{\infty} \frac{f_n}{\rho^{n+1}} \tag{49}$$

*The zero and first order terms in W and U are equal due to the choice of the reference surface and coordinate system.

Then $$\frac{\partial T}{\partial \rho} = -\sum_{n=2}^{\infty} \frac{(n+1)\, f_n}{\rho^{n+2}} \tag{50}$$

From (44) we obtain

$$-\Delta g = \frac{1}{\rho} \sum_{n=2}^{\infty} \frac{2\, f_n}{\rho^{n+1}} - \sum_{n=1}^{\infty} \frac{(n+1)\, f_n}{\rho^{n+2}}$$

i.e. $$\Delta g = \sum_{n=2}^{\infty} \frac{(n-1)\, f_n}{\rho^{n+2}} \tag{51}$$

Applying the Fourier technique to determine f_n we obtain:

$$f_n = \frac{2n+1}{4\pi} \int_{\sigma} \frac{\rho^{n+2}}{n-1}\, \Delta g\, P_n\, d\sigma \tag{52}$$

Here ρ is the radius vector to the physical surface, $d\sigma$ is the surface element of unit sphere and Δg is the gravity anomaly at Q (Figure 7).

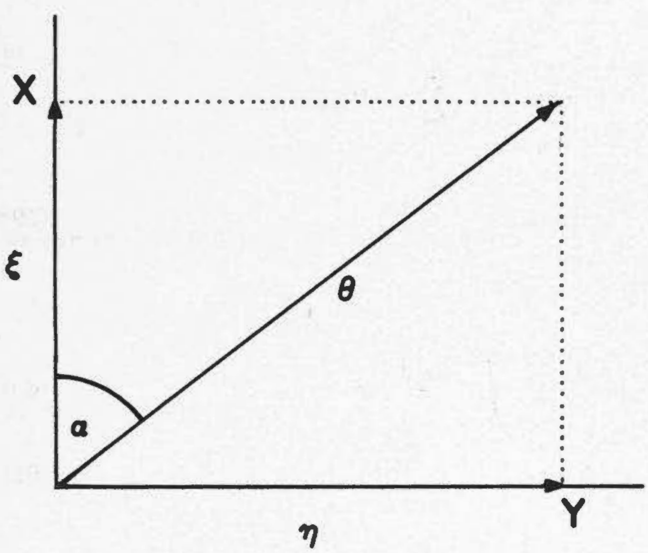

Figure 6

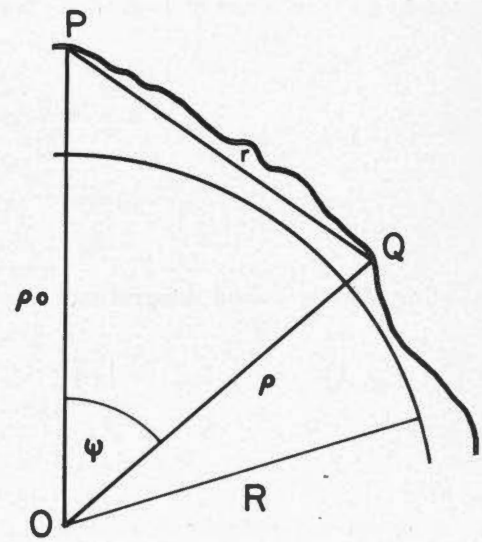

Figure 7

If the disturbing potential is required at point P whose radius vector is ρ_0, then

$$T = \sum_{n=2}^{\infty} \frac{f_n}{\rho_0^{\,n+1}}$$

and using (52) we obtain:

$$T = \sum_{n=2}^{\infty} \frac{2n+1}{4\pi} \int \frac{\rho^{n+2}}{\rho_0^{\,n+1}}\, \frac{1}{n-1}\, \Delta g\, P_n\, d\sigma$$

or $$T = \frac{\rho_0}{4\pi} \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \int \left(\frac{\rho}{\rho_0} \right)^{n+2} P_n\, \Delta g\, d\sigma$$

Interchanging the order of summation and integration,

$$\mathrm{T} = \frac{\rho_0}{4\pi} \int \sum_{n=2}^{\infty} \frac{2n+1}{n-1}\, P_n \left(\frac{\rho}{\rho_0} \right)^{n+2} \Delta g\, d\sigma \tag{53}$$

Letting $$x = \frac{\rho}{\rho_0} \tag{54}$$

and using the identity

$$\frac{2n+1}{n-1} = 2 + \frac{3}{n-1} \tag{55}$$

(53) becomes $$T = \frac{\rho_0}{4\pi} \int \sum_{n=2}^{\infty} \left(2 + \frac{3}{n-1} \right) P_n \, x^{n+2} \, \Delta g \, d\sigma \tag{56}$$

From Figure 7 $$\begin{aligned} r &= (\rho_0^2 + \rho^2 - 2\,\rho\,\rho_0 \cos\psi)^{1/2} \\ &= \rho_0 \, (1 + x^2 - 2\,x \cos\psi)^{1/2} \\ &= \rho_0 \, X \end{aligned} \tag{57}$$

$$\text{where } X = (x^2 - 2\,x\cos\psi + 1)^{1/2} = (a\,x^2 + b\,x + c)^{1/2} \tag{58}$$

Expanding $1/r$ in series of Legendre's polynomials P_n, we obtain:

$$\frac{1}{r} = \frac{1}{\rho_0 X} = \frac{1}{\rho_0} \sum_{n=2}^{\infty} x^n P_n = \frac{1}{\rho_0} \left(1 + x\cos\psi + \sum_{n=2}^{\infty} x^n P_n \right)$$

or $$\sum_{n=2}^{\infty} x^n P_n = \frac{1}{X} - 1 - x\cos\psi \tag{59}$$

Dividing (59) by x^2 and integrating

$$\int \sum_{n=2}^{\infty} x^{n-2} P_n \, dx = \int \frac{dx}{x^2 X} - \int \frac{dx}{x^2} - \int \frac{\cos\psi}{x} \, dx \tag{60}$$

i.e. $$I = I_1 + I_2 + I_3 \tag{61}$$

Carrying out the integration we obtain

$$I = \sum_{n=2}^{\infty} \frac{x^{n-1}}{n-1} \, P_n$$

$$I_1 = - \frac{X}{cx} + \frac{b}{2c} \quad \frac{1}{\sqrt{c}} \, ln \left(\frac{2\,\sqrt{c}\,X}{x} + \frac{2c}{x} + b \right) + C_1$$

$$I_2 = \frac{1}{x} \qquad I_3 = - \cos\psi \, ln\, x$$

Substituting the values of these integrals into (60) and the coefficients b and c from (58), we obtain

$$\sum_{n=2}^{\infty} \frac{x^{n-1}}{n-1} \, P_n = - \frac{X}{x} - \cos\psi \, ln \left(\frac{2X}{x} + \frac{2}{x} - 2\cos\psi \right) + C_1 + \frac{1}{x} - \cos\psi \, ln\, x$$

or

$$\sum_{n=2}^{\infty} \frac{x^{n-1}}{n-1} \, P_n = \frac{1 - X}{x} - \cos\psi \, ln\, 2 \left(1 + X - x\cos\psi \right) + C_1 \tag{62}$$

To determine the constant of integration let $x = 0$. To find the first term of the right hand side we must evaluate the limit when x approaches zero:

$$\lim_{x \to 0} \frac{1 - X}{x} = \lim_{x \to 0} \frac{-\dfrac{x - \cos\psi}{X}}{1} = \cos\psi$$

Then from (62):

$$C_1 = -\cos\psi + \cos\psi \, ln\, 4 \tag{63}$$

Using this value (62) takes the final form:

$$\sum_{n=2}^{\infty} \frac{x^{n-1}}{n-1} P_n = \frac{1 - X}{x} - \cos\psi \left(1 + ln \frac{1 + X - x\cos\psi}{2}\right) \tag{64}$$

Multiplying (59) by $2\,x^2$, (64) by $3\,x^3$ and then substituting them into (56), we get the following expression for T:

$$T = \frac{\rho_0}{4\pi} \int\left\{2x^2\left(\frac{1}{X} - 1 - x\cos\psi\right) + 3x^3\left(\frac{1 - X}{x} - \cos\psi\left(1 + ln \frac{1 + X - x\cos\psi}{2}\right)\right)\right\} \Delta g\, d\sigma \tag{65}$$

or

$$T = \frac{\rho_0}{4\pi} \int x^2\left\{\frac{2}{X} + 1 - 3\,X - 5x\cos\psi - 3x\cos\psi\, ln \frac{1 + X - x\cos\psi}{2}\right\} \Delta\, g d\sigma \tag{66}$$

Remembering that

$$X = \frac{r}{\rho_0}$$

we can rewrite (66) into the following form:

$$T = \frac{\rho_0}{4\pi} \int x^2 \left\{\frac{2\,\rho_0}{r} + 1 - \frac{3r}{\rho_0} - 5x\cos\psi - 3x\cos\psi\, ln \frac{\rho_0 + r - \rho_0\, x\cos\psi}{2\,\rho_0}\right\} \Delta\, g\, d\sigma \tag{67}$$

Then letting $\rho = R$ we have

$$T = \frac{R}{4\pi} \int R \left\{\frac{2}{r} + \frac{1}{\rho_0} \frac{3r}{\rho_0^2} - \frac{5R}{\rho_0^2} \cos\psi - \frac{3R}{\rho_0^2} \cos\psi\, ln \frac{\rho_0 + r - R\cos\psi}{2\,\rho_0}\right\} \Delta\, g\, d\sigma \tag{68}$$

Letting $\rho_0 = R$, then $r = 2R \sin \frac{1}{2}\,\psi$ and we obtain

$$T = \frac{R}{4\pi} \int\left[\frac{1}{\sin\frac{1}{2}\psi} + 1 - \sigma \sin\tfrac{1}{2}\psi - 5\cos\psi - 3\cos\psi\, ln\, (\sin\tfrac{1}{2}\psi + \sin^2\tfrac{1}{2}\psi)\right] \Delta\, g\, d\sigma \tag{69}$$

where the expression in the square bracket is the well known Stokes function. If $d\sigma$ is the surface element of the unit sphere and dS is that of a sphere of radius R, then

$$d\sigma = \frac{dS}{R^2}$$

and using this relation (68) becomes

$$T = \frac{1}{4\pi} \int\left[\frac{2}{r} + \frac{1}{\rho_0} - \frac{3r}{\rho_0^2} \cdot \frac{5R}{\rho_0^2} \cos\psi - \frac{3R}{\rho_0^2} \cos\psi\, ln \frac{\rho_0 + r - R\cos\psi}{2\,\rho_0}\right] \Delta\, g\, dS \tag{70}$$

or

$$T = \frac{1}{4\pi} \int K(\rho_0, \psi)\, \Delta\, g\, dS \tag{71}$$

where $$K(\rho_0, \psi) = \frac{2}{r} + \frac{1}{\rho_0} - \frac{3r}{\rho_0^2} - \frac{5R}{\rho_0^2} \cos\psi - \frac{3R}{\rho_0^2} \cos\psi \, ln \frac{\rho_0 + r - R\cos\psi}{2\,\rho_0} \tag{72}$$

From (45) $$\theta = \frac{1}{\gamma} \frac{\partial T}{\partial \psi} = \frac{1}{4\pi\gamma} \frac{\partial}{\partial \psi} \int K(\rho_0, \psi) \, \Delta \, dgS \tag{73}$$

Formally the solution is given as

$$\theta = \frac{1}{4\pi\gamma} \int K'(\rho_0, \psi) \, \Delta \, g \, dS \tag{74}$$

where $$K'(\rho_0, \psi) = \lim_{\Delta\psi \to 0} \frac{K(\rho_0, \psi + \Delta\psi) - K(\rho_0, \psi)}{\rho_0 \, \Delta\psi}$$

that is, the derivative is taken in a direction perpendicular to ρ_0,

i.e. $$K'(\rho_0, \psi) = \frac{\partial K(\rho_0, \psi)}{\rho_0 \, \partial\psi} \tag{75}$$

Differentiating (72) term by term and remembering that r is a function of ψ, we obtain

$$K'(\rho_0, \psi) = -\frac{2}{r^3} R\sin\psi - \frac{3}{\rho_0^2 r} R\sin\psi + \frac{5}{\rho_0^3} R\sin\psi$$

$$+ \frac{3}{\rho_0^3} R\sin\psi \, ln \frac{r + \rho_0 - R\cos\psi}{2\,\rho_0} - \frac{3}{\rho_0^3} R\cos\psi \frac{R\rho_0 \sin\psi + rR\sin\psi}{r(r + \rho_0 - R\cos\psi)} \tag{76}$$

From this expression Vening Meinesz's formula results by letting $\rho_0 = R$. In (74) we need

$$K'(\rho_0, \psi) \, dS = K'(\rho_0, \psi) \, R^2 \sin\psi \, d\,\psi \, d\theta \tag{77}$$

and making use of (76) we obtain, for a sphere of radius R, Vening Meinesz's formula:

$$K'(R, \psi) R^2 \sin\psi = -\cos^2 \tfrac{1}{2}\psi \left[\frac{1}{\sin \frac{1}{2}\psi} + \sigma \sin \tfrac{1}{2}\psi - 20 \sin^2 \tfrac{1}{2}\psi \right.$$

$$\left. - 12 \sin^2 \tfrac{1}{2}\psi \, ln\left(\sin \tfrac{1}{2}\psi + \sin^2 \tfrac{1}{2}\psi \right) + 3\cos\psi \frac{1 + 2\sin \frac{1}{2}\psi}{1 + \sin \frac{1}{2}\psi} \right] \tag{78}$$

The last term T_5 of this expression can be put into a simpler form. Using the identity

$$\cos\psi = 1 - 2\sin^2 \tfrac{1}{2}\psi$$

then $$T_5 = \frac{3 - 6\sin^2 \frac{1}{2}\psi + 6\sin \frac{1}{2}\psi - 12\sin^3 \frac{1}{2}\psi}{1 + \sin \frac{1}{2}\psi}$$

Adding to and subtracting $6 \sin^2 \frac{1}{2}\psi$ from the numerator, we obtain

$$T_5 = \frac{3 - 6\sin^2 \frac{1}{2}\psi + 6\sin^2 \frac{1}{2}\psi + 6\sin \frac{1}{2}\psi - 12\sin^3 \frac{1}{2}\psi}{1 + \sin \frac{1}{2}\psi}$$

$$= \frac{3 - 12 \sin^2 \frac{1}{2}\psi \left(1 + \sin \frac{1}{2}\psi\right) + 6 \sin \frac{1}{2} \left(1 + \sin \frac{1}{2}\psi\right)}{1 + \sin \frac{1}{2}\psi}$$

i.e.

$$T_5 = \frac{3}{1 + \sin \frac{1}{2}\psi} - 12 \sin^2 \tfrac{1}{2}\psi + 6 \sin \tfrac{1}{2}\psi \tag{79}$$

Substituting this value back into (78), we have:

$$D(R, \psi) = K'(R, \psi) R^2 \sin \psi = \cos^2 \tfrac{1}{2}\psi \left[\frac{1}{\sin \frac{1}{2}\psi} + 12 \sin \tfrac{1}{2}\psi - 32 \sin^2 \tfrac{1}{2}\psi \right.$$

$$\left. + \frac{3}{1 + \sin \frac{1}{2}\psi} - 12 \sin \tfrac{1}{2}\, ln \left(\sin \tfrac{1}{2}\psi + \sin^2 \tfrac{1}{2}\psi \right) \right] \tag{80}$$

The expression in the square bracket is given on page 10.

From (74)

$$\theta = \frac{1}{4\pi\gamma} \int \Delta g\, D(\rho_0, \psi)\, d\psi d\alpha \tag{81}$$

where

$$D(\rho_0, \psi) = K'(\rho_0, \psi) R^2 \sin \psi$$

If we expand $D(\rho_0, \psi)$ about a sphere of radius R, we have:

$$D(\rho_0, \psi) = D(R, \psi) + \left. \frac{\partial D(\rho_0, \psi)}{\partial \rho_0} \right|_R H + \left. \frac{\partial^2 D(\rho_0, \psi)}{\partial \rho_0^2} \right|_R \tfrac{1}{2} H^2 + \dots \tag{82}$$

where

$$H = \rho_0 - R$$

Since the value of $D(R, \psi)$ at $\psi = 1°$ is

$$D(R, \psi) = \frac{206,265}{980,000} \left[\frac{1}{.0087} + 12(.0087) - 32(.0087)^2 + \frac{3}{1.0087} - 12(.0087)\, ln\, (.0087 + (.0087)^2) \right] \doteq 25$$

and an upper bound for the coefficient of H is about $-\dfrac{1}{100}$.

Taking the maximum H as 10 km, the contribution due to the second term of the right hand side of (82) would be of the order of

$$-\frac{1}{10},$$

i.e. in the ratio 1:250. As our calculation is not intended to reach this order of accuracy, at present only the first term on the right hand side of (82) is used.

To summarize the result: generalized forms to calculate the height corrections and plumb-line deflections at the physical surface have been obtained. These forms reduce to those of Stokes and Vening Meinesz respectively for the sphere of radius R. When the topographic model is known for the area under study, then the corrections due to the non-spherical properties of the earth can be obtained for the calculation of deflection. This correction is small for low elevations. For practical purposes Stokes' and Vening Meinesz's formulae still can be used and usually satisfy the required accuracy. However, we want to emphasize that,

a) the values obtained by using either Stokes' or Vening Meinesz's formula belong to the physical surface. Therefore the gravimetrically determined deflection values are directly comparable with the astro-geodetic deflections without reducing the latter to mean sea level;

b) the values obtained by using either Stokes' or Vening Meinesz's formula are only first approximations. To obtain corrections to this approximate solution requires a great deal of labour: to each point, weight must be attached which depends on the elevation of the point. On the other hand, uncertainties are being introduced owing to incomplete knowledge of the gravity anomalies over the whole earth, and therefore it is questionable whether the corrections for elevation have any significance as yet.

It is clear that in the study of the shape of the earth the theoretical development is more advanced than the practical application of it. The problem is now to apply the results of the theoretical studies and to work out practical procedures for the calculation. The procedure should be flexible in order that different systems of weightings may be incorporated easily. Because of the large amount of data-handling the problem is well suited for a digital computer: with a standard program, the calculation can be repeated in minutes or hours depending on the equipment; changes can be made according to request, and calculations can keep pace with the receipt of new data. By using machine procedures the plotting of points and the contouring are eliminated.

In the following pages some details about the programming of the problem are given.

PART III

Outline of Area

The area for which the calculations were carried out is shown in Figure 8.

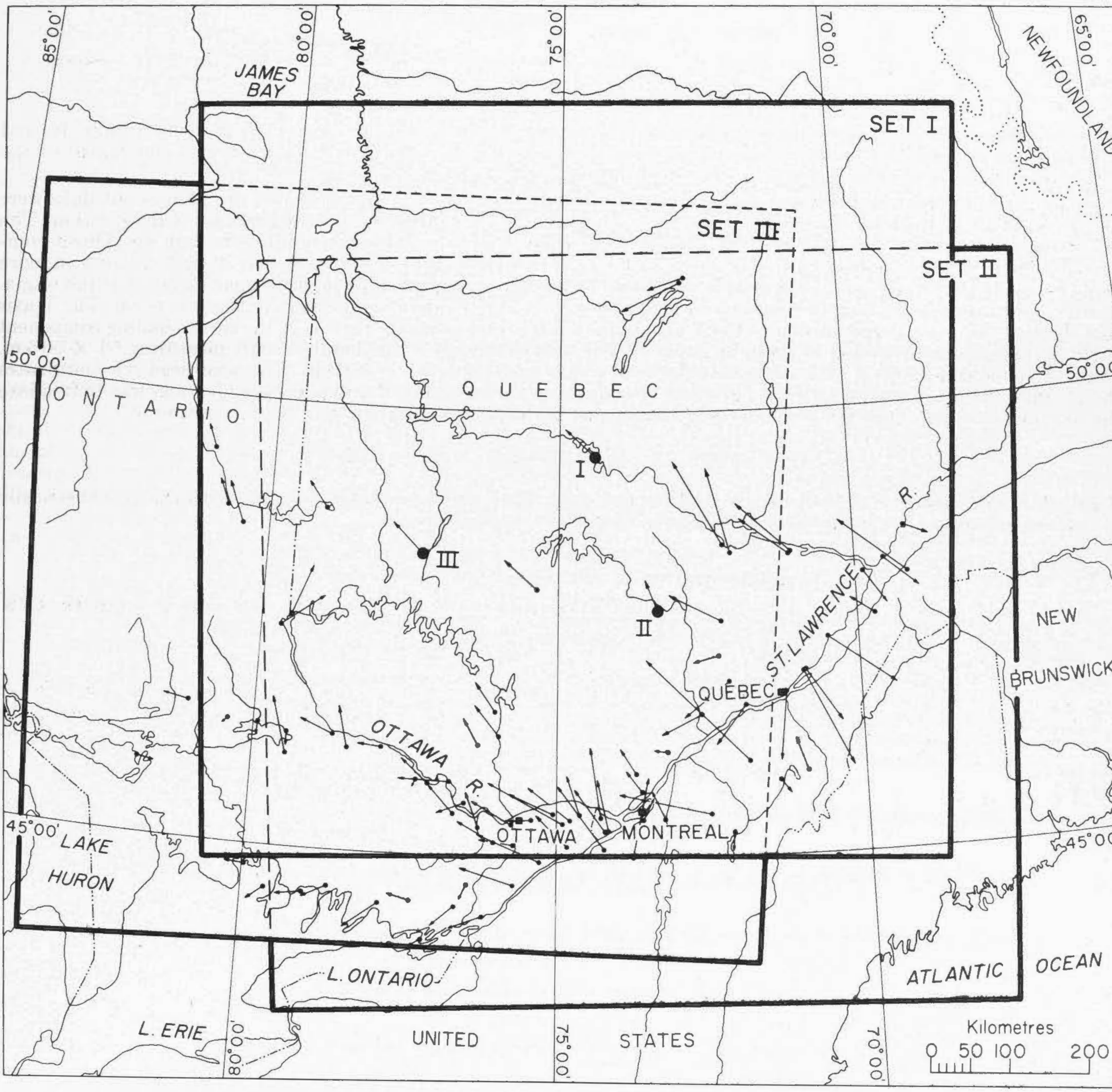

Figure 8

Three overlapping sets were used, Sets I, II and III. The coordinates and the deflections at the three origins are given in Table I.

TABLE I

No.	Set	φ		λ		ξ	η
306	I	40° 30′	14″.80	74° 23′	02″.34	2″.34	2″.38
150	II	47° 45′	44″.14	73° 19′	17″.70	2″.44	″.23
128	III	48° 23′	23″.72	77° 20′	40″.01	2″.40	2″.54

The first column gives the station numbers and the second one identifies the origins. The geodetic coordinates and the deflection components follow. These values refer to Clarke's spheroid of 1866. The east-west component of the deflection is positive towards the west.

The three different sets serve as a check on the calculations; by shifting the origins, the basic input data were regrouped. This introduced changes in the number of points per unit area and affected the point distribution. The central part of the area—around the three origins—should give basically the same result from each set. Direct comparison of any two sets is possible. The same coordinate system was used for each set as is described below in more detail. A plane Cartesian coordinate system was used. This plane is tangential to international ellipsoid at the origin. The X coordinate axis is along the meridian, which passes through the origin and is positive towards north. The Y axis goes through the origin perpendicular to the X axis and is positive eastward (the reverse of the corresponding component of the astro-geodetic deflection as given in Table I). The area of any set is divided into units measuring 50 × 50 km. The number of grid-points is 19 × 19 so that the total area of each set is 950 × 950 km. To avoid negative coordinates the origin of the plane system (after the geodetic coordinates were transformed into plane coordinates) was shifted into the southwest corner. Thus the coordinates of the origins of the sets in the plane coordinate system are

$$X_i = 500 \text{ km} \qquad Y_i = 500 \text{ km} \qquad i = \text{I, II, III}$$

Eighty-five astro-geodetic deflections are used for this area. They are shown later, together with the gravimetrically calculated deflections. There are about 6,000 gravity stations for the area.

Transformation of Astro-geodetic Deflections

In Canada the astro-geodetic deflections refer to Clarke's spheroid of 1866, the parameters of which are given below:

$$a = 6\,378\,206.4 \text{ metres} \qquad 1/\alpha = 294.98$$

The gravimetric deflections refer to the international ellipsoid. For purposes of comparison the two sets must be referred to the same reference system. Since the international ellipsoid is recommended for future work for all countries the astro-geodetic values were transformed. The first transformation takes into account the change in the parameters of the two ellipsoids. These formulae were derived by Vening Meinesz (1950b) and are given below:

$$\begin{aligned}
\Delta\xi_1'' = {} & \rho''\left[\sin(\varphi_1-\varphi_0) - 2\cos\varphi_0 \sin\varphi_1 \sin^2\tfrac{1}{2}(\lambda_1-\lambda_0)\right]\Delta\beta - 4\rho''\cos\varphi_1\cos\tfrac{1}{2}(\varphi_1-\varphi_0)\sin\tfrac{1}{2}(\varphi_1-\varphi_0)\,\Delta\alpha \\
& + \rho''\Big[\Big(2 - 3\sin^2\varphi_1 - \cos(\varphi_1+\varphi_0)\Big)\sin(\varphi_1-\varphi_0) - 2\cos\varphi_0\sin\varphi_1\,(2+\sin^2\varphi_0 \\
& - 3\sin^2\varphi_1)\sin^2\tfrac{1}{2}(\lambda_1-\lambda_0) - \sin 2\varphi_0 \sin^2\tfrac{1}{2}(\varphi_1-\varphi_0) - 2\cos\varphi_1\cos\tfrac{1}{2}(\varphi_1+\varphi_0)\sin\tfrac{1}{2}(\varphi_1-\varphi_0)\Big]\alpha\Delta\beta \\
& + \rho''\Big[\tfrac{1}{2}\sin 4\varphi_0\Big(\sin^2\tfrac{1}{2}(\varphi_1-\varphi_0) + \sin\varphi_0\sin\varphi_1\sin^2\tfrac{1}{2}(\lambda_1-\lambda_0)\Big) \\
& - \sin\varphi_0\sin\tfrac{3}{2}(\varphi_1+\varphi_0)\sin\tfrac{3}{2}(\varphi_1-\varphi_0) - \sin\varphi_0\sin\tfrac{1}{2}(\varphi_1+\varphi_0)\sin\tfrac{1}{2}(\varphi_1-\varphi_0) \\
& - 4\cos\varphi_1\cos\tfrac{1}{2}(\varphi_1+\varphi_0)\sin\tfrac{1}{2}(\varphi_1-\varphi_0) + 4\sin 2\varphi_1\cos^2\tfrac{1}{2}(\varphi_1+\varphi_0)\sin^2\tfrac{1}{2}(\varphi_1-\varphi_0)\Big]\alpha d\alpha
\end{aligned} \tag{83}$$

$$\begin{aligned}
\Delta\eta_1'' = {} & -\rho''\cos\beta_0\sin(\lambda_1-\lambda_0)\,\Delta\beta + \rho''\cos\varphi_0\sin(\lambda_1-\lambda_0)\Big[\sin(\varphi_1-\varphi_0)\sin(\varphi_1+\varphi_0)\,\alpha\Delta\varphi \\
& + \tfrac{1}{4}\operatorname{tg}\varphi_0\sin 4\varphi_0\,\alpha\Delta\alpha\Big]
\end{aligned} \tag{84}$$

where $\Delta\xi_1''$ and $\Delta\eta_1''$ are the corrections to the north-south and east-west deflection components
φ_0, λ_0 and φ_1, λ_1 are the geodetic coordinates of the origin and of the point where the transformed value is required,

and
$$\Delta\beta = \frac{a - a'}{a} + \sin\varphi_0\,\Delta\alpha$$

where a' is the major axis of the international ellipsoid
a is the major axis of Clarke's spheroid

and

$$\Delta\alpha = \alpha' - \alpha = \frac{b}{a} - \frac{b'}{a'} = \frac{b}{a}\left(\frac{\Delta a}{a} - \frac{\Delta b}{b}\right) = (1-\alpha)\left(\frac{\Delta a}{a} - \frac{\Delta b}{b}\right)$$

Their numerical values are:

$$\Delta\alpha = .000,\ 023,\ 057$$

$$\Delta\beta = \frac{181.6}{6,\ 378,\ 206.4} + \Delta\alpha \sin 39^\circ\ 13'\ 26''\!.686 = .000,\ 019,\ 242$$

Secondly it was necessary to take into account the change in the deflection component at the origin. As is well known, it is customary to assume that the geodetic and astronomic coordinates of the origin are the same, i.e. the deflection is zero. Nowadays it seems possible to accept one reference surface for world-wide use. In this case it is advisable to eliminate as many assumptions as possible. The deflection components for any point,—hence for the origin of geodetic datum—can be calculated from gravity data. These values refer to the international ellipsoid and are sometimes called the *absolute* deflections. Of course these values also depend upon the reference surface, and therefore the term is misleading. The calculated values will differ from the assumed zero value at the origin of the geodetic coordinate system. The necessary forms to take this into account are due to Vening Meinesz (1950a) and the transformation was made according to his derivation. If $\Delta\xi_2$ and $\Delta\eta_2$ designate the corrections at a point of the geodetic net due to the changes of the deflection components and the geoidal height at the origin, then rearranging Vening Meinesz's forms

$$\begin{aligned}\Delta\xi''_2 = {} & (W_1^3/W_0^3)\Big\{\cos\varphi_0\cos\varphi_1 + \sin\varphi_0\sin\varphi_1\cos\lambda_0\cos\lambda_1 + \sin\varphi_0\sin\varphi_1\sin\lambda_0\sin\lambda_1\Big\}\xi''_0 + \Big(W_1^3/W_0\,[1-e^2]\Big) \\ & \Big\{\sin\varphi_1\sin\lambda_1\cos\lambda_0 - \sin\varphi_1\cos\lambda_1\sin\lambda_0\Big\}\,\eta''_0 - \Big(W^3{}_1/a\,[1-e^2]\Big)\Big\{\sin\varphi_0\cos\varphi_1 \\ & - \sin\varphi_1\cos\varphi_0\cos\lambda_1\cos\lambda_0 + \sin\varphi_1\cos\varphi_0\sin\lambda_0\sin\lambda_1\Big\}\,\Delta N_0 \qquad (85)\end{aligned}$$

$$\begin{aligned}\Delta\eta''_2 = {} & -\Big(W_1\,[1-e^2]\Big)/W_0^3\Big\{\sin\varphi_0\sin\lambda_1\cos\lambda_0 - \sin\varphi_0\cos\lambda_1\sin\lambda_0\Big\}\,\xi''_0 + (W_1/W_0)\Big\{\cos\lambda_1\cos\lambda_0 \\ & + \sin\lambda_1\sin\lambda_0\Big\}\,\eta''_0 - (W_1/a)\Big\{\cos\varphi_0\sin\lambda_1\cos\lambda_0 - \cos\varphi_0\cos\lambda_1\sin\lambda_0\Big\}\,\Delta N_0 \qquad (86)\end{aligned}$$

$$\text{where } W_i = \sqrt{1 - e^2\sin^2\varphi_i} \qquad \text{for } i = 0,\ 1$$

$$e^2 = \frac{a^2 - b^2}{a^2} = \alpha\,(2-\alpha)$$

and ΔN_0 is the change of geoidal height at the origin. Let ξ''_c and η''_c be the corrected deflections values, then

$$\xi''_c = \xi'' + \Delta\xi''_1 + \Delta\xi''_2 \qquad (87)$$

$$\eta''_c = \eta'' + \Delta\eta''_1 + \Delta\eta''_2 \qquad (88)$$

Both sets of transformation formulae (83), (84) and (85), (86), respectively, were programmed. In the calculation, floating point arithmetic was used. First, the transformations (83) and (84) were programmed. The input was obtained from the paper by Ney (1952), *Geodetic Operations in Canada*, January 1, 1954 to December 31, 1956 and privately from J. E. Lilly, Dominion Geodesist, Geodetic Survey of Canada, personal communication. The actual input was:

Word 1..........Identification number and year of observation

2..........geodetic latitude, φ

3..........geodetic longitude, λ

4..........north-south component of the deflection, ξ''

5..........east-west component of the deflection η''

On the output the first three words have not been changed.

Word 4..........$\xi'' + \Delta\xi''_1$

5..........$\eta'' + \Delta\eta''_1$

6 and 7....$\Delta\xi''_1$ and $\Delta\eta''_1$ in fixed point notation

This output served as an input to the second transformation. The output of this calculation: no change in the first three words:

Word 4..........$\xi''_c = \xi'' + \Delta\xi''_1 + \Delta\xi''_2$

5..........$\eta''_c = \eta'' + \Delta\eta''_1 + \Delta\eta''_2$

6 and 7....$\Delta\xi''_2$ and $\Delta\eta''_2$ in fixed point notation

It is emphasized here that the values ξ''_0 and η''_0 are the changes of the corresponding values at the origin of NAD 1927. The numerical values used in (85) and (86) are: $\xi''_0 = -1''.3$ and $\eta''_0 = -1''.9$ respectively. In the first transformation (because of the assumed tangency of Clarke's spheroid and the international ellipsoid) they were kept fixed. From gravimetric data, using free air anomalies Rice (1952) obtained: $\xi''_R = -''.42$ and $\eta''_R = -1''.64$. Recently Szabó (1960) found the following values from isostatic anomalies as referred to the geoid: $\xi''_{Sz} = -''.97$ and $\eta''_{Sz} = -1''.90$. ΔN_0 corresponds to the geoidal height at Meade's Ranch (when the deflection values were calculated on Clarke's spheroid, zero geoidal height was assumed). This value for ΔN_0 was taken from a geoid contour map by Fischer (1960) and was found to be 6 metres. Using the following arbitrary values:

$$\Delta\xi''_0 = -1.''3 \qquad \Delta\eta''_0 = -.''3 \qquad \Delta N_0 = 6 \text{ metres}$$

the second transformation was carried out. It gives the order of 1" to 2" corrections in ξ, much less in η and it is not very sensitive for ΔN_0 (as can be seen from (86)). Appendix A gives the result of the first transformation. The result of the second transformation was not used. First some agreement must be obtained on what values to use for the origin because small variations can introduce large errors in this transformation due to great distances involved.

Two additional important corrections are mentioned briefly:

(a) reduction to FK3 system.

(b) correction for variation of Pole.

These corrections can reach the order of .5" (see Rice (1952)). Unfortunately the necessary information (time of the measurement) has not been made available and the corrections can not be made.

To summarize: an attempt has been made to make proper transformation of deflection values, but because of the difficulties mentioned above, it is doubtful if high significance can be attached to the astro-geodetic deflection values when compared with the gravimetric deflections.

Transformation from Ellipsoid to Plane

As briefly mentioned earlier, for the calculation of gravimetric deflections a plane coordinate system was introduced. The geodetic coordinates φ and λ of each point were converted into plane coordinates X and Y respectively. The transformation was carried out by using the formulae of the Transverse Mercator Projection with origin at Explorer, Que., No. 306. These forms of transformation were recently derived by Tárczy-Hornoch and Hristow (1959) and the first few terms of the series are given below:

$$X = B + \tfrac{1}{2} N \Delta\lambda^2 \sin\varphi \cos\varphi - \tfrac{1}{24} N \Delta \lambda^4 \sin^3 \varphi \cos \varphi \,(1 - 5 \cot^2 \varphi - 9\, \eta^2 \cot^2\varphi) + \ldots \tag{89}$$

or

$$X = B + D + F \tag{90}$$

$$Y = N \Delta \lambda \cos \varphi - \tfrac{1}{6} N \Delta \lambda^3 \sin^2 \varphi \cos \varphi (1 - \cot^2 \varphi - \eta^2 \cot^2 \varphi)$$
$$+ \tfrac{1}{120} N \Delta \lambda^5 \sin^4 \varphi \cos \varphi (1 - 18 \cot^2 \varphi + 5 \cot^4 \varphi) + \ldots . \tag{91}$$

or
$$Y = A + C + E \tag{92}$$

Before the meaning of the different symbols is given, it is noted that the geodetic coordinates of a point where gravity measurement is available are given to the nearest one-tenth of a minute of arc. Therefore the terms in the expression of transformation were selected to secure this accuracy only, and for more accurate transformation higher-order terms must be included. In the above expression B is the length of the meridional arc from the equator to the point in question and is given as:

$$B = a(1 - e^2)\left[1 + \tfrac{3}{4} e^2 + \tfrac{45}{64} e^4 + \ldots\right]\varphi$$
$$- a(1 - e^2)\left[\tfrac{3}{8} e^2 + \tfrac{30}{64} e^4 + \ldots\right]\sin 2\varphi +$$
$$+ a(1 - e^2)\left[\tfrac{51}{256} e^4 + \ldots .\right]\sin 4\varphi + \tag{93}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where φ is in radian.

The other symbols are:

$$N = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}} = \frac{a}{W}$$

$\Delta\lambda_{\theta} = (\lambda_1 - \lambda_0)$ in radian

where λ_0 refers to the origin,

λ_1 refers to a particular point.

$$\eta^2 = e'^2 \cos \varphi$$

$$\text{where } e'^2 = \frac{a^2 - b^2}{b^2} = \frac{e^2}{1 - e^2}$$

Before the programming of the transformation formulae some of the terms were rearranged as follows:

$$C = \tfrac{1}{6} N \Delta\lambda^3 \sin^2 \varphi \cos \varphi \left(1 - \cot^2 \varphi - \eta^2 \cot^2 \varphi\right)$$
$$= -\tfrac{1}{6} N \Delta\lambda^3 \sin^2 \varphi \cos \varphi \left(\frac{\sin^2 \varphi}{\sin^2 \varphi} - \frac{\cos^2 \varphi}{\sin^2 \varphi} - \frac{e^2}{1 - e^2} \cos^2 \varphi \frac{\cos^2 \varphi}{\sin^2 \varphi}\right)$$
$$= \tfrac{1}{6} N \Delta\lambda^3 \cos \varphi \left(\left[\frac{e^2}{1 - e^2} \cos^2 \varphi + 2\right] \cos^2 \varphi - 1\right) \tag{94}$$

$$D = \tfrac{1}{120} N \Delta\lambda^5 \cos \varphi \sin^4 \varphi \left(1 - 18 \cot^2 \varphi + 5 \cot^4 \varphi\right)$$
$$= \tfrac{1}{120} N \Delta\lambda^5 \cos \varphi \sin^2 \varphi \left(1 - \cos^2 \varphi - 18 \cos^2 \varphi + 5 \frac{\cos^4 \varphi}{\sin^2 \varphi}\right)$$
$$= \tfrac{1}{120} N \Delta\lambda^5 \cos \varphi \sin^2 \varphi \left([5 \cot^2 \varphi - 19] \cos^2 \varphi + 1\right) \tag{95}$$

and finally

$$F = \tfrac{1}{24}\, N\, \Delta\lambda^4 \sin\varphi \cos\varphi \left(\left[9\,\frac{e^2}{1-e^2}\cos^2\varphi + 6\right]\cos^2\varphi - 1\right) \qquad (96)$$

The input to this calculation is the format of the Gravity Division (1960) shown in Figure 9 with all the information. The output of this calculation is shown in Figure 10.

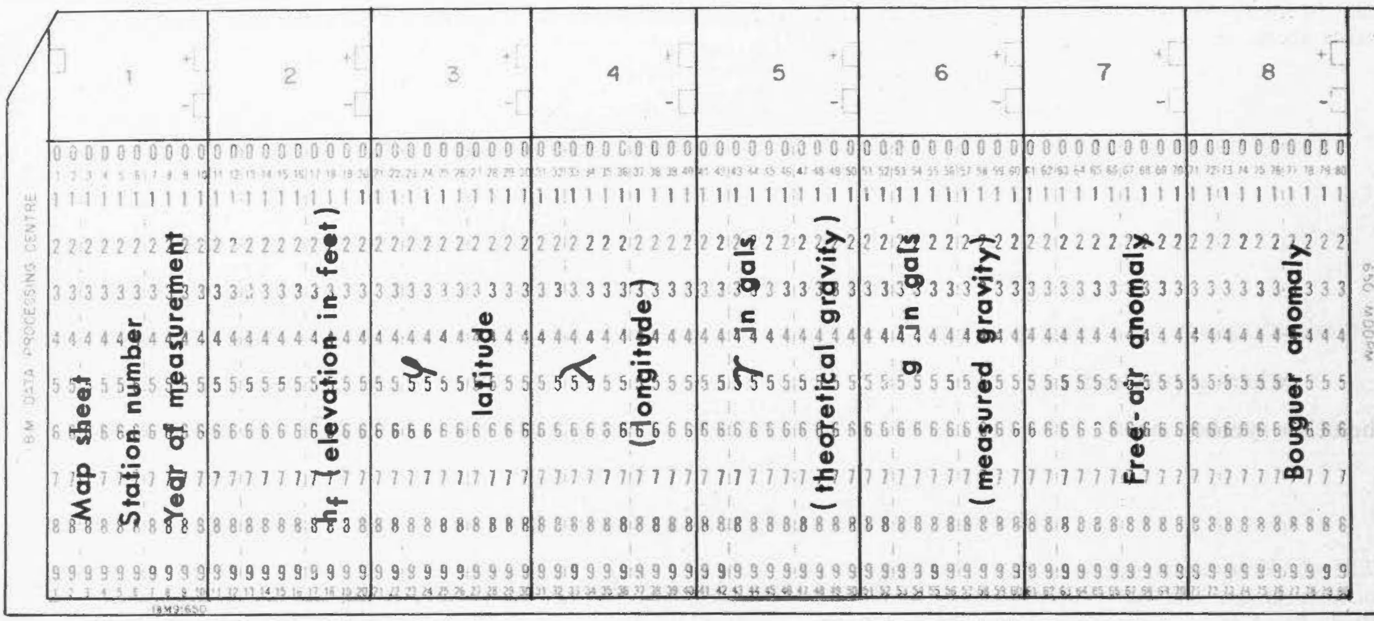

Figure 9

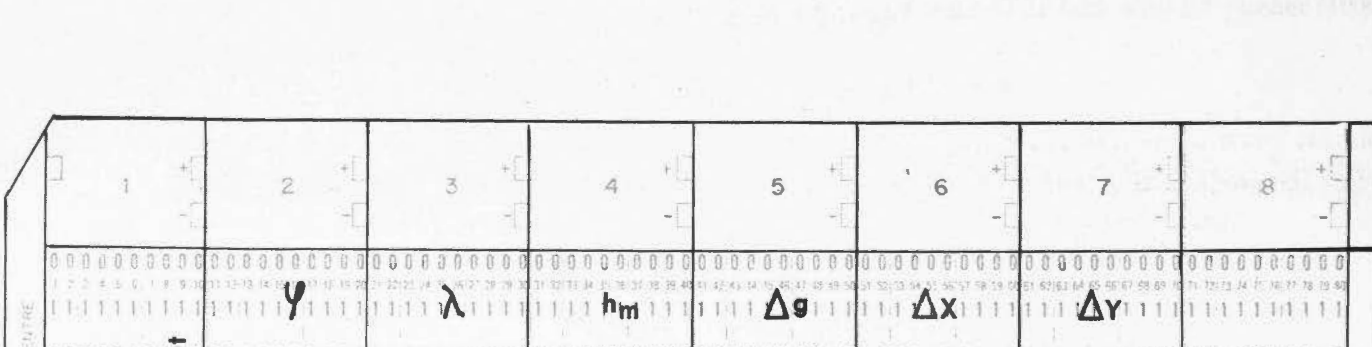

Figure 10

Word 1 .... no change has been made
2 and 3 .... are identical with words 3 and 4 of the input card,
4 .... is the elevation in metres
5 .... is the fee air gravity anomaly in milligals
6 .... ΔX is the x coordinate of a point referring to the unit area in which the point is located, i.e. $\Delta X = X \pmod{50}$, where all the values are in km.
7 .... $\Delta Y$is the y coordinate of the same point
8 .... this identification number was planned for two purposes:
a) to make an easy sorting possible
b) to find the actual coordinates of any point if required.

It is composed of four parts:

1	X X	0 0 0 0 0	Y Y

The first part is the area code. It is 1 for this area. The second part is the integer part of $X/50$, i.e. it gives the number of unit areas to be passed along the X axis to reach the unit area where the point in question is located. Its ΔX coordinate referring to that area is given in word 6. The last part of this word gives similar value for Y. The third part is the station number which was reproduced from word 1, of the input.

Thus by the above transformation all the points can be identified uniquely by the identification number (word 8) and by ΔX and ΔY. The function value is supplied by Δg. The coordinate system and the number of points in each unit area are shown in Figure 11 for one out of the three sets for which the calculations have been carried out.

X																			
18					13	11	18	13	11	11	14	23	10	4	2		2	3	3
17					11	10	11	12	10	11	18	22	14	7	3	3	3	3	4
16	2				4	4	3	10	9	9	10	11	10	4	7	4	1	6	8
15					2	6	9	2	10	8	10	7	6	4	4	7	3	6	7
14	1	3		2		2	4	3	4	4	4	5	3	2	3	7	10	31	9
13	2				1	4	3	3	4	1	3	3	1	1	3	4	5	9	9
12	1				3	1	2	4	2	2	3	1	1		1	4	4	2	4
11	3	4	6	1	1	4	2	3	3	4	1	3	1		2	1	2	2	4
10	5	3	2	2	3	2	4	2	3	5	3	3	1	3	2	6	2	4	2
09	16	6	2	5	1	2	4	3	3	3	3	6	1	3	3	5	41	2	9
08	30	10	5	7	5	2	4	3	4	3	3	3	6	7	2	4		8	2
07	5	23	17	7	5	16	9	6	4	4	2	3	1	5	16	5	9	6	5
06	8	29	6	1	3	8	2	1	7	7	6	8	3		15	12	5	6	3
05	2	4	27	2	3	10	13	7		1	3	3	17	1	10	13	9	5	10
04	31	9	11	4	3	2	6	3	1	3	4	3	11	1	12	33	1		
03	67	14	9	6	3	4		7	12	3	12	5	6	31			50		
02	20		4	1	4	6	3	3	8	21	10	35	48	79	150	94	10		
01			4	7	4	6	15	100	102	355	138	102	112	130	63	48			
00			6	1	4		2	494	413	421	176	127	196	144	47	4			
Y	00	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18

Figure 11

The total number of points for this area is 5491. As can be seen from Figure 11, for 50 units no information is available. Fortunately they are mostly at the edges of the area. The number of points per unit is between 1 and 5 or over 50 for 195 units and is between 6 and 50 for 116 units.

In the calculation an estimate for the gravity anomaly to represent each unit must be obtained. When no information is available zero value is assumed. If the number of points is between 1 and 5, or over 50, the average value is taken as the estimate. If the number of points is 6 or over but not more than 50 the coefficients of a second order surface of the form:

$$\Delta\hat{g} = g_0 + g_1 x + g_2 y + g_3 xy + g_4 x^2 + g_5 y^2 \tag{97}$$

are determined and this surface is integrated over the unit and divided by the area of the unit surface. Also estimated, was the elevation $\hat{h}$ of the same form:

$$\hat{h} = h_0 + h_1 x + h_2 y + h_3 xy + h_4 x^2 + h_5 y^2 \tag{98}$$

In (97) and (98) x and y are identical with the symbols ΔX and ΔY respectively. The coefficients in both (97) and (98) are calculated if $n = 6$ (where n is the number of points). If $6 < n \leq 50$ then the coefficients in (97) and (98) are obtained by the principle of least squares.

Let Δg_i be the given value and $\Delta\hat{g}_i$ the estimate at the same point. Then it is required that the sum of the squares of these deviations over a unit area be a minimum, i.e.:

$$G = \sum^{n} (\Delta\hat{g}_i - \Delta g_i)^2 = \text{MIN.}$$

or

$$\sum^{n} (g_0 + g_1 x_i + g_2 y_i + g_3 x_i y_i + g_4 x^2_i + g_5 y^2_i - \Delta g_i)^2 = \text{MIN.} \tag{99}$$

To get the minimum we differentiate G with respect to each coefficient and equate the result to zero.

$$\begin{aligned} \frac{\partial G}{\partial g_0} &= \sum (g_0 + g_1 x_i + g_2 y_i + g_3 x_i y_i + g_4 x^2_i + g_5 y^2_i - \Delta g_i)\cdot 1 = 0 \\ \frac{\partial G}{\partial g_1} &= \sum (g_0 + g_1 x_i + g_2 y_i + g_3 x_i y_i + g_4 x^2_i + g_5 y^2_i - \Delta g_i)\, x_i = 0 \\ &\vdots \\ \frac{\partial G}{\partial g_5} &= \sum (g_0 + g_1 x_i + g_2 y_i + g_3 x_i y_i + g_4 x^2_i + g_5 y^2_i - \Delta g_i)\, y^2_i = 0 \end{aligned} \tag{100}$$

Carrying out the summation and taking the known term to the right hand side, (100) takes the form (because of the symmetry, only the main diagonal underlined and the upper triangle of the matrix is given):

$$\begin{aligned} \underline{ng_0} + \Sigma x_i g_1 + \Sigma y_i g_2 + \Sigma x_i y_i g_3 + \Sigma x_i^2 g_4 + \Sigma y_i^2 g_5 &= \Sigma \Delta g_i \\ \underline{\Sigma x_i^2 g_1} + \Sigma x_i y_i g_2 + \Sigma x_i^2 y_i g_3 + \Sigma x_i^3 g_4 + \Sigma x_i y_i^2 g_5 &= \Sigma x_i \Delta g_i \\ \underline{\Sigma y_i^2 g_2} + \Sigma x_i y_i^2 g_3 + \Sigma x_i^2 y_i g_4 + \Sigma y_i^3 g_5 &= \Sigma y_i \Delta g_i \\ \underline{\Sigma x_i^2 y_i^2 g_3} + \Sigma x_i^3 y_i g_4 + \Sigma x_i y_i^3 g_5 &= \Sigma x_i y_i \Delta g_i \\ \underline{\Sigma x_i^4 g_4} + \Sigma x_i^2 y_i^2 g_5 &= \Sigma x^2_i \Delta g_i \\ \underline{\Sigma y_i^4 g_5} &= \Sigma y_i^2 \Delta g_i \end{aligned} \tag{101}$$

or in matrix notation:

$$\mathbf{Ag} = \mathbf{G}$$

where

$$\mathbf{A} = \begin{bmatrix} n & \Sigma x_i & \Sigma y_i & \Sigma x_i y_i & \Sigma x_i^2 & \Sigma y_i^2 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i y_i & \Sigma x_i^2 y_i & \Sigma x_i^3 & \Sigma x_i y_i^2 \\ \Sigma y_i & \Sigma x_i y_i & \Sigma y_i^2 & \Sigma x_i y_i^2 & \Sigma x_i^2 y_i & \Sigma y_i^3 \\ \Sigma x_i y_i & \Sigma x_i^2 y_i & \Sigma x_i y_i^2 & \Sigma x^2_i y_i^2 & \Sigma x_i^3 y_i & \Sigma x_i y_i^3 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x^2_i y_i & \Sigma x_i^3 y_i & \Sigma x_i^4 & \Sigma x_i^2 y_i^2 \\ \Sigma y_i^2 & \Sigma x_i y_i^2 & \Sigma y_i^3 & \Sigma x_i y_i^3 & \Sigma x_i^2 y_i^2 & \Sigma y_i^4 \end{bmatrix} \quad \mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} \Sigma \Delta g_i \\ \Sigma x_i \Delta g_i \\ \Sigma y_i \Delta g_i \\ \Sigma x_i y_i \Delta g_i \\ \Sigma x_i^2 \Delta g_i \\ \Sigma y_i^2 \Delta g_i \end{bmatrix} \tag{102}$$

The matrix **A** was obtained by a simple matrix operation. If we define the coefficient matrix **C** as follows:

$$\mathbf{C} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 & x_1^2 & y_1^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & y_n & x_n y_n & x_n^2 & y_n^2 \end{bmatrix} \tag{103}$$

then

$$\mathbf{A} = \mathbf{C}^T\mathbf{C} \tag{104}$$

where $\mathbf{C}^T$ is the transpose of **C**. The solution is obtained by inverting **A** and is given as

$$g = \mathbf{A}^{-1}\,\mathbf{G}$$

where $\mathbf{A}^{-1}$ is the inverse such that

$$\mathbf{AA}^{-1} = \mathbf{I}$$

i.e. multiplying the original matrix **A** by the inverse $\mathbf{A}^{-1}$, the unit matrix **I** results (ones in the main diagonal, zeros elsewhere).

This procedure was followed when this problem was programmed. The general flow diagram is shown in Figure 12.

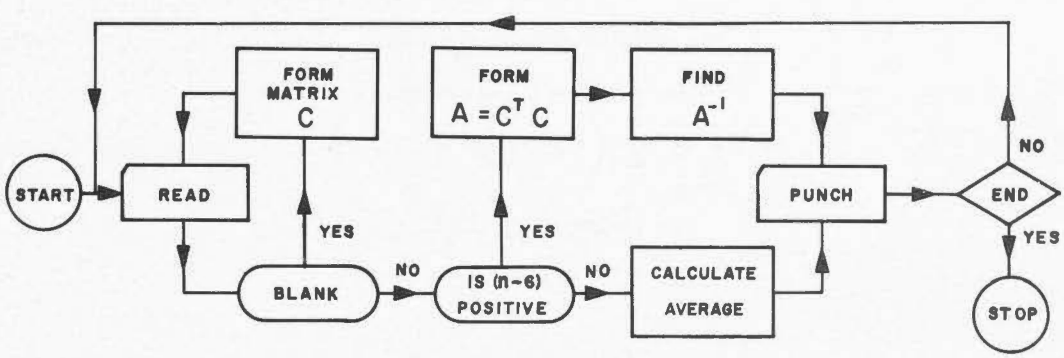

Figure 12

The output of this calculation is as follows (3 cards per unit area)
Output I: the coefficients of $\Delta\hat{g}$ of (97)
Output II: the coefficients of $\hat{h}$ of (98)
Output III: Word 1: identification number

2: number of points for the unit area

3: $\bar{h} = \overset{n}{\Sigma}\, h_i/n$ for $0 \leq n \leq 5$ or $n > 50$

4: $\hat{h}$ at the centre for $6 \leq n \leq 50$, $\bar{h}$ for all other values of n

5: $\overline{\Delta g} = \overset{n}{\Sigma}\, \Delta g_i/n$ for $0 \leq n \leq 5$ or $n > 50$

6: $\int_0^1 \int_0^1 \Delta\hat{g}\, dxdy$ for $6 \leqslant n \leqslant 50$, $\overline{\Delta g}$ for all other values of n

7: $d\xi''$ of (119) for $6 \leqslant n \leqslant 50$, zero for all other values of n

8: $d\eta''$ of (120) for $6 \leqslant n \leqslant 50$, zero for all other values of n

The values $d\xi''$ and $d\eta''$ are the contributions to the components of the deflection at the centre due to the unit area in question. More about this in the next section.

Effect of the Near Region

In this part we examine the effect of the unit area upon its centre and derive the formulae to calculate the contribution to the components of deflection making use of equation (97).

Combining the first term on the right hand side of (82) with (81) we obtain:

$$\xi = \frac{1}{4\pi\gamma} \int \Delta g\, D\,(R, \psi)\, d\psi \cos\alpha\, d\,\alpha \tag{105}$$

where $D\,(R, \psi)$ is given by (78). As can be seen from (78) this integral becomes infinite for $\psi = 0$. Therefore it is necsary to investigate the value of the integral for this case and to obtain an expression to find its value. We split (105) into two integrals:

$$\xi = I_1 + I_2 \tag{106}$$

where

$$I_1 = \frac{1}{4\pi\gamma} \int_{\psi=0}^{\Delta\psi} \int_{\alpha=0}^{2\pi} \Delta g\, D\,(R, \psi)\, d\psi \cos\alpha\, d\,\alpha \tag{107}$$

and

$$I_2 = \frac{1}{4\pi\gamma} \int_{\psi=\Delta\psi}^{\pi} \int_{\alpha=0}^{2\pi} \Delta g\, D\,(R, \psi)\, d\,\psi \cos\alpha\, d\,\alpha \tag{108}$$

The second integral presents no difficulty. We now try to evaluate I_1 when ψ is small.

When ψ is small, then (see Figure 13)

$$\mathrm{tg}\Delta\psi \doteq \Delta\psi = \frac{r}{R}$$

and

$$\sin\Delta\psi = \frac{r}{R}\left(1 - \frac{r^2}{6R^2} + \ldots\ldots\right)$$

taking as

$$r = 25 \text{ km}$$

$$R = 6\,371 \text{ km}$$

then

$$\epsilon = \frac{r^2}{6R^2} \doteq 1 \times 10^{-8}$$

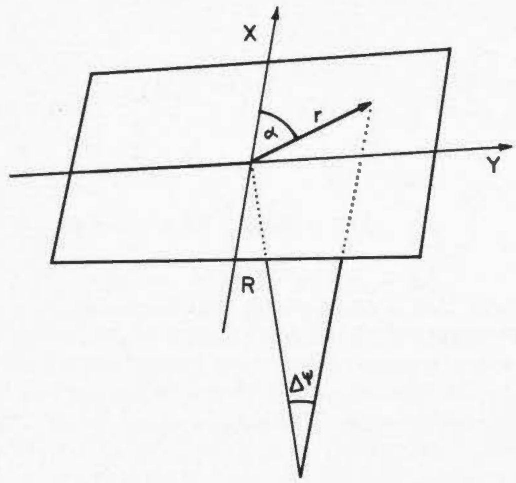

Figure 13

Allowing for this error, then simply

$$\sin \Delta\psi = \frac{r}{R} \tag{109}$$

Then taking only the first, second and fourth terms of (78) we obtain:

$$D(R, \Delta\psi) = \frac{1}{\sin \frac{1}{2}\Delta\psi} + 12 \sin \tfrac{1}{2}\Delta\psi + \frac{3}{1 + \sin \frac{1}{2}\Delta\psi}$$

i.e.

$$D(R, \Delta\psi) = \frac{1}{\frac{r}{2R}} + \frac{12r}{2R} + \frac{3}{1 + \frac{r}{2R}} \equiv \frac{2}{\Delta\psi} + 6\Delta\psi + 3 \tag{110}$$

On the other hand we want to expand Δg around the centre of the unit area:

$$\begin{aligned} \Delta g(\Delta\psi, \alpha) = \Delta g(0,0) &+ \left.\frac{\partial \Delta g}{\partial x}\right|_{0,0} \Delta x + \left.\frac{\partial \Delta g}{\partial y}\right|_{0,0} \Delta y + \tfrac{1}{2} \left.\frac{\partial^2 \Delta g}{\partial x^2}\right|_{0,0} \Delta x^2 \\ &+ \tfrac{1}{2} \left.\frac{\partial^2 \Delta g}{\partial y^2}\right|_{0,0} \Delta y^2 + \tfrac{1}{2} \left.\frac{\partial^2 \Delta g}{\partial x \partial y}\right|_{0,0} \Delta x \Delta y + \ldots\ldots. \end{aligned} \tag{111}$$

since $\qquad r = R\,\Delta\psi$

and $\qquad \Delta x = r \cos\alpha = R\,\Delta\psi \cos\alpha$

$$\Delta y = r \sin\alpha = R\,\Delta\psi \sin\alpha$$

(111) takes the following form:

$$\begin{aligned} \Delta g(\Delta\psi, \alpha) = \Delta g(0,0) &+ \frac{\partial \Delta g}{\partial x} R\Delta\psi \cos\alpha + \frac{\partial \Delta g}{\partial y} R\Delta\psi \sin\alpha + \tfrac{1}{2} \frac{\partial^2 \Delta g}{\partial x^2} R^2\Delta\psi^2 \cos^2\alpha \\ &+ \tfrac{1}{2} \frac{\partial^2 \Delta g}{\partial y^2} R^2\Delta\psi^2 \sin^2\alpha + \tfrac{1}{2} \frac{\partial^2 \Delta g}{\partial x \partial y} R^2\Delta\psi^2 \sin\alpha \cos\alpha + \ldots. \end{aligned} \tag{112}$$

Using (110) and (112) then (107) can be written as:

$$I_1 = \frac{1}{4\pi\gamma} \int_0^{\Delta\psi} \int_0^{2\pi} \left[\Delta g\,(0{,}0) + \frac{\partial \Delta g}{\partial x} R\Delta\psi \cos\alpha + \frac{\partial \Delta g}{\partial y} R\Delta\psi \sin\alpha + \ldots\ldots \right]$$

$$\left(\frac{2}{\Delta\psi} + 3 + 6\Delta\psi \right) d\psi \cos\alpha \, d\,\alpha \tag{113}$$

We have 18 integrals altogether to evaluate. But it can be seen that because of the symmetry property of the function, only the second term in the square bracket gives some contribution,

$$i_1 = \frac{1}{4\pi\gamma} \int_0^{\Delta\psi} \int_0^{2\pi} \frac{\partial \Delta g}{\partial x} \frac{2}{\Delta\psi} R\Delta\psi \cos^2 \alpha d\alpha d\psi = \frac{R}{2\pi\gamma} \frac{\partial \Delta g}{\partial x} \int_0^{\Delta\psi} d\psi \int_0^{2\pi} \cos^2 \alpha d\alpha$$

i.e.

$$i_1 = \frac{R}{2\gamma} \frac{\partial \Delta g}{\partial x} \Delta\psi \Big|_0^{\frac{r}{R}} = \frac{1}{2\gamma} \frac{\partial \Delta g}{\partial x} r \tag{114}$$

Similarly

$$i_2 = \frac{1}{4\pi\gamma} \int_0^{\Delta\psi} \int_0^{2\pi} \frac{\partial \Delta g}{\partial x} 3\, R\Delta\psi \cos^2 \alpha d\alpha d\psi$$

i.e.

$$i_2 = \frac{3}{8\gamma R} \frac{\partial \Delta g}{\partial x} r^2 \tag{115}$$

and

$$i_3 = \frac{1}{4\pi\gamma} \int_0^{\Delta\psi} \int_0^{2\pi} \frac{\partial \Delta g}{\partial x} 6\, \Delta\psi^2 \cos^2 \alpha d\alpha d\psi$$

$$i_3 = \frac{1}{2\gamma R^2} \frac{\partial \Delta g}{\partial x} r^3 \tag{116}$$

Then

$$d\xi = I_1 = i_1 + i_2 + i_3$$

i.e.

$$d\xi = \frac{1}{2\gamma} \frac{\partial \Delta g}{\partial x} r + \frac{3}{8\gamma R} \frac{\partial \Delta g}{\partial x} r^2 + \frac{1}{2\gamma R^2} \frac{\partial \Delta g}{\partial x} r^3$$

or

$$d\xi = \frac{1}{2\gamma} \frac{\partial \Delta g}{\partial x} \left[1 + \frac{3}{4} \frac{r}{R} + \left(\frac{r}{R} \right)^2 \right] r \tag{117}$$

Making use of (97) we obtain:

$$\frac{\partial \Delta g}{\partial x} r = g_1 + g_3\, y + 2\, g_4\, x \tag{118}$$

Since at the programming stage the unit area actually was scaled to 1×1 therefore at the centre $x = y = \frac{1}{2}$. Then combining (118) with (117) we obtain:

$$d\xi'' = \frac{\rho''}{2\gamma} \left(g_1 + \tfrac{1}{2}\, g_3 + g_4 \right) \left(1 + \frac{3}{4} \frac{r}{R} + \frac{r^2}{R^2} \right)$$

with $r = 25$ km, $R = 6{,}371$ km, $\rho'' = 206{,}\ 265$ and $\gamma = 980{,}\ 000$ mgal we obtain

$$d\xi'' = \frac{206,265}{(2)\,980,000}\left(1 + \frac{3}{4}\,\frac{25}{6,371} + \frac{25^2}{6,371^2}\right)\left(g_1 + \tfrac{1}{2}\,g_3 + g_4\right)$$

i.e.

$$d\xi'' = .1055\,(g_1 + \tfrac{1}{2}\,g_3 + g_4) \tag{119}$$

Similarly for $d\eta''$ we obtained

$$d\eta'' = .1055\,(g_2 + \tfrac{1}{2}\,g_3 + g_5) \tag{120}$$

These values were calculated and punched out in words 7 and 8 of the program for surface fitting.

So far we are able to attach an estimate to the unit area and to calculate the contribution of the unit area upon the components of deflection. Before proceeding, a few words should be said about the estimate $\Delta\hat{g}$.

It is quite clear that the accuracy of this estimate depends on the number of observations available for this area, the point distribution, and the property of this local gravity field. Although it would have been possible to obtain some measure of the accuracy of the fit and to try to weight accordingly, our primary purpose was to get away from contouring and manual calculational procedures. It is felt that the problem of how to obtain an estimate of the gravity anomaly for each unit area from a set of measured values, is not solved. It is necessary to study the problem in detail using different sample size, area and weighting, and by practice and experimenting to gain more information about the properties of the gravity anomaly field. For this purpose the flexible procedure programmed seems to be very useful.

In the next part is outlined the method of finding the components of the deflection due to the region outside of the unit area.

Effect of the Outer Region

By outer region is meant the region outside of the unit area but still inside of the local area of 950 x 950 km. It is clear that some errors are introduced into the calculation near the edge of the area.

To calculate the components of the deflection ξ'', we rewrite (106) into the following form:

$$\xi'' = d\xi'' + \delta\xi'' \tag{121}$$

where $d\xi''$ is given by (119) and
$\delta\xi''$ corresponds to (108)

Equation (108) can be written as:

$$\delta\xi'' = I_2 = \frac{\rho''}{4\pi\gamma}\sum\sum W_{ij}\cos\alpha\,\Delta g_{ij}\,dxdy \tag{122}$$

where Δg_{ij} is the gravity anomaly corresponding to the surface element $dxdy$ at i, j,

W_{ij} is the weighting function by which the gravity anomaly Δg_{ij} must be multiplied.

Following the notations used in programming the problem and letting

$$a = \Delta g \qquad b = W\cos\alpha\,dx\,dy \tag{123}$$

we have for:

$$I_2 = S_{K+9,\;L+9} = \frac{\rho''}{4\pi\gamma}\sum\sum a_{IM+50JM}\,b_{IM-K+20+50JM-50\,L} \qquad \begin{matrix} K = 9, 8, \cdots 0, \cdots -9 \\ L = 9, 8, \cdots 0, \cdots -9 \end{matrix} \tag{124}$$

For the east-west component of the deflection η'' similarly we can write

$$\eta'' = d\eta'' + \delta\eta'' \tag{125}$$

with

$$\delta\eta'' = \rho''\sum\sum a\,b' \tag{126}$$

where

$$b' = W\sin\alpha\,dx\,dy$$

Derivation of the Weighting Function

The previous section outlines how the calculation is carried out when the weighting function is known. Now we derive this function from (108).

Since the area for which this procedure is developed is relatively small compared with the dimensions of the earth, some approximations may be introduced into (108). It is desired to approximate $D(R, \psi)$ in the range $\frac{1}{2}° \leqslant \psi \leqslant 7°$ and it is necessary to decide on the order of approximation. The number of significant figures of the gravity anomalies indicates that the second and higher order terms of the series expansion can be ignored. This eliminates, at once, the third term of (180) in the square bracket. Then, using the Taylor expansion of each function, we can write:

$$D(R, \psi) = \left(1 - \frac{\psi^2}{8} + \dots\right)^2 \left[\frac{1}{\frac{\psi}{2} - \frac{1}{6}\left(\frac{\psi}{2}\right)^3 + \dots} + 12\left(\frac{\psi}{2} - \dots\right) + \right.$$

$$\left. + \frac{3}{1 + \frac{\psi}{2} - \dots} - 12\left(\frac{\psi}{2} - \dots\right) ln\left(\frac{\psi}{2} - \dots\right)\right]$$

i.e. $$D(R, \psi) = 1\left[\frac{1}{\frac{\psi}{2}\left(1 - \frac{\psi^2}{24}\right)} + 6\psi + \frac{3}{1 + \frac{\psi}{2}} - 6\psi\, ln \frac{\psi}{2}\right]$$

or $$D(R, \psi) = \frac{2}{\psi}\left(1 + \frac{\psi^2}{24}\right) + 6\psi + 3\left(1 - \frac{\psi}{2}\right) - 6\psi\, ln \frac{\psi}{2}$$

and collecting the like powers:

$$D(R, \psi) = \frac{2}{\psi} + \left(\frac{1}{12} + 6 - \frac{3}{2} - 6\, ln \frac{\psi}{2}\right)\psi + 3 \tag{127}$$

In the bracket we have to approximate $ln\frac{1}{2}\psi$ in the given range. Since it is multiplied by ψ it must be constant over the range, so as not to introduce second order quantities. As is well known, the mean value of a function $f(x)$ of one variable over the interval $b - a$ is given as

$$M = \frac{1}{b - a}\int_a^b f(x)\, dx$$

Applying this to $ln \frac{\psi}{2}$ we obtain

$$M = \frac{1}{\psi_2 - \psi_1}\int_{\psi_1}^{\psi_2} ln \frac{\psi}{2}\, d\psi$$

i.e.

$$M = \frac{2}{\psi_2 - \psi_1}\left[\frac{\psi}{2}\, ln \frac{\psi}{2} - \frac{\psi}{2}\right] \tag{128}$$

where ψ_1 and ψ_2 are given in radian.

Substituting the limits

$$\psi_1 = \tfrac{1}{2}° = .008,727$$

$$\psi_2 = 7° = .122,173$$

we have

$$M = \frac{2}{(.122,173 - .008,727)}\left[.061,086\,(-\,2.7953) - .061,086 - .004,363\,(-\,5.4353 - .004,363)\right]$$

i.e. $$M \doteq -3.746 \doteq -\frac{15}{4}$$

Using this value for $ln\frac{\psi}{2}$ in the given range, (127) takes the form:

$$D(R,\psi) = \frac{2}{\psi} + 3 + \frac{1}{12}\left(1 + 72 - 18 + 270\right)\psi$$

or $$D(R,\psi) = \frac{2}{\psi} + 3 + \frac{325}{12}\psi \tag{129}$$

Substituting (129) back into (108) we obtain

$$I_2 = \frac{1}{4\pi\gamma}\int_{\psi_1}^{\psi_2}\int_0^{2\pi}\Delta g\left[\frac{2}{\psi} + 3 + \frac{325}{12}\psi\right]\cos\alpha\, d\alpha d\psi \tag{130}$$

where ψ_2 is the upper limit of integration.

By change of the variable we can obtain I_2 in cylindrical coordinate system.

Using the relation $$\psi = \frac{r}{R}$$

then $$d\psi = \frac{dr}{R}$$

and $$I_2 = \frac{1}{4\pi\gamma}\int_{r_1/R}^{r_2/R}\int_0^{2\pi}\Delta g\left[2R\frac{1}{r} + 3 + \frac{325}{12R}r\right]\frac{\cos\alpha\, d\alpha dr}{R} \tag{131}$$

or expressing I_2 in seconds of arc we have

$$I''_2 = \frac{\rho''}{2\pi\gamma}\int_{r_1/R}^{r_2/R}\int_0^{2\pi}\Delta g\left[\frac{1}{r} + \frac{3}{2R} + \frac{325}{24R}r\right]\cos\alpha\, d\alpha dr \tag{132}$$

Finally transforming (132) into Cartesian coordinate system we obtain

$$I''_2 = \frac{\rho''}{2\pi\gamma}\iint\Delta g\left[\frac{1}{\sqrt{x^2+y^2}} + \frac{3}{2R} + \frac{325}{24R^2}\sqrt{x^2+y^2}\right]\frac{xdxdy}{x^2+y^2} \tag{133}$$

If we assume that Δg is constant for the surface element $dxdy$ and that the corner points of the surface element are given by the pair of coordinates (x_1, y_1); (x_1, y_2); (x_2, y_1) and (x_2, y_2) respectively (see Figure 14) then the weight function $W_{ij\xi}$ which corresponds to this unit area can be obtained from the following integral:

$$W_{ij\xi} = \frac{\rho''}{2\pi\gamma}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\left[\frac{1}{\sqrt{(x^2+y^2)^3}} + \frac{3}{2R(x^2+y^2)} + \frac{325}{24R^2\sqrt{x^2+y^2}}\right]xdxdy \tag{134}$$

Integrating term by term we obtain

$$\mathscr{I}_1 = \frac{\rho''}{2\pi\gamma}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\frac{xdxdy}{\sqrt{(x^2+y^2)^3}} = \frac{\rho''}{2\pi\gamma}\int_{y_1}^{y_2}dy\;\frac{-1}{\sqrt{x^2+y^2}}$$

$$= \frac{\rho''}{2\pi\gamma} \int_{y_1}^{y_2} \left[\frac{1}{\sqrt{x_2^2 + y^2}} - \frac{1}{\sqrt{x_1^2 + y^2}} \right] dy$$

$$\mathscr{I}_1 = \frac{\rho''}{2\pi\gamma} \left[\text{Arsh}\ \frac{y_1}{x_2} + \text{Arsh}\ \frac{y_2}{x_1} - \text{Arsh}\ \frac{y_1}{x_1} - \text{Arsh}\ \frac{y_2}{x_2} \right] \tag{135}$$

where $\quad \text{Arsh}x = \sinh^{-1}x = ln\,(x + \sqrt{x^2 + 1}\,)$

The second term of (133) gives:

$$\mathscr{I}_2 = \frac{3\rho''}{4\pi\gamma R} \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{xdxdy}{x^2 + y^2} = \frac{3\rho''}{4\pi\gamma R} \int_{y_1}^{y_2} \left[ln\,(x^2{}_2 + y^2) - ln\,(x^2{}_1 + y^2) \right] dy =$$

$$= \frac{3\rho''}{4\pi\gamma R} \left[\tfrac{1}{2}\, y_2 \left\{ ln\,(x_2^2 + y_2^2) - ln\,(x_1^2 + y_2^2) \right\} + \tfrac{1}{2}\, y_1 \left\{ ln\,(x_1^2 + y_1^2) - ln\,(x_2^2 + y_1^2) \right\} \right.$$

$$\left. + x_2 \left\{ \arctan \frac{y_2}{x_2} - \arctan \frac{y_1}{x_2} \right\} + x_1 \left\{ \arctan \frac{y_1}{x_1} - \arctan \frac{y_2}{x_1} \right\} \right] \tag{136}$$

and finally from the last term of (133) we obtain:

$$\mathscr{I}_3 = \frac{325}{48\pi\gamma R^2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{xdxdy}{\sqrt{x^2 + y^2}} = \frac{325}{48\pi\gamma R^3} \int_{y_1}^{y_2} \left[\sqrt{x_2^2 + y^2} - \sqrt{x_1^2 + y^2} \right] dy$$

$$= \frac{325}{96\pi\gamma R^2} \left[y_2 \left\{ \sqrt{x_2^2 + y_2^2} - \sqrt{x_1^2 + y_2^2} \right\} + y_1 \left\{ \sqrt{x_1^2 + y_1^2} - \sqrt{x_2^2 + y_1^2} \right\} \right.$$

$$\left. + x_2^2 \left\{ \text{Arsh}\ \frac{y_2}{x_2} - \text{Arsh}\ \frac{y_1}{x_2} \right\} + x_1^2 \left\{ \text{Arsh}\ \frac{y_1}{x_1} - \text{Arsh}\ \frac{y_2}{x_1} \right\} \right] \tag{137}$$

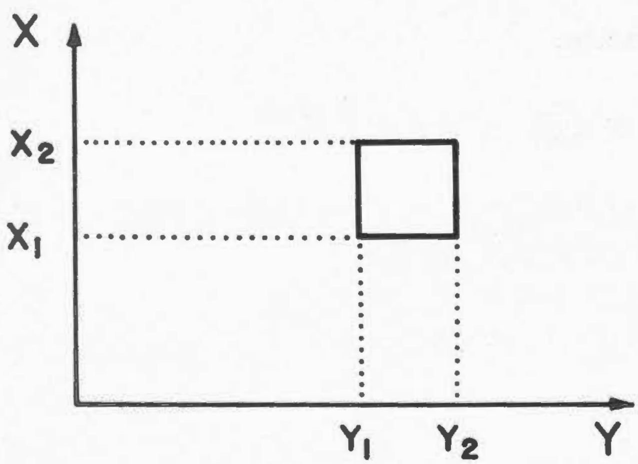

Figure 14

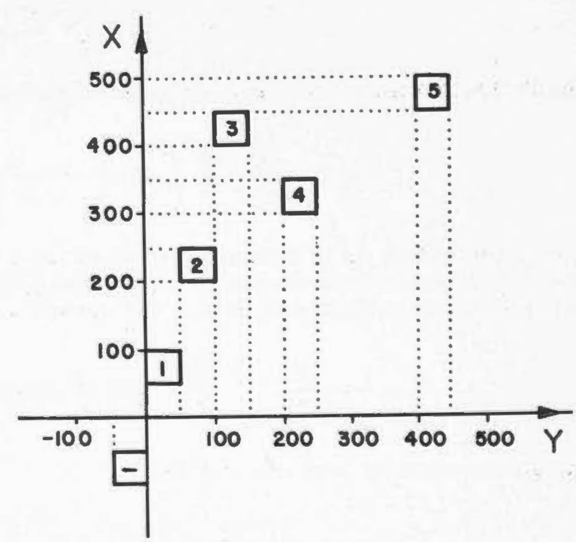

Figure 15

Then using (135), (136) and (137) the weighting function $W_{ij\xi}$ corresponding to the unit surface element shown in Figure 14 is given by the following form:

$$W_{ij\xi} = \frac{\rho''}{2\pi\gamma}\left[\text{Arsh}\ \frac{y_1}{x_2} + \text{Arsh}\ \frac{y_2}{x_1} - \text{Arsh}\ \frac{y_1}{x_1} - \text{Arsh}\ \frac{y_2}{x_2}\right]$$

$$+ \frac{3\rho''}{4\pi\gamma R}\left[\tfrac{1}{2}\, y_2 \left\{ln\,(x_2^2 + y_2^2) - ln\,(x_1^2 + y_2^2)\right\} + \tfrac{1}{2}\, y_1 \left\{ln\,(x_1^2 + y_1^2) - ln\,(x_2^2 + y_1^2)\right\}\right.$$

$$\left. + x_2 \left\{\arctan\ \frac{y_2}{x_2} - \arctan\ \frac{y_1}{x_2}\right\} + x_1 \left\{\arctan\ \frac{y_1}{x_1} - \arctan\ \frac{y_2}{x_1}\right\}\right]$$

$$+ \frac{325\,\rho''}{96\pi\gamma R^2}\left[y_2 \left\{\sqrt{x_2^2 + y_2^2} - \sqrt{x_1^2 + y_2^2}\right\} + y_1 \left\{\sqrt{x_1^2 + y_1^2} - \sqrt{x_2^2 + y_1^2}\right\}\right.$$

$$\left. + x_2^2 \left\{\text{Arsh}\ \frac{y_2}{x_2} - \text{Arsh}\ \frac{y_1}{x_2}\right\} + x_1^2 \left\{\text{Arsh}\ \frac{y_1}{x_1} - \text{Arsh}\ \frac{y_2}{x_1}\right\}\right] \tag{138}$$

This weighting function corresponds to the symbol b in (123).

i.e.
$$W_{ij\xi} \equiv b \equiv W \cos\alpha\, dxdy \tag{139}$$

Now the calculation of the component ξ of the deflection is simple: the sum of the products of the weighting coefficients and the corresponding gravity anomaly all over the area give the required component. The weighting function which gives the east-west component has been obtained similarly and is given below:

$$W_{ij\eta} = \frac{\rho''}{2\pi\gamma}\left[\text{Arsh}\ \frac{x_1}{y_2} + \text{Arsh}\ \frac{y_1}{x_2} - \text{Arsh}\ \frac{x_1}{y_1} - \text{Arsh}\ \frac{x_2}{y_2}\right]$$

$$+ \frac{3\rho''}{4\pi\gamma R}\left[\tfrac{1}{2}\, x_2 \left\{ln\,(x_2^2 + y_2^2) - ln\,(x_2^2 + y_1^2)\right\} + \tfrac{1}{2}\, x_1 \left\{ln(x_1^2 + y_1^2) - ln\,(x_1^2 + y_2^2)\right\}\right.$$

$$\left. + y_2 \left\{\arctan\ \frac{x_2}{y_2} - \arctan\ \frac{x_1}{y_2}\right\} + y_1 \left\{\arctan\ \frac{x_1}{y_1} - \arctan\ \frac{x_1}{y_1}\right\}\right]$$

$$+ \frac{325\,\rho''}{96\pi\gamma R}\left[x_2 \left\{\sqrt{x_2^2 + y_2^2} \quad \sqrt{x_2^2 + y_1^2}\right\} + x_1 \left\{\sqrt{x_1^2 + y_1^2} \quad \sqrt{x_1^2 + y_2^2}\right\}\right.$$

$$\left. + y_2^2 \left\{\text{Arsh}\ \frac{x_2}{y_2} - \text{Arsh}\ \frac{x_1}{y_2}\right\} + y_1^2\ \text{Arsh} \left\{\frac{x_1}{y_1} - \text{Arsh}\ \frac{x_2}{y_1}\right\}\right] \tag{140}$$

which corresponds to b′ of (126).

As can be seen the functions given by (138) and (140) are complicated to evaluate. Therefore an attempt was made to find some approximation to them sufficient for the present purpose. One such approximation can be obtained by taking the point value of the weighting function at the centre of the unit area instead of integrating over the unit area. If l and m are the grid distances along x and y respectively then starting from (134) we can approximate (138) by the following form;

$$W_{ij\xi} = \frac{lm\rho''}{2\pi\gamma}\ \frac{\bar{x}}{\sqrt{(\bar{x}^2 + \bar{y}^2)^3}} + \frac{3lm\rho''}{4\pi\gamma R}\ \frac{\bar{x}}{(\bar{x}^2 + \bar{y}^2)} + \frac{325\, lm\rho''}{48\pi\gamma R^2}\ \frac{\bar{x}}{\sqrt{(\bar{x}^2 + \bar{y}^2)}} \tag{141}$$

where $\bar{x} = \tfrac{1}{2}(x_1 + x_2)$, $\bar{y} = \tfrac{1}{2}(y_1 + y_2)$

Similarly for the east-west component of the deflection, we have the approximation for the weighting function:

$$W_{ij\eta} \doteq \frac{lm\rho''}{2\pi\gamma} \frac{\bar{y}}{\sqrt{(\bar{x}^2+\bar{y}^2)^3}} + \frac{3lm\rho''}{4\pi\gamma R} \frac{\bar{y}}{(\bar{x}^2+\bar{y}^2)} + \frac{325\, lm\rho''}{48\pi\gamma R^2} \frac{\bar{y}}{\sqrt{\bar{x}^2+\bar{y}^2}} \tag{142}$$

For comparison (138) and (141) were evaluated for five differently situated unit areas, the locations of which are shown in Figure 15. Table II gives the coordinates of the unit areas.

TABLE II

No.	x_1	x_2	y_1	y_2	$\frac{y_1}{x_1}$	$\frac{y_1}{x_2}$	$\frac{y_2}{x_1}$	$\frac{y_2}{x_2}$
1	50	100	0	50	0	0	1	.500
2	200	250	50	100	.25	.200	.500	.400
3	400	450	100	150	.25	.200	.375	.300
4	300	350	200	250	.60	.571	.830	.714
5	450	500	400	450	.80	.800	1	.400

The following numerical values of constants were used

$$\rho'' = 206,265 \qquad \gamma_{40°} = 980,180 \text{ mgal}$$
$$R = 6,371 \text{ km} \qquad l = m = 50 \text{ km}$$
$$2\pi = 6.283\,185$$

$$\frac{\rho''}{2\pi\gamma} = 3.''349\,190 \times 10^{-2}\text{ mgal}^{-1}$$

$$\frac{3\rho''}{4\pi\gamma R} = 7.''885\,4 \times 10^{-6}\text{ mgal}^{-1}\text{ km}^{-1}$$

$$\frac{325\,\rho''}{48\pi\gamma R^2} = 1.''17 \times 10^{-8}\text{ mgal}^{-1}\text{ km}^{-2}$$

The numerical values of the three terms (a, b, and c) of (138) and (141) are given in Table III. In the last two columns of the table the sum of the terms i.e. the value of the weighting functions are given. (The values are given in the unit of the 6th decimal).

TABLE III

No.	197			200			(197)	(200)
	a	b	c	a	b	c		
1	13402	337	27	12709	237	28	13766	12974
2	1420	79	28	1412	79	28	1527	1519
3	470	42	28	409	43	28	477	480
4	440	41	24	440	41	24	505	505
5	149	23	22	153	23	22	194	198

The difference between the two sets of weighting function is the greatest the closer the unit area is to the centre of origin. The maximum difference for the above case is:

$$D_{\max.} = (.013,766 - .012,974)\,''\Delta g_1 = ''.000\,792\,\Delta g_1$$

This difference means an error of the order of a tenth of a second of arc if Δg_1 is taken as 100 milligal. This justifies the use of the point value of the weighting function in our calculations.

PART IV

Sequence of Calculations

The input for calculating the plumb-line deflections consists of two sets:

Astro-geodetic data were punched from publications (see p. 28). The number of astro-geodetic deflections used in this study for the area covering all three SETS I, II and III is 85. These deflections are transformed from Clarke's spheroid into the international ellipsoid. The formulae of transformation (83) and (84) are given on p. 26. This calculation takes about 4 seconds per card. The format of input and output is described on p. 28. Although formulae (85) and (86) of the second transformation were programmed, the results (for reasons explained on p. 28) were not used. The output of the first transformation is plotted on a map and is shown, together with the gravimetrically calculated deflections for each SET. The output is also given in numerical form in Appendix A.

Gravity data were made available for this study by the Dominion Observatory, Gravity Division. The information is given in the format of the Division and the number of points is about 6,000. This format (Figure 9, p. 30) was converted into another one (Figure 10, p. 30) which was used for the calculations. On the output card (p. 31) Word 8 was designed to facilitate fast sorting according to unit areas. Formulae (89) and (91) were used to make the transformation from the geodetic to the plane coordinate system. These transformations were carried out separately for each SET. The output cards were then sorted, using the identification number (Word 8). The sorting requires 4 passes and takes about $1\frac{1}{2}$ hours per SET. By sorting, the points belonging to the same unit areas are grouped together. A blank card separates the different unit areas, in order to transfer control in the program when the calculation is finished for a unit area. The sorted cards give the input to the surface-fitting routine. In addition to the average values, when the number of points is between 6 and 50, second-order surfaces in two variables, given by (97) and (98) on p. 32, are fitted by the least-squares procedure to the gravity anomalies Δg and the elevations h. The calculation of the coefficients for both surfaces requires 40 seconds when $n = 6$ and almost 2 minutes when $n = 50$. These estimates include input and the output of results. In the calculation the Matrix Package of the Computation Centre was used. It requires a fast excess-memory of 60 ten-digit words and 3 magnetic tapes. One tape, the library type, has the matrix package routine on it. The other two serve as working tapes to store the intermediate results of matrix manipulations. The largest matrix which can be handled by the matrix package is of the order of 37×50. Because no information was available for about 15% of the unit areas, it did not seem worthwhile to spend the great amount of machine time involved in partitioning the matrices in cases where $n > 50$; the average values were therefore used. On the average one SET requires $3\frac{1}{2}$ hours of machine time. The format of the output is given on p. 34. These cards can be referred to as Δg cards. Before reading them in again, the locations reserved for this array are set to zero, so if no information is available for some of the unit areas, zero value will be used. After the reading is completed the weighting function, (141) on p. 41 to give the x component of the deflection, is generated and stored as a square array, which takes only 3 minutes. The next step is to carry out the double summation, the result of which is actually an estimate for the deflection component at the grid point given by its identification number. A brief description of this procedure is as follows:

Let us imagine that the two square array of numbers—i.e. the gravity anomalies at the grid points Δg_{ij} representing the unit areas, and the weighting function W_{ij}—are put on two separate pages. Then the centre of the W_{ij} array is placed over one corner of the Δg_{ij} array and the sum of the products of the corresponding members gives the required result—the deflection component. Then the centre of the W_{ij} array moves to the next point, and the procedure is repeated for all grid points. The program, which generates the weighting coefficients for the x and y components of the deflection, and carries out the double summation, has 237 instructions plus the square-root routine. In the double summation routine, most of the instructions are used to control the calculations. Actually only two arithmetic codes are used (multiply and add the result to the previous sum). The result of the double summation is punched for each grid point, giving the identification number and the deflection component. The minimum number of multiplications is 100 for the corner points and the maximum is 361 ($= 19 \times 19$) for the centre. The running times are about 5 and 15 seconds respectively. After the x components of the deflections are calculated, the control in the program goes back to generate the weighting coefficients for the y component, and then the double summation starts anew. The calculation of the x and y components of the deflections for all the 361 grid points requires 1 hour and 50 minutes. The calculations were carried out using, firstly, the average value estimates for Δg_A, and secondly, where available, the integral mean values Δg_S (*see* A*verage* and S*urface* in Appendix B). For the three SETS, the program was run six times. The information from the 5 output cards was compiled into a single card, the numbers given in floating point representation were converted, for the convenience of the reader, into fixed-point format; they were then rounded off to the nearest tenth of a second of arc and given in Appendix B.

Discussion of Results

In the procedure developed in the previous part of the paper for calculating the deflections from gravimetric data, the reference surface is a plane which is tangent to the international ellipsoid at the origin of the system. This means that all the calculated values are relative to this origin and the proper orientation of the plane must be given by some other means, as the calculation cannot supply it. This other means is the knowledge of the astro-geodetic deflection at the chosen origin of the plane system. Figure 16 gives the deflections at the centre of each unit area for SET I as they are obtained directly from the calculations (referring to the plane system). In all the maps the letters A and S after the words SET I, II or III mean that the basic input data for the calculations of the deflection values were obtained either from Average values or as the result of Surface fitting. For those units where surface fitting was not possible*, the average values were used. The small diagram on the right of each map shows the point distribution. In order

*for SETS designated by S.

to be able to estimate the magnitude of errors involved, for Set IA the closures were calculated, and are shown in Figure 16. The closures are expressed in seconds of arc. Since the final purpose is to obtain the height corrections, it is interesting to see the order of error in the height determination due to these closures. The error which corresponds to 1″ is

$$\epsilon_{\Delta h} = 50 \text{ km} \times \sin 1'' = 24 \text{ cm}$$

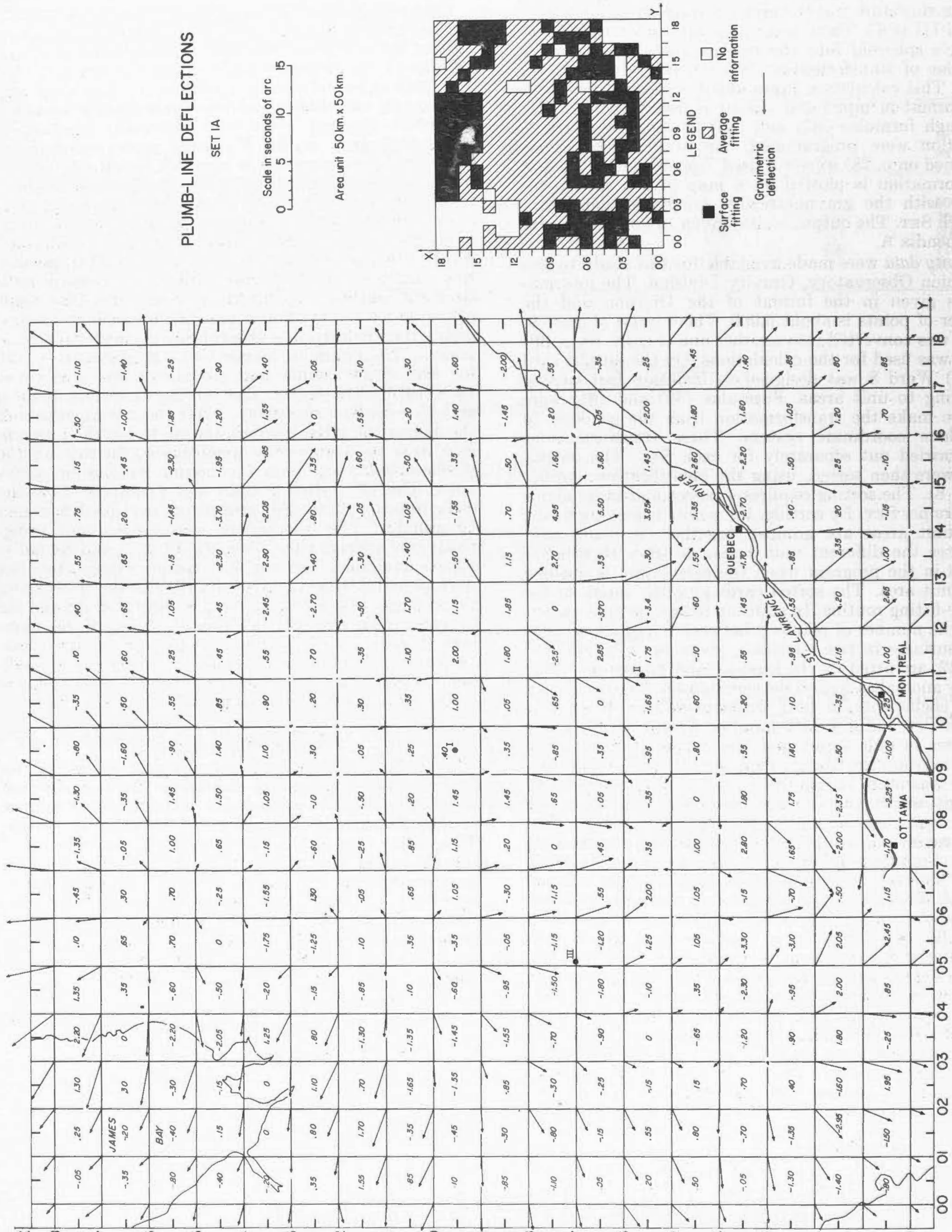

Figure 16

that is 1″ error in the closure corresponds to a 24-cm error in the height determination when 50-km grid distances are used. Figure 17 gives the percentage occurrence of the absolute values of the closures grouped together in half-second intervals. The closures in centimetres are also shown.

As can be seen, the absolute values of the closures are less than $\frac{1}{2}''$ for 39% of the area. For more than 60% of the area they remain less than 1″. In order to appreciate the accuracy of the calculations it is emphasized that SET IA is the result of a procedure where the Δg's representing the unit areas were obtained from simple averaging. In all these cases the contribution of the near region was not taken into account (this is possible in this procedure only when a surface is fitted the gravity to anomaly values). This contribution can be quite appreciable as shown later, but is far from systematic, since it depends on the local gravity field. The maximum of the closure is 5″.5 at $X = 06$, 07 and $Y = 14$, 15 which corresponds to 1.2 metres error in height determination. A comment on this is given later.

As indicated earlier, the orientation of the plane is obtained from the astro-geodetic deflection at the origin. Giving this constant correction to all the calculated values of a SET, the gravimetric deflections become directly comparable with the astro-geodetic values. This has been done for SET I S, SET II A and SET III S. (See pocket). The plotted deflections are so reproduced that any SET can be directly compared with any other.

SET I S is examined first.

On the map the astro-geodetic deflections are also shown. Unfortunately the locations of the astro-geodetic deflections are not the most favourable for comparison purposes. They are scattered near the southern edge of the area and along the St. Lawrence River. In some cases the agreement is excellent (89, 91, 106, 108, 109, 128, 150)*. In other cases it is very poor (307, 308, 141, 125, 144) but on the average the general trend of the astro-geodetic and gravimetric deflections does agree. It may be noticed that the points with poor agreement are also situated near the edge of the area.

Now it is possible to comment on the large closures around the astro-geodetic points 82 and 83. As described previously, Appendix B also gives the results of calculations which are graphically given in the maps. Taking the figures for the point $X = 07$ and $Y = 14$, the components of deflection from surface fitting are as follows:

$$\xi_S = 1''.8 \qquad \eta_S = -2''.1$$

and the correction

$$\Delta\xi = 6''.7 \qquad \Delta\eta = 2''.8$$

thus the components of the deflection are

$$\xi = 8''.5 \qquad \eta = ''.7$$

It is apparent that the correction which is the contribution due to the near region, is too large. Although the number of points is 16 for this unit area, the point distribution (shown in Figure 18) indicates that extra heavy weighting was given for the region to the right. In fitting a surface to this unit the inherent error due to the uneven distribution is enlarged when the directional derivatives are taken to obtain the corrections to the deflection components. This may explain why the corrections are large. Of course another reason could be that the gravity field is a complex one. Since no more surface fittings could be made in the neighbourhood of this particular unit area, no further conclusion can be drawn.

Examination of SET II A shows a similar pattern. The agreement is fairly good for the central part of the area.

*Station numbers of astro-geodetic deflections as listed in Appendix A.

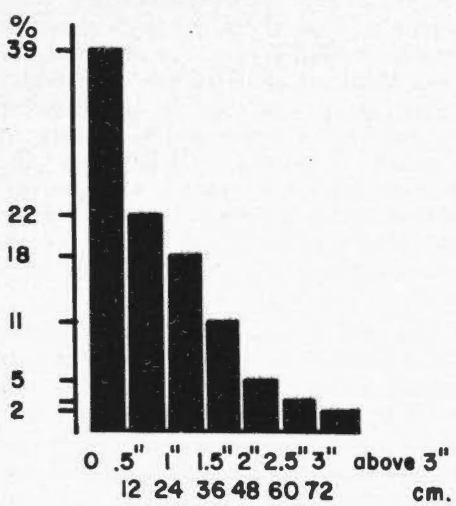

Figure 17

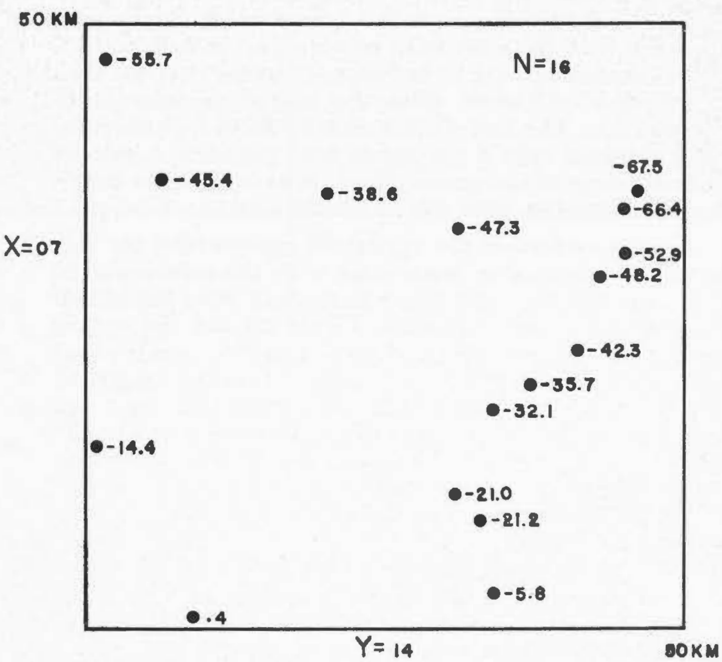

Figure 18

SET III S was chosen as an example to discuss some of the details of the various relationships which exist among the different quantities: the degree of agreement between astro-geodetic and gravimetric deflections; the size of the contribution of the near region as a function (i) of the *gravity field*, (ii) of the *point distribution*, and (iii) of the *number of points per unit area*.

In Figure 19 the point distributions for twelve selected unit areas are shown. The number of points per unit area varies from 6 to 50, and also different types of distributions are shown. In Table IV some information concerning the selected unit areas are given.

TABLE IV

N	$\lvert \Delta g_A - \Delta g_S \rvert$ in mgal	$\theta = (\xi^2 + \eta^2)^{\frac{1}{2}}$ in seconds	Distribution	Identification number	
44	.4	1.1	a	01	12
36	7.1	3.5	a	04	03
15	24.9	13.6	p	04	11
23	1.9	.8	a	04	15
30	2.5	3.6	a	05	05
50	13.0	4.0	p	06	03
6	18.3	9.5	p	09	08
6	.6	1.2	g	10	10
9	6.7	3.8	p	10	13
8	51.1	3.7	p	10	17
9	.1	.6	g	18	13
11	.2	.4	g	18	14

In the first column N is the number of points per unit area. The second column gives the absolute value of the difference between the two estimates—which are the average Δg_A and the integral mean Δg_S which represent the unit area, given by the identification number in the last column. In the next column θ is the contribution to the deflection due the unit area upon its centre. The fourth column arbitrarily classifies the point distribution into three broad groups: poor, average or good. A short inspection of Table IV shows that when the number of points is large and the distribution is good then reasonable values for θ can be expected, that is in the order of few seconds of arc. A more careful examination indicates that the most decisive factor might be the point distribution alone, since for the unit areas 10-10, 18-13 and 18-14 where the distributions are good, reasonable values were obtained with the number of points equal to 6, 9, and 11, which cannot be said to be large. It seems that an even distribution of points gives the maximum amount of information. The best distribution for fixed N could have been obtained only if the points were placed in a pattern characteristic of the gravity field, so as to give the maximum information with the minimum number of points.

To explore further the statement made about the role of the distribution in connection with the calculation of the contribution, some more unit areas were examined. The facts are summarized in Figure 20 and the details are given in Table V. It can be seen from this sample that when the point distribution is good θ is always small, in the present case less than 2″. Unfortunately the basic gravity data used for this study were much less favourable than expected and no detailed analyses could be made.

Conclusions

It is verified by the results that the plane coordinate system introduced for a local area of the size of 950 × 950 km is sufficient to use. The simplification achieved by the use of this system needs no comment. On the other hand the extension of the procedure for larger areas using other coordinate systems is just a question of an increase in computing time.

The astro-geodetic deflections have been transformed from Clarke's spheroid into the international ellipsoid. Only one transformation has been carried out and used because of the lack of information to carry out the other transformation and corrections. Thus even the transformed astro-geodetic deflections given in Appendix A contain systematic errors. It is estimated to be of the order of ±1″.

In the gravimetric deflections the largest error (apart from the region near the edges, say about 100-150 km) is caused by not taking into account the effect of the unit area upon its centre. A good average figure for this error is ±2″. The error caused by ignoring the effect of the region outside of the limits of the integration is much less; thus the value ±2″ can be used as a measure of the error in the gravimetrically calculated deflections. Keeping in mind the error bounds of the astro-geodetic and gravimetric deflections, a comparison indicates that very good agreement was obtained. This agreement could be made better if, instead of the one value at the origin, all the astro-geodetic deflections were used to obtain the best orientation of the gravimetric deflections. This could be done by minimizing the sum of the squares of the deviations between the astro-geodetic and gravimetric deflections. The gravimetric deflection can be obtained at a geodetic station by two-dimensional linear interpolation.

The length of the grid distance in this calculation serves as a filter. The smaller the grid distance, the more detail can be obtained. The choice of 50 km as the grid distance seems suitable; enough details can be obtained. Much smaller grid distances would be impracticable since it would require many more gravity stations to maintain the same point distribution per unit area. On the other hand, for studies involving a much larger area and for seeking general information, greater grid distance may serve the purpose better.

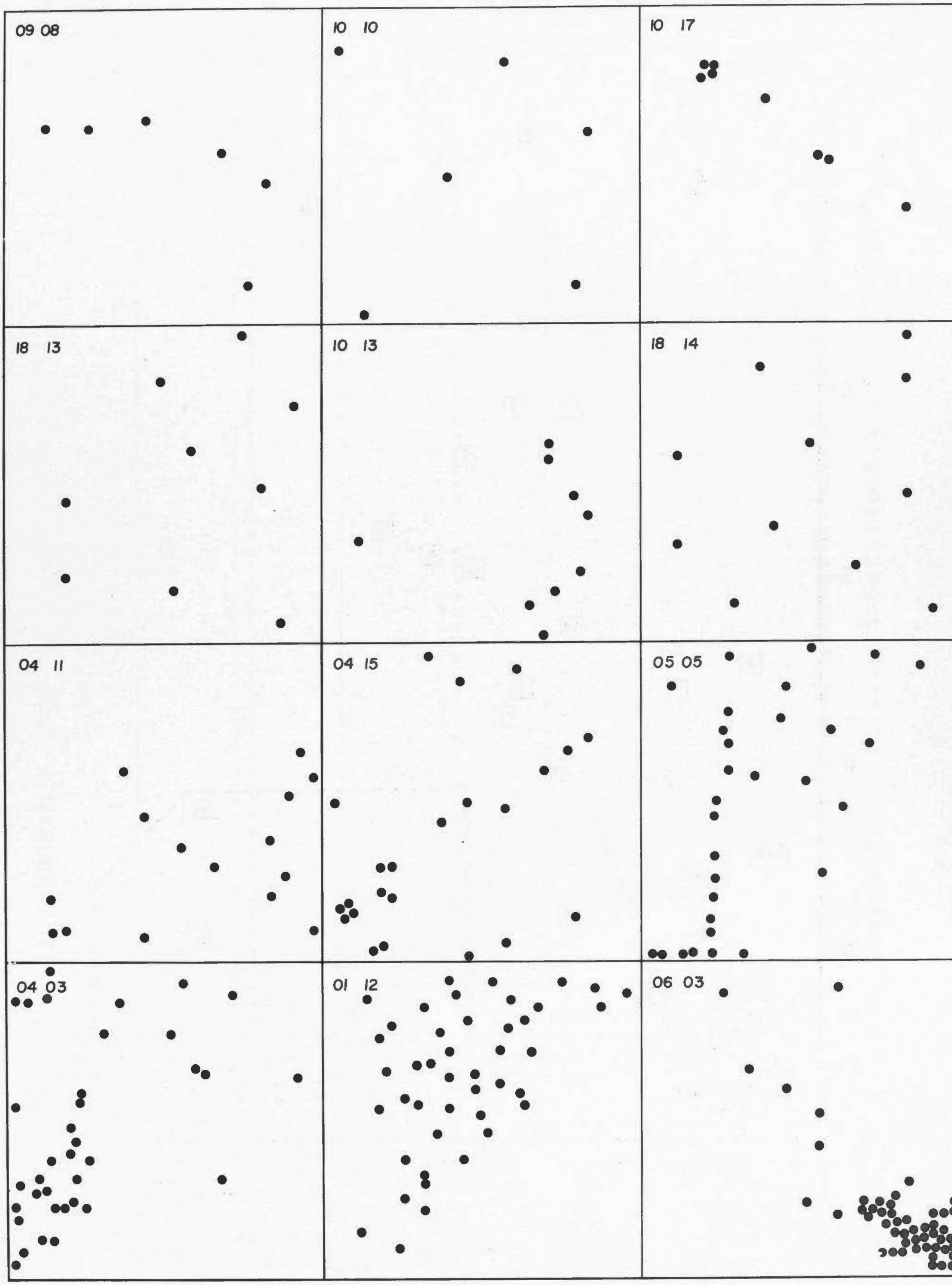

Figure 19

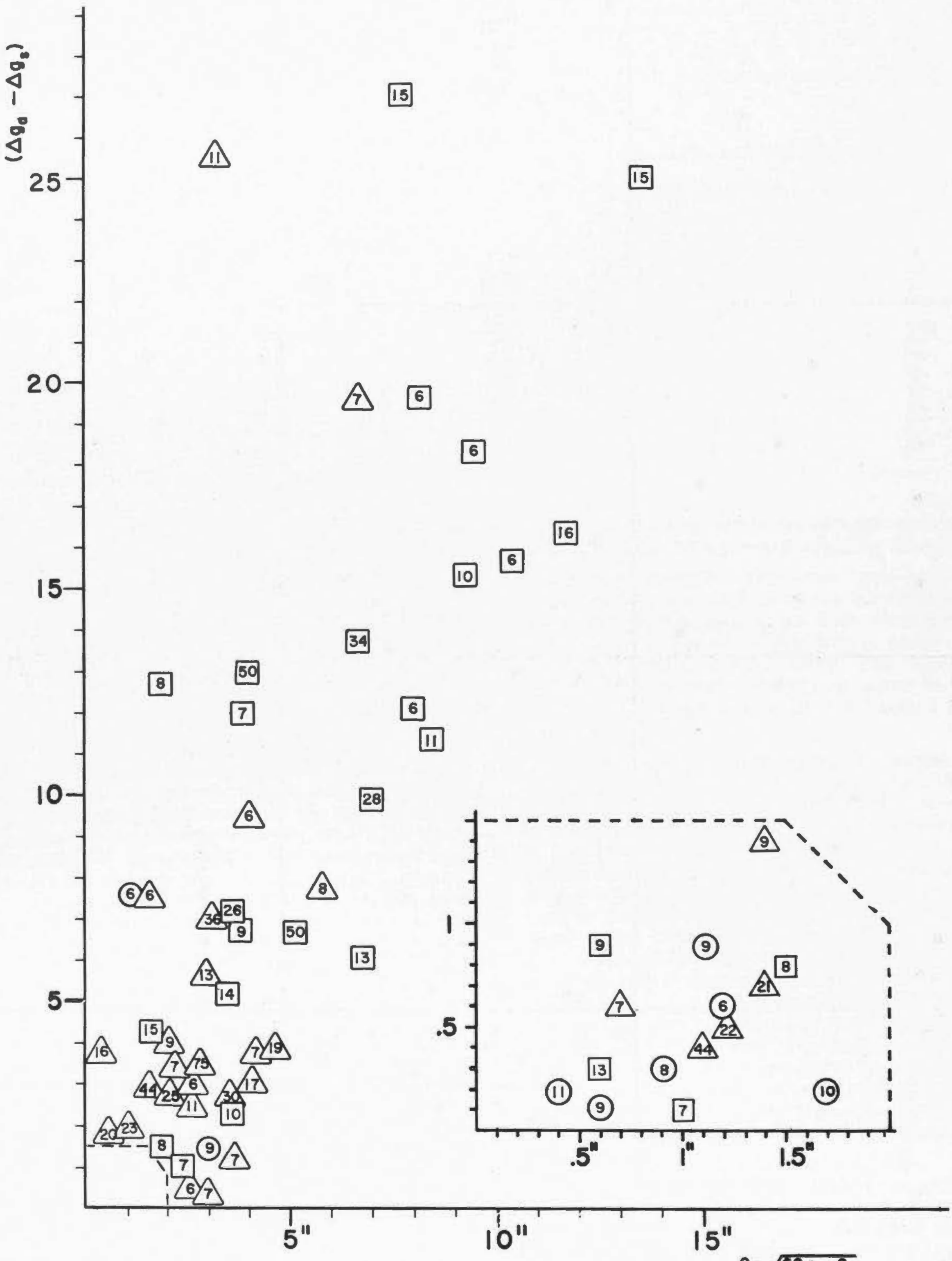

Figure 20

TABLE V

N	$\lvert \Delta g_A - \Delta g_B \rvert$	θ	Distribution	Identification
6	15.6	10.4	p	09 11
	8.6	11.2	p	02 11
	19.6	8.2	p	06 17
	12.1	8.0	p	08 08
	9.5	4.0	a	06 16
	.5	2.7	a	11 17
	3.1	2.6	a	18 18
	7.6	1.2	g	07 11
	7.6	1.3	a	01 06
7	.1	1.0	p	08 07
	12.0	3.9	p	00 08
	1.0	2.4	p	18 10
	3.7	4.3	a	10 18
	19.6	6.7	a	00 10
	3.5	2.2	p	01 11
	.3	2.9	a	03 08
	.6	.7	a	17 13
	1.2	3.6	a	05 07
8	.3	.9	g	17 14
	.8	1.5	b	05 14
	12.7	1.9	b	09 12
	1.5	1.8	b	08 10
	7.7	5.8	a	09 09
9	.9	1.1	g	18 15
	1.4	3.0	g	18 16
	.9	.6	b	06 07
	3.9	2.1	a	05 06
	1.4	1.4	a	09 13
10	2.3	3.6	b	12 04
	.2	1.7	g	18 22
	15.3	9.3	b	12 03
11	25.8	3.2	a	02 11
	11.3	8.4	b	09 07
	2.6	2.6	a	07 10
13	5.7	3.0	a	05 15
	6.1	6.8	p	05 16
	.3	.6	p	09 10
14	5.2	3.5	p	07 16
15	3.5	2.5	a	08 16
	4.3	1.9	p	04 04
	27.0	7.7	p	08 05
16	16.3	11.7	p	04 12
	3.7	.3	a	05 12
17	3.1	4.1	a	08 06
19	3.7	4.6	a	09 05
20	1.8	.6	a	11 04
21	.7	1.4	a	10 04
22	.5	1.2	a	10 05
25	2.8	2.1	a	09 06
26	7.2	3.6	p	06 18
28	9.9	7.0	p	07 06
34	13.7	6.7	p	02 15
44	2.9	1.6	a	05 17
50	6.7	5.2	p	05 01

The approximation of the gravity field over a unit area by a second-order surface in two variables is adequate. The minimum number of points required to determine the coefficients is six. The third-order surface would require ten points. It is also known that the higher the order of the surface, the larger the number of points are needed (in excess of the minimum requirements) to control the fitting and not to introduce unwanted oscillations. Since not many more than ten points can be expected per unit area, the second-order surface may provide a good choice.

The necessary weighting functions were derived for the plane system. Then approximations for them were obtained. No systematic error due to the approximations can be observed. The approximations greatly simplify the calculations.

The test calculations indicate that the distribution of points over the unit areas plays an important role in estimating the gravity anomaly which represents the given unit area. It is concluded that on the average 10 points, evenly distributed over the unit area may supply very good input information. This would require about 4,000 points to the size of the area used in this study. Although the number of points available for the test calculations is about 6,000 it is believed that less favourable results were obtained than would have been possible from 4,000 points of even distribution.

Acknowledgments

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APPENDIX A

Astro-Geodetic Deflections

Station Number	Geodetic Coordinates						Deflection Components	
	φ			λ			ξ''	η''
71	46°	03′	13.98	71°	53′	18.38	12.8	-13.0
73	46	51	33.04	71	13	08.53	- 8.4	- .8
74	47	02	32.74	70	53	23.05	-11.6	- 5.5
77	45	27	05.45	73	17	32.42	3.0	- 1.4
78	48	27	33.31	72	05	20.49	10.7	- 2.6
82	48	24	59.20	71	06	50.94	4.5	- 5.8
83	48	25	44.40	71	03	16.74	6.1	- 6.7
86	45	17	24.95	74	12	07.89	5.5	4.1
87	45	17	29.35	74	10	35.50	8.4	- 1.8
88	45	01	04.68	73	47	45.82	9.2	- 2.4
89	48	00	55.92	74	21	25.58	3.7	- 4.0
90	46	12	37.01	73	12	01.31	.3	- 1.4
91	46	16	09.25	76	19	15.98	3.3	- 1.6
95	45	50	33.56	76	44	13.95	.5	- 6.5
96	45	50	03.34	71	22	55.51	- .8	.8
98	48	14	45.56	69	32	49.36	- 3.5	4.7
100	45	05	33.60	74	15	15.30	8.3	- 6.2
102	47	16	17.28	72	18	14.06	- .7	- 2.9
103	47	39	32.02	72	16	23.99	3.3	- 6.4
106	46	38	32.73	76	01	37.61	3.4	2.3
107	45	16	06.56	72	09	22.70	- 2.0	- .3
108	48	49	22.00	79	03	06.73	3.4	- 2.4
109	46	22	50.22	75	57	14.41	3.7	2.8
110	45	44	48.30	73	36	08.50	2.7	- 2.2
113	45	30	18.71	73	34	42.91	2.1	1.1
116	47	00	30.50	70	52	18.82	- 6.2	3.8
121	46	42	39.28	71	53	31.86	- 5.0	- 3.8
122	48	36	34.82	69	05	22.76	- 1.3	2.3
124	45	27	02.76	74	17	54.91	8.7	- 1.4
125	46	59	21.46	72	10	50.18	- 3.6	- 5.0
127	48	30	54.17	72	13	25.89	10.6	- 5.9
*128	48	23	23.72	77	20	40.01	3.6	- 3.6
129	45	36	15.29	76	29	37.82	- 1.1	- 3.4
132	45	29	08.26	72	31	43.48	2.5	-11.2
134	47	40	24.35	69	43	46.59	2.0	- 2.6
136	45	56	21.59	70	56	32.93	4.0	- 1.2
140	45	57	45.14	73	42	51.25	3.1	- 1.7
141	47	22	16.36	70	24	30.58	- 5.6	7.4
143	48	08	32.21	69	42	59.20	- 4.0	2.7
144	46°	36′	11.61	72°	37′	10.61	- 2.4	- 4.9
146	48	06	31.50	69	09	16.94	6.8	- 8.0
*150	47	45	44.14	73	19	17.70	3.5	- 1.5
152	46	24	23.40	80	24	59.29	- .6	- .5
153	45	09	05.27	76	01	54.85	2.3	- 4.6
154	46	33	20.00	81	05	14.80	.5	- 3.0
155	44	27	43.09	77	50	21.72	- 3.5	- 4.5
156	45	07	32.83	74	50	06.11	3.8	- 3.8
158	48	46	41.17	80	46	41.08	3.3	- 1.1
159	46	09	12.50	78	28	32.27	5.2	- 1.5
164	44	41	22.23	76	26	04.28	- 1.8	- .7
167	44	57	33.35	75	16	56.02	3.4	- 8.0
168	44	41	22.60	75	42	18.99	2.1	- 6.6
172	50	07	35.05	81	38	31.69	6.2	- 1.4
173	44	19	30.14	76	10	01.63	- 1.4	- 4.1
176	44	01	59.27	77	06	19.04	- .6	- 1.5
180	44	30	42.39	79	02	25.59	- .4	- 3.3
181	48	32	01.30	80	27	54.31	5.0	- 2.0
182	46	18	39.80	78	42	16.35	1.7	- 4.1
183	45	17	43.27	74	48	49.36	3.9	- 5.2
185	45	26	05.77	75	23	47.21	.4	3.3
186	47	30	39.19	79	40	47.44	3.7	- 5.2
187	46	03	33.79	79	27	04.12	6.8	- 1.8
188	46	18	42.89	79	27	58.21	- 1.6	- .1
189	45	08	56.34	75	42	49.33	.6	3.9
192	44	31	51.18	79	39	33.63	- 1.5	- 1.9
193	45	23	36.77	75	42	52.70	2.6	- 5.2
194	45	20	02.89	76	17	31.26	3.7	- 1.7
195	45	49	04.78	77	05	23.94	2.9	- 3.4
196	45	51	36.69	77	18	37.56	- .9	- .2
197	45	30	05.57	75	03	21.41	2.9	- 6.9
198	44	25	59.10	76	38	19.60	- 2.4	- 2.9
199	45	28	15.45	76	40	53.62	3.6	- 3.3

Station Number	Geodetic Coordinates						Deflection Components	
	φ			λ			ξ''	η''
202	45	51	26.62	81	50	59.71	− .3	− 2.2
203	44	36	48.07	78	39	25.77	− 1.6	− 2.7
205	46	23	57.10	79	56	07.25	− 1.4	− .1
206	45	23	07.74	74	54	22.93	5.1	− 9.5
208	46	12	50.76	77	53	43.44	.3	− 2.8
210	44	46	01.66	79	58	57.71	.1	− 2.2
213	45	30	33.82	74	39	58.85	4.5	− 1.2
215	47	36	16.71	79	31	44.66	6.3	2.3
216	44	28	43.93	77	19	59.72	.7	− 2.7
218	49	24	33.00	81	03	46.54	2.6	− 1.0
*306	49	30	14.77	74	23	02.25	3.6	− 3.6
307	51	25	55.92	72	52	19.15	− 3.2	− 7.4
308	51	30	21.35	72	47	17.01	− 1.1	− 4.5

*origins of sets.

APPENDIX B

Gravimetrically Calculated Deflections

SET I

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
00	00	5.1	-4.2	5.2	-4.1			
00	01	4.8	-4.2	6.9	-4.0			
00	02	4.5	-4.0	7.3	-4.1	- 1.3	- 1.0	6
00	03	4.1	-4.2	7.8	-4.4			1
00	04	3.6	-3.8	3.5	-3.9			4
00	05	3.8	-2.0	3.5	-1.9			
00	06	5.0	-3.9	5.2	-3.7			2
00	07	4.6	-5.6	4.7	-5.6			494
00	08	4.0	-3.8	4.1	-3.8			413
00	09	3.5	-3.7	8.2	-3.7			421
00	10	3.1	-4.2	8.7	-4.2			176
00	11	3.1	-3.1	3.2	-3.1			127
00	12	3.9	-1.7	4.0	-1.8			196
00	13	5.6	-2.9	6.1	-3.0			144
00	14	5.3	-3.7	6.4	-3.8	- 2.1	- 1.0	47
00	15	4.9	-3.4	4.9	-3.4			4
00	16	4.9	-3.5	4.8	-3.5			
00	17	5.0	-3.2	5.0	-3.2			
00	18	5.2	-3.2	5.2	-3.2			
01	00	4.7	-4.5	4.8	-4.5			
01	01	4.3	-4.7	4.3	-4.7			
01	02	3.0	-4.8	2.8	-4.9			4
01	03	1.8	-4.7	1.8	-4.8	- 1.0	3.0	7
01	04	2.4	-3.9	9.2	-4.1			4
01	05	3.3	-1.3	3.6	-1.0	10.2	- 6.0	6
01	06	5.0	-2.6	5.1	-2.2	0.3	- 6.9	15
01	07	6.4	-5.6	6.5	-5.7			100
01	08	4.8	-5.0	6.9	-5.1			102
01	09	3.0	-4.7	3.0	-4.7			355
01	10	2.4	-4.2	2.5	-4.1			138
01	11	2.6	-2.8	9.0	-2.8			102
01	12	2.3	-0.5	2.5	-0.5			112
01	13	3.4	-1.6	8.2	-1.6			130
01	14	4.9	-4.2	5.1	-4.4			63
01	15	4.0	-3.9	4.0	-4.1	1.0	1.7	48
01	16	4.0	-3.1	3.9	-3.2			
01	17	4.7	-2.9	7.2	-2.9			
01	18	5.0	-3.0	5.0	-2.9			
02	00	5.3	-4.9	5.3	-5.0	0.6	1.3	20
02	01	4.5	-5.9	4.3	-5.9			
02	02	3.0	-6.6	3.0	-6.6			4
02	03	2.6	-4.5	2.7	-4.5			1
02	04	3.9	-2.4	4.1	-2.2			4
02	05	4.3	-0.1	4.8	0.1	- 4.2	- 3.0	6
02	06	3.9	-1.0	3.9	-1.0			3
02	07	4.2	-5.8	4.3	-5.9			3
02	08	3.8	-6.9	4.0	-7.1			8
02	09	3.6	-5.5	3.7	-5.5	3.3	4.8	21
02	10	4.1	-4.3	4.2	-4.0	0.9	1.0	10
02	11	3.8	-3.3	4.1	-3.2	2.1	0.8	35
02	12	3.1	-1.3	3.6	-1.2	0.2	0.1	48
02	13	1.9	-0.3	2.5	-0.3			79
02	14	2.8	-3.2	3.6	-3.5			150
02	15	3.6	-4.4	3.8	-5.0			94
02	16	4.0	-2.6	3.4	-2.8			10
02	17	4.5	-2.2	4.2	-2.1			
02	18	4.8	-2.7	4.8	-2.6			
03	00	5.8	-5.2	6.1	-5.5			67
03	01	5.2	-6.9	5.4	-7.1	- 2.7	2.2	14
03	02	5.2	-6.8	4.9	-6.6	- 2.7	2.0	9
03	03	5.9	-3.8	5.8	-3.7			6
03	04	5.5	-2.2	5.5	-2.0			3
03	05	4.7	-1.8	4.6	-1.6			4
03	06	2.3	-2.7	2.3	-2.6			
03	07	1.8	-5.2	1.9	-5.1			7
03	08	3.6	-5.7	3.7	-5.7	1.4	1.7	12
03	09	5.0	-4.4	5.3	-4.4			3
03	10	5.7	-3.8	5.9	-3.6			12
03	11	5.9	-3.5	6.2	-3.1			5
03	12	5.8	-2.1	6.4	-1.9	- 3.6	- 0.1	6
03	13	5.9	-1.5	5.8	-1.7	- 3.2	1.3	31
03	14	4.3	-3.0	3.2	-3.4			
03	15	3.3	-3.6	3.2	-4.9			
03	16	2.9	-2.4	2.9	-2.7	- 3.0	10.4	50
03	17	4.0	-1.9	3.9	-1.2			
03	18	4.6	-2.3	4.6	-2.2			

IDENT.		PLUMB-LINE DEFLECTION				CORRECTIONS		NUMBER OF POINTS
X	Y	AVERAGE		SURFACE				
04	00	5.3	-5.3	5.4	-5.3	- 5.4	0.9	31
04	01	5.6	-6.8	5.8	-7.5	- 4.0	2.3	9
04	02	6.1	-6.1	6.0	-6.2	- 2.6	- 5.7	11
04	03	6.4	-4.1	6.4	-3.6			4
04	04	5.5	-3.0	5.6	-2.8			3
04	05	5.1	-2.8	5.2	-2.7			2
04	06	4.5	-3.7	4.7	-3.6			6
04	07	4.7	-4.2	5.1	-4.2			3
04	08	5.9	-4.1	6.0	-4.2			1
04	09	6.1	-4.0	6.2	-3.9			3
04	10	5.9	-3.6	6.0	-3.4			4
04	11	5.9	-3.5	6.0	-2.9			3
04	12	6.1	-1.1	6.0	-1.3	0.1	- 6.9	11
04	13	7.4	-1.0	6.9	-2.8			1
04	14	7.7	-5.6	6.8	-5.7	- 4.3	-10.7	12
04	15	4.9	-6.2	4.7	-5.7	- 4.6	3.5	33
04	16	3.1	-2.0	3.8	-2.3			1
04	17	3.5	-0.5	3.7	-0.4			
04	18	3.8	-1.7	3.8	-1.7			
05	00	4.3	-5.2	4.1	-5.3			2
05	01	4.5	-6.3	4.5	-6.7			4
05	02	4.9	-5.8	5.2	-5.9	- 0.2	1.4	27
05	03	4.7	-4.2	4.9	-3.9			2
05	04	4.8	-3.4	5.0	-3.1			3
05	05	4.8	-2.9	5.1	-2.7	- 0.9	- 1.0	10
05	06	5.7	-3.4	5.9	-3.0	- 2.3	- 1.0	13
05	07	6.7	-3.6	6.8	-3.5	- 0.6	- 1.6	7
05	08	6.8	-4.0	6.8	-4.3			
05	09	6.5	-4.4	6.6	-4.3			1
05	10	5.9	-4.0	6.3	-3.9			3
05	11	5.0	-3.7	4.9	-3.4			3
05	12	4.8	-1.2	4.2	-1.4	3.5	- 7.4	17
05	13	3.7	0.1	3.7	-1.0			1
05	14	4.8	-5.0	5.4	-5.4			10
05	15	5.8	-7.7	5.1	-7.5	- 1.5	- 4.8	13
05	16	5.4	-3.5	5.3	-3.4	- 0.1	3.3	9
05	17	3.5	-1.1	3.5	-1.3			5
05	18	3.8	-0.9	3.9	-0.9	- 1.0	- 1.0	10

IDENT.		PLUMB-LINE DEFLECTION				CORRECTIONS		NUMBER OF POINTS
X	Y	AVERAGE		SURFACE				
06	00	4.2	-5.8	4.2	-5.8	4.5	2.4	8
06	01	4.5	-5.9	4.5	-6.2	3.9	- 2.1	29
06	02	4.8	-5.8	4.8	-5.8			6
06	03	4.7	-4.2	4.7	-4.0			1
06	04	4.5	-3.3	4.5	-2.9			3
06	05	4.0	-2.6	4.1	-2.4	- 0.9	- 4.6	8
06	06	3.9	-2.1	4.2	-2.2			2
06	07	5.0	-3.0	4.7	-3.1			1
06	08	5.1	-4.1	5.0	-4.2			7
06	09	5.0	-4.6	5.0	-4.2	- 1.9	2.6	7
06	10	4.9	-5.0	4.9	-5.0	4.2	1.0	6
06	11	4.3	-4.4	4.2	-4.7	- 1.7	3.2	8
06	12	3.1	-2.7	2.8	-2.9			3
06	13	0.5	-1.5	0.5	-1.9			
06	14	0.1	-3.5	0.2	-4.4			15
06	15	3.1	-5.5	3.1	-5.5	33.7	-42.9	12
06	16	4.9	-4.3	4.9	-3.6			5
06	17	4.4	-2.0	4.5	-1.9			6
06	18	4.5	0.0	4.7	0.0			3
07	00	5.4	-5.9	5.3	-5.8			5
07	01	5.1	-5.6	5.4	-5.9	- 2.0	1.3	23
07	02	4.9	-5.8	5.0	-5.8	0.9	- 1.4	17
07	03	4.9	-4.6	4.9	-4.3			7
07	04	4.5	-4.1	4.4	-3.8			5
07	05	3.4	-3.7	3.2	-3.3	- 2.7	- 1.1	16
07	06	2.4	-2.4	2.3	-2.4	- 2.7	- 3.8	9
07	07	2.1	-2.4	2.0	-2.7			6
07	08	2.3	-4.1	2.2	-4.2			4
07	09	2.3	-4.6	2.2	-4.4			4
07	10	2.7	-4.5	2.3	-4.5			2
07	11	3.5	-5.0	3.4	-5.1			3
07	12	1.8	-4.9	2.0	-5.0			1
07	13	0.7	-3.0	0.9	-3.3			5
07	14	1.4	-1.7	1.8	-2.1	6.7	2.8	16
07	15	4.8	-2.5	5.7	-2.6			5
07	16	6.2	-3.2	6.6	-3.0			9
07	17	5.9	-2.3	6.1	-2.3			6
07	18	5.3	0.0	5.4	0.0			5

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS	IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
08	00	5.4	-6.3	5.4	-6.0	- 1.0	- 2.9	30	10	00	7.2	-6.5	7.2	-6.4			5
08	01	4.7	-6.4	4.8	-6.4	- 4.1	0.2	10	10	01	7.4	-6.7	7.3	-6.7			3
08	02	4.4	-6.1	4.6	-6.3			5	10	02	7.0	-6.7	6.9	-6.7			2
08	03	4.1	-4.6	4.1	-4.5			7	10	03	5.8	-6.3	5.7	-6.3			2
08	04	3.6	-4.6	3.5	-4.5			5	10	04	5.2	-5.8	5.1	-5.8			3
08	05	2.9	-4.4	2.7	-4.2			2	10	05	5.3	-4.9	5.3	-4.9			2
08	06	3.0	-3.1	2.6	-3.1			4	10	06	5.1	-2.9	5.0	-2.9			4
08	07	2.2	-2.9	2.0	-3.0			3	10	07	5.7	-2.4	5.6	-2.4			2
08	08	2.2	-3.8	2.0	-3.8			4	10	08	6.4	-2.7	6.3	-2.7			3
08	09	2.9	-4.3	2.7	-4.3			3	10	09	6.6	-3.3	6.6	-3.2			5
08	10	3.1	-3.8	3.0	-3.7			3	10	10	6.8	-3.9	6.8	-3.9			3
08	11	3.1	-5.3	3.0	-5.0			3	10	11	7.3	-3.8	7.2	-3.8			3
08	12	3.3	-5.6	3.2	-5.6	5.4	12.5	6	10	12	8.1	-3.4	7.9	-3.4			1
08	13	4.3	-2.2	4.3	-2.6	- 1.3	- 1.9	7	10	13	8.9	-1.4	8.8	-1.5			3
08	14	5.5	0.9	5.6	0.7			2	10	14	8.5	0.4	8.4	0.4			2
08	15	7.3	-0.3	7.6	-0.3			4	10	15	7.7	-0.7	7.6	-0.5	- 3.2	1.9	6
08	16	7.6	-3.9	8.2	-3.8				10	16	7.4	-2.7	7.1	-2.6			2
08	17	6.7	-3.3	7.0	-3.4			8	10	17	7.9	-2.8	7.8	-3.0			4
08	18	6.5	-0.2	6.7	-0.3			2	10	18	7.8	-1.8	7.8	-1.8			2
09	00	6.0	-6.8	6.0	-6.7	- 1.1	- 2.9	16	11	00	6.0	-5.8	6.0	-5.8			3
09	01	5.8	-6.6	5.5	-6.6	- 6.3	- 3.7	6	11	01	6.9	-6.6	6.9	-6.6			4
09	02	5.8	-6.2	5.8	-6.3			2	11	02	7.3	-7.5	7.3	-7.5	1.5	1.0	6
09	03	5.3	-5.4	5.2	-5.4			5	11	03	7.1	-7.4	7.1	-7.4			1
09	04	4.3	-5.4	4.3	-5.4			1	11	04	6.2	-5.9	6.1	-5.9			1
09	05	3.9	-4.4	3.8	-4.4			2	11	05	6.0	-4.5	5.9	-4.4			4
09	06	3.7	-3.1	3.6	-3.0			4	11	06	6.4	-2.8	6.3	-2.8			2
09	07	4.2	-3.2	4.1	-3.2			3	11	07	6.5	-1.9	6.4	-1.9			3
09	08	4.4	-3.3	4.3	-3.3			3	11	08	6.7	-2.4	6.6	-2.4			3
09	09	5.3	-3.5	5.2	-3.5			3	11	09	6.5	-3.1	6.4	-3.1			4
09	10	5.4	-4.2	5.3	-4.2			3	11	10	6.6	-3.8	6.6	-3.8			1
09	11	5.5	-4.9	5.3	-4.8			6	11	11	6.7	-3.8	6.6	-3.8			3
09	12	7.0	-4.1	6.6	-4.1			1	11	12	6.8	-2.2	6.8	-2.2			1
09	13	7.5	-1.5	7.3	-1.6			3	11	13	7.6	-0.9	7.6	-0.9			
09	14	7.6	1.2	7.6	1.2			3	11	14	7.3	-0.2	7.3	-0.2			2
09	15	6.7	-0.1	6.8	0.3			5	11	15	7.0	-1.1	7.0	-1.0			1
09	16	7.2	-3.7	7.3	-3.7	3.4	- 2.8	41	11	16	7.3	-3.2	7.3	-3.2			2
09	17	8.1	-3.2	8.1	-3.7			2	11	17	7.0	-2.8	7.0	-2.8			2
09	18	7.7	-0.6	7.8	-0.7			9	11	18	6.8	-2.0	6.9	-2.0			4

IDENT.		PLUMB-LINE DEFLECTION				CORRECTIONS		NUMBER OF POINTS
X	Y	AVERAGE		SURFACE				
12	00	5.5	-5.5	5.5	-5.5			1
12	01	6.2	-5.3	6.2	-5.3			
12	02	7.0	-6.5	7.1	-6.5			
12	03	7.8	-7.6	7.8	-7.6			
12	04	7.3	-7.0	7.3	-7.0			3
12	05	7.0	-4.6	7.0	-4.6			1
12	06	6.7	-2.6	6.7	-2.6			2
12	07	6.4	-2.0	6.4	-2.0			4
12	08	6.1	-2.7	6.0	-2.7			2
12	09	5.8	-3.4	5.8	-3.3			2
12	10	5.8	-3.6	5.8	-3.5			3
12	11	6.0	-3.7	6.0	-3.6			1
12	12	5.5	-2.6	5.5	-2.5			1
12	13	4.3	-1.1	4.3	-1.0			
12	14	4.6	-0.6	4.6	-0.6			1
12	15	4.8	-0.4	4.8	-0.4			4
12	16	6.3	-2.4	6.3	-2.3			4
12	17	6.1	-3.4	6.2	-3.3			2
12	18	5.6	-2.4	5.9	-2.4			4
13	00	6.2	-5.5	6.2	-5.5			2
13	01	6.0	-5.1	5.9	-5.1			
13	02	6.2	-6.2	6.2	-6.2			
13	03	5.9	-7.3	5.9	-7.3			
13	04	7.3	-6.6	7.3	-6.6			1
13	05	7.2	-4.9	7.2	-4.9			4
13	06	6.2	-3.5	6.2	-3.5			3
13	07	5.2	-2.5	5.2	-2.5			3
13	08	5.0	-2.9	5.0	-2.9			4
13	09	5.2	-3.3	5.2	-3.3			1
13	10	5.6	-3.5	5.6	-3.4			3
13	11	5.7	-3.7	5.7	-3.7			3
13	12	6.0	-3.8	6.0	-3.8			1
13	13	3.9	-2.0	3.9	-2.0			1
13	14	3.2	-0.1	3.2	-0.1			3
13	15	3.3	0.3	3.2	0.4			4
13	16	3.3	-2.0	3.3	-1.8			5
13	17	3.9	-3.6	4.4	-3.2	- 1.2	0.1	9
13	18	4.7	-2.1	5.7	-2.0			9
14	00	6.5	-5.6	6.5	-5.6			1
14	01	6.4	-5.0	6.4	-5.0			3
14	02	6.4	-6.5	6.4	-6.5			
14	03	6.4	-6.7	6.4	-6.7			2
14	04	6.2	-5.9	6.2	-5.9			
14	05	6.1	-5.8	6.1	-5.8			2
14	06	5.6	-4.5	5.6	-4.5			4
14	07	5.0	-3.1	5.0	-3.1			3
14	08	5.1	-2.5	5.1	-2.5			4
14	09	5.9	-2.6	5.9	-2.6			4
14	10	6.6	-3.1	6.5	-3.1			4
14	11	7.1	-2.9	7.0	-2.9			5
14	12	7.5	-4.2	7.5	-4.2			3
14	13	7.1	-4.0	7.1	-4.0			2
14	14	5.3	-1.3	5.3	-1.3			3
14	15	3.4	-0.7	3.4	-0.6	9.1	3.9	7
14	16	2.4	-1.7	2.4	-1.2	4.6	- 2.2	10
14	17	4.1	-2.2	4.2	-0.9	1.5	- 3.5	31
14	18	4.7	-1.2	5.7	-1.2	- 5.8	- 3.3	9
15	00	6.2	-5.3	6.2	-5.3			
15	01	6.8	-5.1	6.8	-5.0			
15	02	6.9	-6.1	6.9	-6.1			
15	03	7.1	-7.6	7.1	-7.6			
15	04	5.2	-7.0	5.2	-6.9			2
15	05	5.1	-5.5	5.0	-5.4	1.2	0.6	6
15	06	5.4	-4.7	5.4	-4.6	- 0.1	2.1	9
15	07	5.6	-3.1	5.6	-3.1			2
15	08	6.2	-1.9	6.2	-1.9	- 0.7	- 0.9	10
15	09	7.0	-1.8	7.0	-1.8	- 0.7	- 0.2	8
15	10	7.6	-2.4	7.6	-2.5	- 1.8	- 0.1	10
15	11	8.0	-2.8	8.0	-2.8	- 0.1	1.4	7
15	12	7.9	-3.7	7.8	-3.6			6
15	13	8.8	-4.1	8.9	-4.1			4
15	14	8.1	-3.4	8.2	-3.4			4
15	15	6.5	-2.6	6.6	-2.6			7
15	16	5.2	-1.4	4.9	-1.3			3
15	17	4.9	-1.4	3.9	0.0	0.1	- 4.6	6
15	18	5.0	-0.9	4.0	-0.6	- 3.7	2.6	7

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
16	00	6.7	-5.1	6.7	-5.1			2
16	01	6.8	-4.4	6.8	-4.4			
16	02	6.9	-6.0	6.9	-6.0			
16	03	6.8	-8.4	6.8	-8.3			
16	04	6.1	-8.0	6.1	-8.0			4
16	05	5.9	-5.4	5.9	-5.4			4
16	06	6.4	-4.2	6.3	-4.1			3
16	07	6.7	-2.7	6.7	-2.7	0.8	- 0.2	10
16	08	7.0	-1.2	6.9	-1.3	0.1	- 1.5	9
16	09	7.3	-1.2	7.3	-1.2	- 1.1	- 0.6	9
16	10	7.6	-2.0	7.6	-2.0	- 0.1	- 0.3	10
16	11	7.8	-2.8	8.0	-2.8	- 1.8	1.1	11
16	12	8.0	-3.3	8.2	-3.2	- 0.3	0.4	10
16	13	8.4	-3.7	8.8	-3.6			4
16	14	8.7	-3.7	8.8	-3.8	1.5	0.7	7
16	15	8.6	-3.5	8.6	-3.6			4
16	16	7.8	-3.0	7.6	-3.4			1
16	17	6.7	-2.1	6.3	-1.6	- 4.9	8.7	6
16	18	6.2	-0.4	5.0	0.3	1.4	- 1.8	8
17	00	7.7	-4.9	7.7	-4.9			
17	01	7.2	-5.0	7.2	-5.0			
17	02	7.2	-6.0	7.2	-6.0			
17	03	7.4	-8.5	7.4	-8.5			
17	04	7.8	-8.1	7.9	-8.1	0.5	- 0.4	11
17	05	8.4	-5.0	8.4	-5.0	- 0.7	- 0.9	10
17	06	8.5	-3.9	8.5	-3.9	0.3	- 0.2	11
17	07	8.4	-2.6	8.3	-2.7	- 0.3	- 0.5	12
17	08	8.1	-1.4	8.1	-1.4	- 0.6	0.1	10
17	09	7.5	-1.4	7.6	-1.2	1.2	- 0.3	11
17	10	7.4	-1.8	7.4	-1.8	1.2	- 0.3	18
17	11	7.4	-2.4	7.5	-2.4	0.5	- 0.8	22
17	12	7.7	-3.0	7.8	-2.7	- 0.2	- 0.4	14
17	13	7.9	-3.3	8.0	-3.3	12.8	- 9.4	7
17	14	8.3	-3.5	8.3	-3.8			3
17	15	8.7	-3.6	8.7	-3.7			3
17	16	8.9	-3.1	8.9	-3.3			3
17	17	8.6	-2.7	8.9	-2.6			3
17	18	7.8	-1.3	7.6	-1.0			4
18	00	6.9	-4.7	6.9	-4.7			
18	01	7.0	-5.0	7.0	-5.0			
18	02	7.3	-5.6	7.3	-5.6			
18	03	8.3	-7.6	8.3	-7.5			
18	04	10.3	-7.0	10.3	-7.0	- 2.7	- 0.3	13
18	05	11.0	-4.7	11.0	-4.7	- 1.0	- 1.2	11
18	06	11.0	-4.1	11.0	-4.1	- 0.6	1.4	18
18	07	10.8	-3.0	10.8	-3.0	0.4	- 0.8	13
18	08	10.1	-2.1	10.2	-2.0	- 0.1	- 0.3	11
18	09	9.3	-1.9	9.3	-1.8	0.6	- 0.2	11
18	10	8.7	-2.1	8.6	-2.0	0.9	- 1.8	14
18	11	8.3	-2.5	8.2	-2.3	1.0	- 0.1	23
18	12	8.1	-2.8	8.0	-2.7	1.3	- 0.3	10
18	13	8.2	-3.0	7.9	-3.1			4
18	14	8.4	-2.9	8.3	-3.1			2
18	15	8.6	-3.2	8.6	-3.2			
18	16	8.8	-3.4	8.7	-3.4			2
18	17	8.5	-3.1	8.5	-3.1			3
18	18	8.3	-2.1	8.2	-2.0			3

SET II

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
00	00	1.8	-2.3	1.8	-2.2			
00	01	1.3	-1.2	1.4	-1.0			
00	02	1.2	-1.2	1.4	-1.1			
00	03	1.5	-1.1	1.8	-0.9			
00	04	1.7	-0.8	2.1	-0.8			
00	05	2.4	-0.7	2.7	-0.8			
00	06	2.7	-0.9	2.9	-1.0			
00	07	3.0	-0.9	3.1	-1.0			
00	08	3.1	-1.0	3.2	-1.1			
00	09	3.2	-1.1	3.3	-1.2			
00	10	3.2	-1.2	3.3	-1.2			
00	11	3.3	-1.2	3.3	-1.3			
00	12	3.3	-1.3	3.4	-1.3			
00	13	3.4	-1.3	3.4	-1.3			
00	14	3.4	-1.3	3.4	-1.3			
00	15	3.4	-1.3	3.4	-1.3			
00	16	3.4	-1.3	3.4	-1.4			
00	17	3.5	-1.3	3.5	-1.4			
00	18	3.5	-1.3	3.5	-1.3			
01	00	3.0	-3.1	3.3	-3.0	0.3	- 1.1	46
01	01	2.5	-1.9	2.7	-1.7			4
01	02	2.2	-1.1	2.5	-0.8			5
01	03	2.2	-0.9	2.9	-0.4			3
01	04	2.1	-0.4	3.4	-0.4			1
01	05	2.0	-0.2	2.6	-0.6			
01	06	2.1	-0.5	2.4	-0.8			
01	07	2.8	-0.6	3.0	-0.7			
01	08	3.2	-0.9	3.3	-1.0			
01	09	3.2	-1.1	3.3	-1.2			
01	10	3.2	-1.2	3.3	-1.3			
01	11	3.2	-1.2	3.3	-1.2			
01	12	3.3	-1.2	3.4	-1.2			
01	13	3.3	-1.3	3.5	-1.3			
01	14	3.3	-1.3	3.5	-1.4			
01	15	3.3	-1.3	3.4	-1.4			
01	16	3.4	-1.3	3.4	-1.4			
01	17	3.4	-1.3	3.4	-1.3			
01	18	3.4	-1.3	3.4	-1.3			

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
02	00	4.1	-2.9	4.4	-2.7	0.7	- 1.6	8
02	01	4.7	-1.8	4.9	-1.7			4
02	02	4.8	-1.4	5.0	-1.0			4
02	03	4.6	-1.2	4.8	0.1	- 1.9	- 0.3	8
02	04	4.2	-1.1	4.4	-1.2	-36.0	2.5	6
02	05	3.0	-0.9	3.2	-2.1	- 3.7	1.4	16
02	06	2.6	-0.1	2.7	-0.5			4
02	07	3.1	0.1	3.1	0.0			
02	08	3.5	-0.8	3.7	-0.9			
02	09	3.4	-1.2	3.5	-1.4			
02	10	2.9	-1.2	2.9	-1.4			
02	11	3.1	-1.1	2.9	-1.1			
02	12	3.1	-1.2	3.3	-1.1			
02	13	3.1	-1.3	3.5	-1.3			
02	14	3.1	-1.3	3.4	-1.4			
02	15	3.2	-1.2	3.2	-1.4			
02	16	3.2	-1.2	3.2	-1.3			
02	17	3.2	-1.2	3.2	-1.3			
02	18	3.3	-1.2	3.3	-1.3			
03	00	4.5	-2.3	4.3	-2.1	- 0.7	- 0.3	9
03	01	4.6	-1.4	4.6	-1.3			
03	02	4.6	-1.3	4.7	-0.9			
03	03	5.0	-0.8	4.9	-0.2			5
03	04	7.0	-1.8	6.3	-1.8			2
03	05	4.4	-2.3	4.4	-2.7	- 0.5	- 0.8	31
03	06	3.3	-0.5	4.0	-0.8	1.6	- 3.6	44
03	07	2.5	0.3	3.3	0.1	- 9.8	2.9	41
03	08	2.2	-0.8	2.7	-0.8			
03	09	1.8	-1.8	2.1	-2.2			
03	10	2.0	-1.1	1.9	-1.5			
03	11	3.2	-0.8	3.6	-0.6	-16.6	1.4	25
03	12	3.0	-1.3	3.6	-0.6			
03	13	3.0	-1.3	4.5	-1.3			
03	14	2.9	-1.3	3.4	-1.8			
03	15	2.9	-1.2	3.0	-1.5			
03	16	3.0	-1.2	3.0	-1.3			
03	17	3.1	-1.2	3.0	-1.2			
03	18	3.2	-1.2	3.1	-1.2			

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
04	00	3.9	-2.1	3.6	-1.8			5
04	01	3.5	-1.9	3.4	-1.6			1
04	02	2.6	-1.7	2.8	-1.3			3
04	03	2.2	1.3	2.5	2.0			2
04	04	3.2	-1.5	4.6	-1.2			1
04	05	3.9	-4.7	6.4	-4.6	- 0.6	- 8.3	45
04	06	2.7	-1.6	4.2	-1.7	2.9	1.3	41
04	07	1.1	-0.8	2.1	-1.3	1.5	- 9.3	33
04	08	-0.0	-1.4	0.7	-1.9	0.1	2.9	40
04	09	-0.0	-1.6	-0.1	-2.6	- 1.8	3.4	35
04	10	0.9	0.4	1.0	0.7	- 3.0	2.0	50
04	11	2.6	0.2	3.8	0.8	- 6.4	3.2	46
04	12	3.2	-1.6	3.4	-0.6	- 0.3	- 0.2	42
04	13	2.7	-1.5	2.6	-1.6	5.7	5.1	10
04	14	2.3	-1.3	2.2	-2.9			
04	15	2.6	-1.0	2.5	-1.4			
04	16	2.8	-1.1	2.7	-1.2			
04	17	2.9	-1.1	2.8	-1.1			
04	18	3.0	-1.1	2.9	-1.1			
05	00	2.3	-2.3	1.8	-2.0			4
05	01	1.9	-2.7	1.7	-2.1			4
05	02	0.6	-2.7	0.5	-2.2			
05	03	0.1	-0.2	-0.0	1.0			4
05	04	-1.1	0.2	-1.0	2.3	3.1	- 4.2	14
05	05	2.6	-2.7	3.0	-2.8	- 7.7	2.7	27
05	06	2.5	-3.3	2.5	-5.2	- 2.4	- 2.9	44
05	07	1.0	-2.2	0.8	-2.8	- 2.1	- 2.4	49
05	08	0.2	-1.7	-0.1	-2.7			
05	09	0.3	-1.2	-0.0	-2.3	- 0.3	0.4	43
05	10	0.2	1.2	0.2	2.2	2.5	0.2	45
05	11	0.1	1.6	-0.4	2.1	3.2	- 5.2	41
05	12	2.1	-1.1	1.6	-1.8	- 5.0	3.5	40
05	13	2.2	-2.3	0.3	-2.5	1.5	0.4	35
05	14	1.9	-1.3	1.1	-1.7			
05	15	2.3	-0.6	2.1	-0.8			
05	16	2.5	-0.9	2.5	-1.0			
05	17	2.6	-1.1	2.5	-1.1			
05	18	2.5	-0.9	2.4	-1.0			

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
06	00	2.2	-3.1	1.6	-2.8	3.4	- 2.0	8
06	01	1.1	-3.4	1.1	-2.6			
06	02	1.7	-2.9	1.4	-2.5			4
06	03	2.0	-0.7	1.3	0.0			5
06	04	1.1	0.8	-0.6	1.7			4
06	05	0.9	-1.2	-1.7	-1.0			5
06	06	1.5	-3.8	0.2	-4.4	6.0	-18.4	8
06	07	1.5	-3.0	0.9	-3.7	2.4	- 0.7	13
06	08	1.9	-1.8	1.4	-2.5	- 0.4	- 0.3	11
06	09	2.0	-1.2	2.6	-1.8	3.1	2.5	20
06	10	1.7	0.0	1.8	0.1	- 0.5	1.8	30
06	11	0.1	1.9	-0.8	2.2	1.0	-2.6	36
06	12	-0.0	0.5	-0.8	-0.1	3.6	- 6.2	37
06	13	1.0	-1.9	0.2	-2.4	- 2.9	9.6	38
06	14	2.0	-1.2	2.0	-0.9	- 1.8	- 0.4	41
06	15	1.9	-0.4	1.9	-0.3			
06	16	2.3	-0.7	2.2	-0.8			
06	17	2.5	-1.2	2.4	-1.3			
06	18	2.4	-0.8	2.3	-0.8			2
07	00	3.2	-4.3	3.2	-3.7	- 2.4	2.0	36
07	01	2.5	-3.4	2.4	-2.3	- 5.3	- 1.0	7
07	02	3.4	-1.6	3.1	-1.6			4
07	03	3.4	-1.4	2.9	-1.1			3
07	04	2.5	-0.8	1.7	-0.6			1
07	05	1.0	-1.6	0.1	-1.4			1
07	06	1.6	-2.7	0.7	-2.9	- 3.8	5.7	14
07	07	3.0	-1.9	2.4	-2.3			3
07	08	3.8	-1.2	3.5	-1.2			3
07	09	3.9	-1.1	3.7	-1.5	2.2	0.6	15
07	10	3.9	-0.4	3.6	-0.8			4
07	11	4.4	0.1	4.0	-0.2	- 4.3	0.3	13
07	12	2.3	-0.3	1.4	-1.2	- 1.2	0.6	32
07	13	1.7	-0.8	1.2	-0.2	4.4	- 2.3	50
07	14	1.0	-0.8	2.1	-0.1	1.3	0.9	50
07	15	1.7	-0.3	1.9	-0.7			2
07	16	2.1	-0.2	2.0	-0.3			
07	17	2.5	-1.0	2.4	-1.1			
07	18	3.0	-0.5	2.8	-0.5			

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
08	00	3.2	-4.1	3.8	-3.9	0.2	1.8	18
08	01	3.9	-3.4	3.6	-2.7	- 1.2	- 1.1	10
08	02	3.4	-1.8	3.1	-1.7			3
08	03	3.4	-1.4	3.0	-1.3			2
08	04	3.5	-1.5	2.9	-1.4			5
08	05	3.4	-1.9	2.6	-1.8			5
08	06	3.8	-1.7	3.1	-1.6			3
08	07	4.4	-1.2	3.9	-1.3			2
08	08	3.9	-1.3	3.5	-1.4			2
08	09	3.6	-1.4	2.9	-1.9			2
08	10	3.5	0.8	3.0	0.3	0.2	1.9	14
08	11	4.6	0.4	3.9	0.0			1
08	12	4.5	-1.5	3.4	-2.8			4
08	13	4.9	-2.7	2.2	-1.3	-11.0	5.0	24
08	14	2.2	-1.8	0.2	-0.4			
08	15	1.7	0.6	0.6	-0.4			
08	16	1.5	0.1	0.9	-0.2			
08	17	2.1	-0.2	1.7	-0.3			
08	18	3.3	-0.1	3.2	0.0	- 2.2	- 3.3	13
09	00	2.9	-3.9	3.9	-3.8			
09	01	3.0	-3.0	3.4	-3.0			3
09	02	3.5	-2.0	3.3	-2.1			4
09	03	3.1	-1.5	2.5	-1.5			4
09	04	3.3	-1.1	2.8	-1.2	- 1.2	0.5	15
09	05	4.4	-1.4	4.0	-1.4	- 1.2	2.4	7
09	06	4.8	-1.4	4.3	-1.2			2
09	07	4.4	-1.7	4.1	-1.6			
09	08	3.7	-2.1	3.6	-2.2			3
09	09	2.9	-1.7	2.7	-2.1			2
09	10	2.4	0.5	2.3	0.0	0.7	0.8	18
09	11	1.3	0.5	1.1	-0.2			1
09	12	2.0	-0.1	1.7	-2.7			
09	13	2.4	-2.4	2.1	-2.7	3.1	8.4	11
09	14	3.2	-4.5	1.7	-1.4	- 3.1	4.9	19
09	15	1.3	-0.9	1.2	-0.1	4.8	- 8.3	9
09	16	1.1	0.9	1.2	0.4	4.0	2.8	10
09	17	2.2	1.2	2.2	1.4	- 0.1	- 2.6	11
09	18	3.1	0.6	3.3	0.9			5

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
10	00	3.3	-4.3	3.7	-4.0	3.0	- 3.0	34
10	01	2.9	-2.7	3.4	-3.4	- 0.6	- 0.1	7
10	02	2.6	-1.9	2.4	-2.7			
10	03	2.2	-1.9	1.2	-2.1	- 1.8	- 2.0	8
10	04	1.9	-0.4	1.5	0.0			4
10	05	2.4	-0.4	2.3	-0.5			1
10	06	2.7	-1.7	2.4	-1.5	- 4.2	7.1	6
10	07	2.0	-2.2	1.8	-1.8	0.1	0.2	8
10	08	1.9	-2.5	1.8	-2.6	- 1.4	0.4	8
10	09	2.2	-2.3	2.2	-2.7	- 2.8	0.8	7
10	10	1.5	-0.9	1.5	-1.5			3
10	11	-1.1	-0.2	-1.4	-0.9			1
10	12	-1.3	-0.6	-1.0	-1.6			3
10	13	-1.7	-1.5	0.1	-1.5			
10	14	1.3	-2.8	2.6	-1.1			3
10	15	2.5	-1.5	2.7	-0.9	- 2.9	1.0	8
10	16	2.0	1.2	2.4	0.7			1
10	17	1.9	1.9	2.3	1.9			
10	18	2.3	0.9	2.5	1.0			1
11	00	3.5	-4.3	2.3	-3.8	- 5.2	1.0	39
11	01	2.9	-3.6	2.6	-4.0	- 5.7	- 2.9	11
11	02	1.9	-2.6	2.3	-4.1			5
11	03	1.4	-2.1	1.7	-2.5	- 6.9	-10.7	9
11	04	1.1	-1.0	1.0	-0.2	- 2.1	2.0	15
11	05	0.1	-0.5	-0.0	-0.5	- 0.6	0.5	6
11	06	0.1	-1.8	0.1	-1.7	2.4	- 0.9	6
11	07	0.4	-2.6	0.3	-2.2			4
11	08	0.7	-1.8	0.6	-1.8			1
11	09	1.7	-1.2	1.8	-1.5			3
11	10	3.1	-3.0	3.3	-3.8			2
11	11	-0.3	-2.5	-0.0	-3.1	- 2.4	2.2	9
11	12	-0.7	-0.1	-0.2	-0.7	6.7	- 2.0	7
11	13	1.5	0.5	2.2	0.8	10.8	5.2	12
11	14	3.6	-0.5	4.1	0.7			
11	15	4.5	-1.3	4.4	-1.0			
11	16	3.1	0.0	3.2	-0.1			4
11	17	1.4	1.5	1.5	1.4			
11	18	1.2	1.7	1.3	1.7			

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
12	00	3.4	-4.8	1.9	-4.5	0.9	3.4	26
12	01	2.7	-4.4	1.7	-3.3			5
12	02	2.0	-3.3	2.0	-3.7			
12	03	1.1	-2.5	1.9	-3.2			2
12	04	1.0	-1.5	1.3	-1.3			4
12	05	0.7	-1.3	0.7	-1.2			3
12	06	0.4	-1.2	0.6	-1.1			3
12	07	1.6	-1.7	1.7	-1.4			2
12	08	2.1	-1.8	2.4	-1.5			3
12	09	1.7	0.3	2.7	0.1			4
12	10	1.7	-3.1	2.3	-3.8			1
12	11	2.5	-4.3	3.3	-4.7			4
12	12	3.4	1.5	4.3	1.5			3
12	13	4.9	2.6	6.6	3.0			2
12	14	5.1	0.3	6.5	0.7			3
12	15	5.4	-1.9	6.1	-2.0			
12	16	5.0	-1.9	5.2	-2.0	5.9	- 9.4	10
12	17	4.1	0.1	4.1	0.0			2
12	18	3.5	2.1	3.5	2.1			5
13	00	3.6	-5.0	3.6	-4.9	- 0.1	- 0.3	14
13	01	3.7	-5.0	3.4	-3.9			3
13	02	3.4	-4.2	2.7	-4.0			3
13	03	2.7	-3.1	2.6	-3.2			1
13	04	2.2	-1.5	2.3	-1.5			2
13	05	2.0	-1.3	2.0	-1.2			5
13	06	2.7	-0.7	2.7	-0.6			2
13	07	3.3	-0.9	3.3	-0.6			4
13	08	3.8	-1.7	3.8	-0.7			4
13	09	3.4	-1.6	3.4	-1.7			
13	10	2.1	-2.4	2.2	-3.5			2
13	11	5.2	-1.3	5.4	-1.6			4
13	12	6.1	2.1	6.4	2.4			4
13	13	5.5	2.0	5.8	2.9	5.1	1.1	6
13	14	4.5	0.2	4.7	0.3	5.5	7.5	39
13	15	5.4	-0.9	5.4	-1.7			
13	16	6.5	-1.5	6.5	-1.9			2
13	17	6.4	-0.4	6.4	-0.5	-22.7	10.2	8
13	18	4.8	1.1	4.8	1.1			

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
14	00	4.3	-4.7	4.5	-4.7	- 4.4	- 0.2	6
14	01	4.8	-4.5	4.8	-3.9			2
14	02	5.2	-4.4	5.0	-4.2			2
14	03	4.6	-3.8	4.5	-3.7			2
14	04	3.8	-2.3	3.7	-2.3			2
14	05	3.9	-0.8	3.9	-0.7			3
14	06	4.7	0.1	4.7	0.2			2
14	07	4.8	-0.5	4.7	-0.3			3
14	08	5.0	-1.5	4.6	-1.1			3
14	09	4.9	-1.9	3.9	-1.9			4
14	10	5.3	-1.4	5.0	-1.8			1
14	11	6.1	-0.1	6.1	-0.3			1
14	12	6.4	1.8	6.3	1.9			2
14	13	5.9	1.2	5.4	1.5			4
14	14	6.0	0.1	5.0	0.1			2
14	15	5.5	-0.2	4.9	-0.5			4
14	16	5.4	-1.1	5.2	-1.4			1
14	17	5.0	-2.0	4.9	-2.1			4
14	18	3.2	-0.2	3.1	-0.2			2
15	00	4.8	-4.2	5.0	-4.2	- 1.7	- 1.5	9
15	01	5.2	-4.2	5.3	-3.7			4
15	02	4.7	-4.5	4.7	-4.4			
15	03	4.9	-4.5	4.9	-4.4			3
15	04	4.9	-2.2	4.9	-2.2			3
15	05	5.7	-0.2	5.6	-0.1			3
15	06	5.5	0.1	5.5	0.2			2
15	07	5.3	-0.7	5.2	-0.6			4
15	08	4.8	-1.7	4.6	-1.6			3
15	09	4.9	-1.9	4.7	-1.9			3
15	10	4.8	-1.1	4.7	-1.2			1
15	11	5.2	0.4	5.1	0.4			1
15	12	5.0	1.7	4.9	1.7			1
15	13	4.5	1.5	4.2	1.5			1
15	14	5.1	0.0	4.7	0.0			2
15	15	5.5	-1.3	5.1	-1.3			3
15	16	4.4	-1.5	4.2	-1.6			3
15	17	4.4	-2.6	4.3	-2.6			3
15	18	4.5	-0.4	4.4	-0.4			1

IDENT.		PLUMB-LINE DEFLECTION				CORRECTIONS		NUMBER OF POINTS
X	Y	AVERAGE		SURFACE				
16	00	4.1	-3.5	4.7	-3.5			
16	01	5.1	-4.4	5.3	-4.2			
16	02	5.2	-4.7	5.2	-4.6			2
16	03	5.6	-3.8	5.6	-3.7			1
16	04	6.0	-1.8	5.9	-1.8			2
16	05	5.8	-0.2	5.7	-0.1			3
16	06	5.2	-0.3	5.2	-0.2			5
16	07	4.7	-1.3	4.7	-1.2			2
16	08	4.4	-1.8	4.4	-1.7			1
16	09	4.7	-1.8	4.6	-1.8			4
16	10	4.2	-1.2	4.2	-1.3			1
16	11	3.1	0.1	3.0	0.0			
16	12	2.4	0.9	2.2	0.7			
16	13	2.7	1.4	2.2	1.1			3
16	14	3.5	1.3	2.4	1.4			4
16	15	3.9	-1.1	3.8	-0.8			2
16	16	3.7	-2.0	3.6	-2.1			4
16	17	4.6	-1.6	4.3	-1.7			3
16	18	5.9	-0.6	5.8	-0.5	-4.6	9.4	6
17	00	4.1	-3.1	4.3	-3.1			
17	01	4.8	-3.2	4.9	-3.2			
17	02	5.4	-4.1	5.4	-4.0			
17	03	6.5	-3.5	6.5	-3.6			4
17	04	6.4	-2.1	6.2	-2.3			3
17	05	5.3	-1.0	4.8	-1.0			2
17	06	4.5	-0.9	4.4	-0.7			2
17	07	4.4	-1.5	4.4	-1.4			3
17	08	4.7	-1.6	4.8	-1.6			1
17	09	4.9	-1.7	4.9	-1.8			2
17	10	5.0	-2.6	5.0	-2.6			3
17	11	3.5	-1.4	3.5	-1.5			1
17	12	2.7	1.2	2.7	1.0			1
17	13	2.7	2.4	2.7	1.3			5
17	14	2.0	1.1	2.0	1.4	7.7	7.7	7
17	15	1.7	-1.3	1.7	-0.2	-0.3	1.2	10
17	16	3.2	-1.1	3.4	-1.5	4.6	-4.7	13
17	17	3.8	-0.1	3.9	-0.2	-9.6	0.1	6
17	18	5.1	-0.3	5.2	0.0			2

IDENT.		PLUMB-LINE DEFLECTION				CORRECTIONS		NUMBER OF POINTS
X	Y	AVERAGE		SURFACE				
18	00	5.0	-2.8	5.1	-2.8			
18	01	5.1	-3.3	5.2	-3.3			
18	02	5.7	-3.0	5.7	-3.0			2
18	03	6.7	-2.5	6.8	-2.6			
18	04	7.0	-2.9	7.1	-1.9			3
18	05	6.7	-1.9	6.8	-3.4	8.6	-4.2	6
18	06	6.3	-1.2	6.4	-0.7			3
18	07	6.3	-1.2	6.3	-0.9			5
18	08	6.5	-1.4	6.5	-1.3	-0.3	1.1	6
18	09	6.6	-1.3	6.6	-1.4	0.5	0.1	6
18	10	7.1	-2.3	7.1	-2.4			5
18	11	7.7	-2.2	7.8	-2.3			1
18	12	7.2	0.3	7.3	0.1			2
18	13	5.6	0.7	6.0	0.3			4
18	14	4.1	0.1	5.1	0.3	4.6	-1.6	8
18	15	4.1	0.1	4.3	0.7	2.8	2.7	21
18	16	4.8	-0.2	5.0	-0.3	2.2	1.9	18
18	17	4.6	0.1	5.0	0.0			5
18	18	4.4	0.0	4.6	0.1			4

SET III

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS	IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
00	00	3.9	-3.6	3.9	-3.6				02	00	3.5	-4.0	3.5	-4.0			
00	01	3.7	-3.8	3.7	-3.8				02	01	3.2	-4.3	3.2	-4.3			
00	02	3.4	-4.0	3.4	-4.1				02	02	2.7	-4.6	2.7	-4.6			
00	03	2.9	-3.6	2.9	-3.4				02	03	3.5	-3.7	3.5	-3.6			1
00	04	3.0	-3.8	3.0	-3.3				02	04	3.2	-2.8	3.1	-2.8			
00	05	2.7	-4.4	2.6	-4.4				02	05	4.3	-3.1	4.2	-3.0			
00	06	2.7	-3.7	2.5	-3.6				02	06	4.8	-3.1	4.9	-3.0			4
00	07	2.7	-3.2	2.7	-2.8				02	07	4.9	-3.2	4.9	-3.0			
00	08	2.7	-2.8	2.8	-3.1	- 3.8	0.8	7	02	08	4.5	-3.1	4.5	-2.9			
00	09	3.0	-2.4	3.1	-3.4			5	02	09	3.6	-2.7	3.8	-2.5			5
00	10	3.1	-2.5	3.3	-2.5	- 4.9	4.7	7	02	10	3.9	-2.3	4.4	-1.6			
00	11	2.9	-1.8	3.3	-1.2			6	02	11	4.9	-2.8	5.9	-2.7	- 2.2	11.0	6
00	12	3.0	-1.6	3.3	-1.5				02	12	4.0	-2.7	4.4	-3.3			118
00	13	3.6	-2.1	3.8	-2.1				02	13	3.0	-1.8	3.2	-1.9			172
00	14	3.8	-2.4	4.0	-2.4				02	14	2.8	-2.0	2.9	-1.6			91
00	15	3.9	-2.6	4.1	-2.6				02	15	2.6	-3.2	2.7	-3.2	5.3	4.1	34
00	16	3.9	-2.6	4.1	-2.7				02	16	2.7	-2.2	2.8	-2.7			36
00	17	4.1	-2.6	4.2	-2.7				02	17	3.9	-1.7	4.1	-1.9			52
00	18	4.2	-2.7	4.3	-2.8				02	18	4.2	-2.7	4.3	-2.8			20
01	00	3.8	-3.8	3.8	-3.8				03	00	2.7	-4.4	2.6	-4.5			
01	01	3.6	-4.0	3.6	-4.0				03	01	2.6	-4.8	2.5	-4.8			
01	02	3.4	-4.5	3.4	-4.5				03	02	3.0	-4.1	3.0	-4.0			1
01	03	3.0	-3.8	3.1	-3.7			4	03	03	3.3	-3.6	3.4	-3.6			
01	04	3.4	-3.5	3.2	-3.4			6	03	04	3.3	-3.0	3.1	-3.0			1
01	05	4.2	-3.9	3.8	-4.1			4	03	05	3.1	-2.3	3.0	-2.2			
01	06	4.9	-3.3	4.6	-3.2	- 0.8	- 1.0	6	03	06	3.0	-2.8	2.9	-2.7			2
01	07	4.9	-3.2	4.5	-2.8			4	03	07	3.2	-3.4	3.2	-3.2			5
01	08	5.0	-2.7	4.6	-2.7			2	03	08	3.0	-4.0	3.0	-3.8	- 0.1	2.9	7
01	09	4.8	-2.4	5.0	-2.5			5	03	09	2.2	-3.1	2.3	-2.7			5
01	10	4.7	-2.7	5.7	-2.4			4	03	10	3.3	-0.9	3.4	0.4			4
01	11	4.5	-2.5	5.3	-2.2	0.3	2.0	7	03	11	5.0	-2.3	5.4	-2.4	- 3.1	0.7	11
01	12	3.6	-1.8	4.0	-1.9	1.0	0.5	44	03	12	4.2	-4.1	3.9	-5.2			698
01	13	3.5	-1.6	3.8	-1.7				03	13	2.7	-2.8	2.6	-2.9			327
01	14	3.9	-2.2	4.2	-2.1				03	14	1.6	-2.6	1.5	-2.5			553
01	15	3.9	-2.7	4.5	-2.7				03	15	1.1	-3.0	0.8	-3.0			133
01	16	3.4	-2.5	3.7	-2.7				03	16	1.8	-1.0	1.9	-1.2			195
01	17	4.0	-2.4	4.2	-2.5				03	17	2.9	-0.6	3.2	-0.7			200
01	18	4.0	-2.6	4.1	-2.7				03	18	4.4	-2.5	4.6	-2.7			129

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
04	00	3.1	-5.1	3.2	-5.2			6
04	01	3.3	-3.9	3.1	-4.0			86
04	02	4.3	-3.8	4.1	-3.6			144
04	03	4.5	-3.3	4.3	-3.4	2.3	- 2.6	36
04	04	4.9	-3.1	4.7	-3.2	- 0.6	1.8	15
04	05	3.8	-3.5	3.6	-3.3			1
04	06	2.2	-3.2	2.2	-3.1			4
04	07	1.4	-3.1	1.4	-3.0			1
04	08	1.7	-4.1	1.7	-3.8			
04	09	2.7	-2.7	2.6	-2.3			2
04	10	3.2	0.5	2.8	1.6			9
04	11	2.7	-2.0	1.9	-2.4	-11.5	- 7.3	15
04	12	2.7	-5.3	2.5	-6.3	4.2	-10.9	16
04	13	2.0	-4.1	2.0	-3.8			18
04	14	1.5	-3.3	1.6	-3.2			63
04	15	1.4	-2.9	1.8	-2.8	0.6	0.5	23
04	16	1.5	-0.7	2.0	-0.7			80
04	17	1.0	0.9	1.5	0.7			107
04	18	2.7	-1.4	3.0	-1.7			115
05	00	4.6	-5.0	4.7	-5.3	2.7	2.5	9
05	01	5.2	-3.6	5.2	-3.9	- 4.7	- 2.3	50
05	02	4.9	-3.1	4.6	-3.0			134
05	03	5.4	-2.9	4.6	-3.0			
05	04	5.2	-3.5	5.0	-3.5			
05	05	4.8	-4.7	4.7	-4.4	- 2.8	2.3	30
05	06	4.0	-5.0	3.9	-4.8	- 1.3	1.7	9
05	07	2.8	-3.8	2.7	-3.8	- 3.6	0.5	7
05	08	2.9	-2.8	2.8	-2.6			5
05	09	3.8	-1.9	3.6	-1.6			5
05	10	2.9	-1.1	2.5	-0.6			2
05	11	1.1	-2.7	0.2	-2.7			
05	12	1.8	-4.7	1.9	-4.9	0.2	0.2	16
05	13	2.5	-4.0	2.6	-4.0			5
05	14	3.1	-3.1	3.1	-2.8	- 0.1	- 1.5	8
05	15	3.3	-2.6	3.5	-2.1	- 1.8	2.4	13
05	16	3.2	-1.6	3.9	-1.4	- 6.6	1.5	13
05	17	2.0	0.3	2.9	0.0	- 1.5	0.5	44
05	18	1.4	0.1	2.0	-0.5			94
06	00	5.3	-4.4	5.1	-4.5	- 2.1	- 1.3	34
06	01	4.9	-3.6	5.0	-3.9			5
06	02	4.8	-3.9	4.7	-4.3			3
06	03	3.3	-3.0	3.1	-3.0	- 1.0	- 4.0	50
06	04	4.1	-3.3	3.9	-2.9			204
06	05	4.6	-4.6	4.5	-4.3			
06	06	4.8	-5.2	4.8	-5.0			5
06	07	5.0	-4.2	4.8	-4.1	- 0.3	0.6	9
06	08	4.8	-2.7	4.7	-2.6			3
06	09	4.4	-2.3	4.3	-2.1			2
06	10	3.8	-2.4	3.6	-2.1			2
06	11	3.2	-3.1	3.1	-3.0			3
06	12	3.6	-3.6	3.4	-3.7			4
06	13	4.6	-3.1	4.5	-3.1			2
06	14	4.9	-2.8	4.9	-2.5			3
06	15	5.0	-2.7	4.8	-2.0			11
06	16	4.8	-1.7	4.7	-1.1	1.6	3.8	6
06	17	5.3	-0.9	5.2	-1.2	- 8.3	0.1	6
06	18	4.0	-0.2	4.0	-1.2	- 3.0	2.0	26
07	00	5.5	-4.8	5.4	-4.9			4
07	01	5.2	-3.8	5.1	-4.0			5
07	02	5.3	-4.6	5.4	-4.8			3
07	03	4.7	-3.7	5.1	-3.9			4
07	04	4.4	-2.4	4.2	-2.5			3
07	05	3.9	-4.5	2.9	-4.1			2
07	06	4.1	-5.1	3.5	-4.7	- 2.5	6.6	28
07	07	4.0	-4.0	3.7	-4.0			4
07	08	3.9	-3.0	3.7	-2.6			3
07	09	3.9	-2.5	4.1	-2.3			5
07	10	4.3	-2.6	4.2	-2.2	- 2.5	0.3	11
07	11	4.6	-3.1	4.5	-2.9	0.7	- 1.2	6
07	12	5.1	-2.9	4.9	-3.1			5
07	13	5.8	-2.7	5.6	-2.6			1
07	14	5.4	-3.0	5.3	-2.8			4
07	15	4.8	-3.0	4.5	-2.5			3
07	16	4.6	-1.1	3.9	-0.8	1.7	3.2	14
07	17	5.5	-0.6	4.7	-0.8			2
07	18	5.0	-1.4	4.6	-1.8			

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
08	00	5.2	-4.0	5.0	-4.1			2
08	01	5.9	-3.7	5.9	-3.8			5
08	02	5.2	-4.5	5.2	-4.6			1
08	03	5.3	-4.0	5.4	-4.3			3
08	04	4.1	-4.0	4.0	-4.9			6
08	05	3.5	-4.8	3.3	-4.8	- 7.5	- 1.6	15
08	06	3.8	-5.0	3.2	-4.1	2.2	3.6	17
08	07	3.7	-4.4	2.9	-4.2	0.8	0.7	7
08	08	3.4	-3.0	2.5	-2.2			2
08	09	3.4	-2.2	2.8	-1.8	3.7	7.1	6
08	10	3.3	-2.0	3.2	-1.9	1.6	- 0.9	8
08	11	4.2	-2.3	4.2	-2.3			4
08	12	5.1	-2.3	4.6	-2.5			1
08	13	5.0	-2.7	4.7	-2.5			1
08	14	4.7	-3.8	4.6	-3.5			5
08	15	4.1	-3.7	4.0	-3.5			4
08	16	3.5	-1.1	3.3	-0.8	1.9	1.7	15
08	17	2.7	-0.5	2.8	-0.5			1
08	18	3.1	-1.7	3.2	-2.0			
09	00	4.7	-3.7	4.7	-3.8	2.1	- 4.3	6
09	01	4.6	-4.1	4.6	-4.0			3
09	02	3.4	-5.2	3.5	-5.3			1
09	03	3.7	-4.7	3.8	-4.8			4
09	04	3.4	-4.1	3.8	-4.5			2
09	05	3.9	-4.5	4.8	-4.5	- 3.0	- 3.6	19
09	06	3.7	-4.8	4.0	-4.8	- 2.1	- 0.3	25
09	07	3.8	-4.4	3.8	-4.8	- 8.2	1.9	11
09	08	3.5	-3.3	3.3	-2.8	9.4	1.9	6
09	09	3.1	-2.8	2.5	-1.9	5.5	- 2.2	8
09	10	2.4	-2.1	2.1	-1.3	0.5	- 0.5	13
09	11	2.3	-1.4	2.1	-1.6	6.5	- 8.3	6
09	12	2.3	-2.2	2.2	-2.6	1.8	0.9	8
09	13	1.7	-3.1	1.6	-2.6	- 1.0	- 1.1	9
09	14	1.8	-3.6	1.7	-3.3			4
09	15	2.5	-3.9	2.5	-3.5			4
09	16	2.1	-2.6	2.6	-1.9			4
09	17	-0.1	-1.0	1.5	-1.0			2
09	18	0.7	-0.3	1.3	-1.0			4
10	00	4.1	-3.6	4.2	-3.7			4
10	01	4.1	-4.8	4.2	-4.8			2
10	02	4.6	-5.1	4.6	-5.1			1
10	03	5.0	-4.1	5.0	-4.2	- 3.6	1.5	9
10	04	4.5	-4.3	4.7	-4.4	- 1.0	- 1.1	21
10	05	4.0	-4.6	4.4	-4.7	- 0.5	- 1.1	22
10	06	3.5	-5.2	3.8	-5.3			4
10	07	3.4	-4.6	4.0	-4.7			5
10	08	3.0	-3.6	3.7	-3.4			6
10	09	2.1	-3.3	2.4	-2.9			7
10	10	2.0	-2.7	1.8	-2.2	0.3	1.2	6
10	11	1.4	-2.0	1.0	-2.0			2
10	12	0.8	-2.7	1.0	-2.7			5
10	13	0.9	-3.3	0.9	-3.0			2
10	14	1.2	-3.0	1.1	-2.7			3
10	15	1.7	-3.0	1.6	-2.5			3
10	16	2.6	-3.9	2.4	-2.1			1
10	17	0.9	-2.3	0.8	-2.2	3.1	2.1	8
10	18	1.1	1.6	1.0	-0.2	3.9	- 2.1	7
11	00	4.1	-3.9	4.2	-4.0			4
11	01	4.4	-3.4	4.4	-3.5			2
11	02	5.9	-4.3	5.8	-4.5			1
11	03	5.6	-4.8	5.1	-4.8			3
11	04	5.5	-4.9	5.4	-4.8	- 0.1	0.7	20
11	05	4.9	-5.3	5.0	-5.3			6
11	06	4.5	-5.4	4.7	-5.4			7
11	07	4.4	-4.8	4.6	-4.7			4
11	08	3.5	-4.7	3.7	-4.6			3
11	09	2.5	-3.8	2.6	-3.6			1
11	10	2.4	-2.6	2.4	-2.4			1
11	11	2.2	-2.4	2.1	-2.3			3
11	12	2.5	-2.6	2.5	-2.5			3
11	13	3.2	-2.7	3.1	-2.5			3
11	14	3.6	-3.0	3.5	-2.8			3
11	15	3.4	-2.5	3.2	-2.1			6
11	16	3.2	-3.7	2.5	-3.1			2
11	17	3.8	-2.9	1.9	-2.8	- 2.7	- 0.6	6
11	18	4.2	2.2	3.4	1.6			3

IDENT.		PLUMB-LINE DEFLECTION				CORRECTIONS		NUMBER OF POINTS	IDENT.		PLUMB-LINE DEFLECTION				CORRECTIONS		NUMBER OF POINTS
X	Y	AVERAGE		SURFACE					X	Y	AVERAGE		SURFACE				
12	00	4.6	-4.0	4.7	-4.1			3	14	00	4.4	-4.4	4.5	-4.4			1
12	01	4.6	-2.9	4.6	-3.1			3	14	01	4.7	-2.6	4.8	-2.7			1
12	02	5.0	-3.6	5.1	-4.1			6	14	02	4.6	-2.4	4.7	-2.4			
12	03	6.1	-4.4	6.2	-4.3	5.7	7.4	10	14	03	4.1	-3.3	4.3	-3.3			1
12	04	6.8	-4.8	6.8	-4.3	- 3.7	0.1	10	14	04	3.9	-4.3	4.0	-4.2			2
12	05	6.7	-5.7	6.8	-5.7			2	14	05	4.7	-5.5	4.7	-5.5			1
12	06	6.1	-5.8	6.2	-5.7			3	14	06	5.8	-5.6	5.9	-5.6			2
12	07	5.4	-5.6	5.5	-5.5			1	14	07	6.3	-6.3	6.3	-6.3			
12	08	4.5	-5.6	4.6	-5.5			2	14	08	6.1	-6.0	6.1	-6.0			2
12	09	4.1	-4.2	4.1	-4.1			2	14	09	6.6	-4.3	6.6	-4.3			3
12	10	4.0	-2.9	4.0	-2.8			5	14	10	6.1	-2.9	6.1	-2.8			2
12	11	4.1	-2.1	4.0	-2.0			3	14	11	6.3	-1.4	6.2	-1.3			4
12	12	4.9	-1.7	4.8	-1.6			3	14	12	6.0	-1.7	5.9	-1.6			4
12	13	5.4	-2.3	5.4	-2.2			3	14	13	5.6	-2.5	5.6	-2.5			3
12	14	5.5	-2.8	5.4	-2.6			3	14	14	5.4	-2.8	5.3	-2.7			2
12	15	5.4	-3.3	5.2	-3.1			3	14	15	6.0	-2.5	5.8	-2.4			2
12	16	5.1	-3.6	4.7	-3.4			2	14	16	6.5	-1.3	6.3	-1.3			2
12	17	6.7	-1.4	6.1	-1.4			2	14	17	6.5	-0.7	6.3	-0.7			
12	18	6.8	2.0	6.3	1.8			2	14	18	6.0	-0.6	5.8	-0.6			1
13	00	4.3	-4.1	4.4	-4.2			7	15	00	4.9	-3.9	4.9	-3.9			1
13	01	4.1	-2.8	4.2	-3.0			5	15	01	5.2	-2.5	5.3	-2.6			
13	02	4.6	-2.6	4.8	-2.8			2	15	02	4.7	-3.0	4.8	-3.0			
13	03	5.4	-3.3	5.9	-3.3			5	15	03	4.4	-3.8	4.5	-3.9			
13	04	5.9	-4.5	6.1	-4.3			2	15	04	4.2	-4.1	4.3	-4.1			3
13	05	6.0	-6.1	6.1	-6.0			2	15	05	4.5	-4.2	4.6	-4.2			
13	06	6.0	-6.4	6.1	-6.4			3	15	06	5.6	-4.9	5.7	-4.9			
13	07	6.0	-6.2	6.1	-6.2			4	15	07	5.9	-6.4	5.9	-6.3			
13	08	5.5	-6.1	5.5	-6.0			2	15	08	6.1	-5.8	6.1	-5.7			1
13	09	5.6	-4.3	5.6	-4.2			3	15	09	6.6	-4.3	6.6	-4.2			3
13	10	5.4	-2.7	5.4	-2.6			3	15	10	6.2	-3.2	6.2	-3.2			3
13	11	5.8	-1.6	5.8	-1.5			1	15	11	5.5	-1.7	5.4	-1.7			2
13	12	6.0	-1.3	6.0	-1.2			3	15	12	4.9	-1.8	4.9	-1.8			2
13	13	5.8	-2.2	5.7	-2.1			3	15	13	5.1	-2.6	5.0	-2.6			2
13	14	5.7	-3.0	5.6	-2.9			5	15	14	5.3	-2.7	5.2	-2.7			2
13	15	5.9	-3.0	5.7	-2.9			3	15	15	5.4	-1.9	5.3	-1.9			3
13	16	6.9	-2.2	6.7	-2.2			1	15	16	5.1	-1.7	5.0	-1.7			
13	17	8.1	-0.8	7.8	-0.8			1	15	17	3.7	-1.3	3.6	-1.3			
13	18	7.3	0.5	7.0	0.4			3	15	18	4.1	-0.6	4.0	-0.6			

IDENT. X	IDENT. Y	PLUMB-LINE DEFLECTION AVERAGE		PLUMB-LINE DEFLECTION SURFACE		CORRECTIONS		NUMBER OF POINTS
16	00	5.2	-4.0	5.2	-4.0			1
16	01	4.0	-2.9	4.0	-2.9			
16	02	4.3	-3.1	4.3	-3.1			
16	03	4.5	-3.6	4.6	-3.6			
16	04	5.0	-4.1	5.0	-4.1			
16	05	5.1	-3.9	5.2	-3.9			4
16	06	5.5	-4.2	5.5	-4.2			
16	07	6.0	-6.0	6.0	-6.0			
16	08	6.7	-5.2	6.7	-5.2			2
16	09	6.1	-4.3	6.1	-4.3			
16	10	5.5	-4.2	5.5	-4.1			5
16	11	4.6	-2.6	4.6	-2.6			4
16	12	4.3	-2.0	4.3	-2.0			3
16	13	4.6	-2.1	4.6	-2.1			1
16	14	5.2	-2.4	5.2	-2.4			2
16	15	5.6	-2.0	5.5	-2.0			3
16	16	6.1	-3.0	6.0	-3.0			3
16	17	4.0	-2.2	3.9	-2.2			2
16	18	3.6	0.5	3.5	0.5			1
17	00	5.2	-4.8	5.3	-4.9			
17	01	4.3	-3.7	4.3	-3.7			1
17	02	3.8	-2.6	3.8	-2.6			
17	03	3.7	-3.4	3.8	-3.4			
17	04	4.0	-3.4	4.0	-3.4			
17	05	5.3	-3.5	5.4	-3.5			
17	06	5.6	-4.1	5.6	-4.1			
17	07	5.8	-5.2	5.8	-5.2			
17	08	5.9	-5.5	5.9	-5.5			
17	09	5.6	-5.2	5.6	-5.2			4
17	10	5.8	-4.6	5.7	-4.5			5
17	11	5.5	-2.9	5.5	-2.9			1
17	12	5.7	-1.7	5.7	-1.7			8
17	13	6.0	-1.7	6.0	-1.7	0.4	0.6	7
17	14	6.4	-2.0	6.4	-2.0	- 0.6	0.8	8
17	15	6.6	-0.9	6.6	-0.9			5
17	16	6.8	-3.2	6.7	-3.2			1
17	17	7.0	-4.0	6.9	-4.0			2
17	18	5.9	0.2	5.8	0.2			3
18	00	5.4	-4.6	5.4	-4.6			
18	01	6.1	-4.4	6.1	-4.4			1
18	02	5.4	-3.4	5.4	-3.4			2
18	03	5.2	-3.5	5.2	-3.5			1
18	04	5.3	-2.7	5.3	-2.7			2
18	05	5.5	-2.6	5.5	-2.6			
18	06	5.7	-3.9	5.7	-3.9			
18	07	6.0	-5.4	6.0	-5.4			
18	08	6.6	-5.3	6.6	-5.3			4
18	09	7.6	-4.3	7.6	-4.4			4
18	10	8.3	-4.3	8.3	-4.3	- 2.2	1.2	7
18	11	8.6	-3.0	8.6	-2.9			5
18	12	8.2	-1.6	8.2	-1.6	1.5	- 1.1	10
18	13	7.8	-1.6	7.8	-1.6	- 0.2	- 0.6	9
18	14	7.6	-1.9	7.6	-1.9	- 0.5	- 0.2	11
18	15	7.1	-1.9	7.1	-1.9	0.1	1.1	9
18	16	6.2	-2.7	6.2	-2.7	1.1	- 2.8	9
18	17	7.8	-3.0	7.7	-3.1			5
18	18	7.6	-1.0	7.5	-1.0	- 2.2	- 1.3	6