PUBLICATIONS of the EARTH PHYSICS BRANCH

VOLUME 42 - NO. 1

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F. PRIMDAHL

DEPARTMENT OF ENERGY, MINES AND RESOURCES

OTTAWA, CANADA 1971

Cat. No.: M70-42/1

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Abstract. To define a system of electromagnetic units uniquely, six basic quantities must be chosen. A suitable choice may include one basic unit and five constants, for example, or four basic units and two constants. Maxwell's equations are derived in a general form, independent of any particular choice of the six basic quantities. It is shown that the MKSA-formulation of the field equations involves three different unit systems.

In the relation $\overline{B} = \mu_0 \overline{H}$, the symbols $\overline{B}$ and $\overline{H}$ are called a Giorgi-pair; other Giorgi-pairs are shown to exist in the MKSA-formulation of electromagnetic theory. Although of different dimensions, the members of a Giorgi-pair always describe the same physical phenomenon, and may thus be used arbitrarily, provided the Haines rule is observed. The general form of Maxwell's equations is derived for material bodies. It is shown that the MKSA-formulation uses two conflicting models for polarizable materials, leading to two electric and two magnetic fields, and the introduction of two units for each field in order to remove the constants from Maxwell's equations. Dimensional equations are replaced by more general expressions, called logometric formulas, including all six basic units and constants; they are used to analyze the existing unit systems.

Résumé. Pour donner une définition unique d'un système d'unités électromagnétiques, il faut choisir six quantités de base. On peut choisir, par exemple, une unité de base et cinq constantes ou quatre unités de base et deux constantes. Les équations de Maxwell ont été établies, sous une forme générale, indépendamment de tout choix particulier des six quantités de base. La présente étude démontre que dans le système M.K.S.A. la formulation des équations du champ électromagnétique fait intervenir trois systèmes d'unités différents.

Dans l'équation $\overline{B} = \mu_0 \overline{H}$, les symboles $\overline{B}$ et $\overline{H}$ sont appelés paire de Giorgi; l'auteur démontre qu'il existe d'autres paires de ce genre dans la formulation en M.K.S.A. de la théorie électromagnétique. Bien que de dimensions différentes, les éléments d'une paire de Giorgi décrivent toujours le même phénomène physique et peuvent donc être appliqués de façon arbitraire, à condition d'observer la règle de Haines. La forme générale des équations de Maxwell est établie pour les corps matériels. L'auteur montre que la formulation selon le système M.K.S.A. emploie deux modèles contradictoires pour les matières polarisables, d'où deux champs électriques et deux champs magnétiques et l'introduction de deux unités pour chacun de ces champs afin d'éliminer les constantes des équations de Maxwell. Les équations de dimensions ont été remplacées par des expressions plus générales que l'auteur nomme formules logométriques, qui font intervenir les six unités et constantes de base; ce sont ces formules qui one été utilisées pour analyser les systèmes d'unités existants.

Introduction

For almost 100 years physicists have argued about units and unit systems in electromagnetism and the growing worldwide adoption of the SI† -units has revived this discussion (Rosser, 1969; Stopes-Roe, 1969; Temperly, 1969; Lovering, 1970).

One of the difficulties is, no doubt, that so many different terminologies exist and hence participants in the discussion do not use the same concepts in the same meaning. Much of the disagreement is thus believed to lie in the lack of proper means of communication.

The purpose of this paper is to provide a common denominator in the form of a set of electromagnetic equations written independently of any unit system, and to indicate the characteristic features of some of the existing systems. The equations are only derived as far as necessary to evaluate their general form and the selection is more or less arbitrary. It is, however, hoped that the examples cover a wide enough area to demonstrate the method and to indicate extensions.

The notation is very close to that of O'Rahilly (1965) whose treatment of electromagnetism, units, and dimensions has been the basic source of information. His viewpoint is that the symbols in the physical equations stand for pure numbers representing the ratios between the measured quantities (measure-ratios) and some chosen unit quantities. He replaces the dimensions by so-called logometric formulas giving the number by which a measure-ratio has to be multiplied if the basic units or constants are changed.

†Système international.

Basic units

An earlier paper (Primdahl, 1970) has described, how two constants δ and γ have to be included in Newton's second law and the mass attraction law to generalize them, and similarly how three constants α, β, and a are necessary to generalize the electromagnetic equations.

$$F = \delta M \frac{d^2 r}{dt^2} \quad \text{Newton II} \qquad \ldots.1$$

$$F = \gamma \frac{MM'}{r^2} \quad \text{Newton's law of gravitational attraction} \qquad \ldots.2$$

$$F = \frac{qq'}{\alpha r^2} \quad \text{Coulomb's law for electric charges} \qquad \ldots.3$$

$$F = \frac{mm'}{\beta r^2} \quad \text{for magnetic poles} \qquad \ldots.4$$

Electric current is the time-rate of charge transported through a cross-sectional area of the conductor

$$I = \frac{dq}{dt} \qquad \ldots.5$$

and magnetic field strength or field intensity is introduced as

$$H = \frac{F}{m} \qquad \ldots.6$$

the ratio between the force on a magnetic pole and its pole strength.

The magnetic field strength $d\overline{H}$ from a current element $I\overline{ds}$ at the distance $r = \overline{r} \cdot \hat{r}$ is proportional to

$$\frac{I\overline{ds} \times \hat{r}}{r^2}$$

so that the field strength at the centre of a circular current loop, radius r, carrying the current I is by integration

$$H = \frac{2 ¶ I}{a \cdot r} \qquad \ldots.7$$

The constants of proportionality $\delta, \gamma, \alpha, \beta$, and a are basic in the same sense as the units for length, mass and time are basic, because they are necessary to uniquely determine the unit system. In mechanics we introduce a unit length, a unit mass, and a unit time. The value of one of the constants δ or γ is then agreed upon internationally, and then the determination of the other is a matter of laboratory measurements. If both the constants δ and γ were fixed by international agreement then only two basic units, e.g. mass and time, would be needed. Similarly, if α and β are chosen in electromagnetism then a and all the electromagnetic units may be determined experimentally, or if a fourth electromagnetic unit is introduced only one of the constants α, β or a has to be chosen.

The seven equations (Equations 1 to 7) contain 13 unknowns so it is necessary to choose six of the 13 unknowns to solve the system. Thus we need, not three or four, but six basic "units" to define uniquely our unit system.

It is emphasized that the letter symbols in the algebraic Equations 1 to 7 represent pure algebraic numbers, e.g. the number M in Equation 1 is thus obtained by comparing the mass in the acceleration experiment to a standard mass. Likewise, if the measure-ratio for one mass can be determined then the measure-ratio for all masses can be determined. Hence for the purpose of counting the number of equations and the number of unknowns all the measure-ratios for mass count for one unknown.

It is interesting to note that as there are only five constants (δ, γ, α, β, and a) we cannot determine a unit system by fixing the values of the constants only; at least one actual physical sample, for example mass, is needed as a starting point. The six basic quantities cannot be chosen at random from the 13 unknowns; only a combination which does not violate any of the Equations 1 to 7 and 28 given below will be legal. An illegal choice will make the system indeterminate, as can be shown by solving for the remaining unknowns.

The claim † that the MKSA-system is defined uniquely by the choice of the four basic units metre, kilogram, second, and ampere is thus not justified. Equation 1 is always stated with $\delta = 1$ giving a fifth basic "unit" and the definition of the ampere is written in a form tacitly stating $a = 4$¶ which then is our sixth basic "unit".

From Equations 1 to 7 the formula for the force between two current elements is derived to be

$$\overline{d^2F} = \frac{\beta}{a^2} \cdot \frac{I_1 \overline{ds_1} \times (I_2 \overline{ds_2} \times \hat{r})}{r^2} \qquad \ldots.8$$

and it is seen that the unit for I may be fixed by specifying the force and the geometry in a standard experiment.

However, in the MKSA-system Equation 8 is normally stated in another form

$$\overline{d^2F} = \frac{\mu_0}{4¶} \frac{I_1 \overline{ds_1} \times (I_2 \overline{ds_2} \times \hat{r})}{r^2} \qquad \ldots..8a$$

where $\mu_0 = \beta/a$, as will be shown later. Implicit in the form of Equation 8a is thus the tacit assumption

$$a = 4¶$$

and this is the sixth basic "unit".

Electrostatics

Starting from Equation 3 electric field strength is defined as the force per unit charge

$$\overline{E} = \frac{\overline{F}}{q} = \frac{q'}{\alpha r^2} \hat{r} \qquad \ldots.9$$

If we change the electrostatic constant in Equation 3 from α to α^* then E will change to E^* and q to q^*. Here and in the following F is unchanged as we do not change the mechanical units.

$$F = \frac{q^* q'^*}{\alpha^* r^2} = \frac{q\, q'}{\alpha\, r^2} \qquad \ldots.10$$

and from this

$$\frac{q^*}{q} = \sqrt{\frac{\alpha^*}{\alpha}} \qquad \ldots.11$$

As

$$F = q \cdot E = q^* E^* \qquad \ldots.12$$

† e.g. Stratton, 1941, p. 18.

we have

$$\frac{E^*}{E} = \frac{q}{q^*} = \sqrt{\frac{\alpha}{\alpha^*}} \qquad \ldots .13$$

The flux of electric field out through a sphere with radius r centred around a charge q is, by integration of Equation 9:

$$\Psi = \int_{\text{sphere}} \overline{E} \cdot \overline{da} = \int_{\text{sphere}} \frac{q}{\alpha r^2}\, da = \frac{4\P}{\alpha}\, q \qquad \ldots .14$$

This is a special case of the more general Gauss' law:

$$\Psi = \frac{4\P}{\alpha} \int_{\text{Vol}} \rho dv \qquad \ldots .15$$

or stated in differential form

$$\nabla \cdot \overline{E} = \frac{4\P}{\alpha}\ \rho \qquad \ldots .16$$

where ρ is the measure-ratio for charge density.

Electric current in a conductor is introduced by Equation 5. If A is the cross-sectional area of the conductor then the current density is

$$J = \frac{I}{A} \qquad \ldots .17$$

and in general a current density vector is introduced as the current per unit area oriented perpendicularly to the direction of flow of charge.

As the current flowing out of a closed surface is equal to the decrease of charge inside the surface the following continuity equation is valid

$$\nabla \cdot \overline{J} = -\frac{\partial \rho}{\partial t} \qquad \ldots .18$$

If we change the electrostatic constant from α to α^* then J becomes J^* and we have

$$\frac{J}{J^*} = \frac{\rho}{\rho^*} = \sqrt{\frac{\alpha}{\alpha^*}} \qquad \ldots .19$$

The same transformation relation is valid for the current I

$$\frac{I}{I^*} = \sqrt{\frac{\alpha}{\alpha^*}} \qquad \ldots .20$$

Magnetostatics

Despite intensive search for magnetic monopoles they have not been found (Fleischer, 1969), and they certainly do not play any role in the magnetostatic effects upon which the theory of electromagnetism is based.

This is often stated as an argument against explicitly writing Equation 4. However, it must be realized that the (mathematical) concept of a magnetic pole is extremely useful in describing forces between magnets, and inevitably every textbook introduces this concept when dealing with magnetized materials. So when the magnetic pole is introduced anyway, why not admit it openly by stating Equation 4 from the beginning taking advantage of its great pedagogic value:

$$F = \frac{mm'}{\beta \cdot r^2} \qquad \ldots .\ 4$$

Changing the magnetostatic constant from β to β^* changes m to m^* and we have the relation

$$\frac{m}{m^*} = \sqrt{\frac{\beta}{\beta^*}} \qquad \ldots .21$$

Magnetic field strength is introduced as

$$H = \frac{F}{m} \qquad \ldots .22$$

which gives

$$\frac{H}{H^*} = \sqrt{\frac{\beta^*}{\beta}} \qquad \ldots .23$$

for a change of β to β^*.

The flux of the magnetic field out through a sphere is

$$\Phi = \int_{\text{sphere}} \overline{H} \cdot \overline{da} = 0 \qquad \ldots .24$$

as magnetic poles always occur in opposite pairs. The differential form of this equation is

$$\nabla \cdot \overline{H} = 0 \qquad \ldots .25$$

Electromagnetics

The field at the centre of a circular conductor can be measured by magnetostatic experiments (Gauss' method) and it is found that

$$H = \frac{2\P}{a} \cdot \frac{I}{r} \qquad \ldots .26$$

where the factor $2\P$ is due to the geometry as stated in the explanation to Equation 7.

This offers a magnetostatic definition of current as opposed to the electrostatic definition in Equation 5:

$$H = 2¶ \frac{I_M}{r} \quad \ldots\ldots 26a$$

If $\alpha = 1$ then $I = I_E$ is the measure-ratio in the electrostatic system and if $\beta = 1$ at the same time then I_M is the measure-ratio in the electromagnetic system; now, from a list of conversion factors between E.M.U. and E.S.U. (ASTM Metric Practice Guide; Maxwell, 1954, Art. 787) it can be verified that for the same current

$$\frac{I_E}{I_M} = c \quad \ldots\ldots 27$$

where c is equal to the speed of light.

If we want to use I_E in Equation 26a we then have to write

$$H_M = \frac{2¶}{c} \frac{I_E}{r} \quad \ldots\ldots 26b$$

where H_M is the measure-ratio for the magnetic field when $\beta = 1$.

For $\alpha = 1$ and $\beta = 1$ we are thus forced to put $a = c$, and this is the characteristic of the Gaussian system.

Changing α from 1 to α^* and β from 1 to β^* we will have to change a from c to a^* and thus

$$H^* = \frac{2¶}{a^*} \cdot \frac{I^*}{r} \quad \ldots\ldots 26c$$

Using Equations 20 and 23 with $I = I_E$, $\alpha = 1$, and $H = H_M$, $\beta = 1$ we get

$$H^* = \frac{H_M}{\sqrt{\beta^*}} = \frac{2¶}{a^*} \cdot \frac{I^*}{r} = \frac{2¶}{a^*} \cdot \frac{I_E}{r} \sqrt{\alpha^*} \quad \ldots\ldots 26d$$

and from this

$$H_M = \frac{\dfrac{2¶ \; I_E/r}{a^*}}{\sqrt{\alpha^* \; \beta^*}} \quad \ldots\ldots 26e$$

By comparison to Equation 26b it is seen that the general limitation on the three constants a, α, and β is

$$\frac{a}{\sqrt{\alpha \; \beta}} = c \quad \ldots\ldots 28$$

It must be remembered that a, α, and β stand for pure algebraic numbers and that c is a number too as it is determined as the ratio between the two measure-ratios I_E and I_M, themselves pure algebraic numbers.

The magnetic field at the centre of a circular conductor of radius r carrying the current I is, by Equation 7 or 26c

$$H = \frac{2¶}{a} \cdot \frac{I}{r} \quad \ldots\ldots 29$$

The force on a magnetic pole, m, would then be

$$F = H \cdot m = \frac{2¶}{a} \cdot \frac{I}{r} \cdot m \quad \ldots\ldots 30$$

and an equal, opposite force would exist on the conductor. This is the classical concept of 'action at a distance' between the magnetic pole and the current-carrying conductor. The force on the conductor may, however, also be considered a direct action from the field around it, in which case

$$F = \frac{2¶}{a} \frac{I}{r} \cdot H' \cdot \beta \cdot r^2 = \frac{\beta}{a} 2¶ \, I \cdot H' \cdot r, \quad \ldots\ldots 31$$

where

$$H' = \frac{m}{\beta \; r^2} \quad \ldots\ldots 32$$

is the field at the circular conductor from the magnetic pole placed at its centre. Equation 31 may be derived assuming that the force $\overline{dF}$ on a current element $I \, \overline{ds}$ in a field $\overline{H}$ is

$$\overline{dF} = \frac{\beta}{a} I \; \overline{ds} \times \overline{H} \quad \ldots\ldots 33$$

At the distance r from a magnetic point-pole m the field is

$$\overline{H} = \frac{m}{\beta \; r^2} \; \hat{r} \quad \ldots\ldots 34$$

If this pole is at the centre of the current loop (current I, radius r) then the field is constant along the conductor and the current element $I \; \overline{ds}$ is perpendicular to $\overline{H}$; using $ds = r d\theta$ in Equation 33 we get

$$F = \frac{\beta}{a} \cdot I \cdot \frac{m}{\beta \; r^2} \int_0^{2¶} r d\theta = \frac{2¶}{a} \frac{I}{r} \cdot m \quad \ldots\ldots 35$$

which is the same as Equation 30, i.e. Equation 33 is a correct form of the force equation between a current element and a magnetic field.

The field $\overline{H}$ in Equation 33 may be generated by another current element $I_1 \; \overline{ds}_1$ at the distance $\overline{r} = \overline{r} \cdot \hat{r}$ from $I \; \overline{ds}$; using Equation 7 and the preceding remarks we get

$$d\overline{H}_1 = \frac{I_1}{a}\,\frac{\overline{ds_1} \times \hat{r}}{r^2} \qquad \ldots.36$$

which inserted into Equation 33 gives

$$d^2\overline{F} = \frac{\beta}{a^2}\,\frac{I\,\overline{ds} \times (I_1\,\overline{ds_1} \times \hat{r}\,)}{r^2} \qquad \ldots.37$$

This is Equation 8 where $\hat{r}$ is a unit vector along $\overline{r}$.

Electrodynamics

From the experiments of Faraday, Maxwell states the induction law as: (Maxwell, 1954, Art. 541)

"The total electromotive force acting round a circuit at any instant is measured by the rate of decrease of the number of lines of magnetic force which pass through it."

The "total electromotive force" is the line-integral of the electric field strength along the closed circuit and the "number of lines of magnetic force which pass through it" is the flux of the magnetic field vector through a surface bounded by the circuit. Stated in mathematical form this is

$$V = \int_{\substack{\text{closed}\\ \text{circuit}}} \overline{E}\cdot\overline{ds} \ \text{proportional to} \ -\frac{d}{dt}\int_{\text{surface}} \overline{H}\cdot\overline{da} \qquad \ldots.38$$

Using Equations 13 and 23 this can be written as

$$\sqrt{\alpha}\int_{\substack{\text{closed}\\ \text{circuit}}} \overline{E}\cdot\overline{ds} = -\,k\cdot\sqrt{\beta}\,\frac{d}{dt}\int \overline{H}\cdot\overline{da} \qquad \ldots.39$$

where the proportionality-factor k is independent of the unit system used, because $\sqrt{\alpha}\cdot E$ and $\sqrt{\beta}\cdot\overline{H}$ are independent of the units. (Throughout this treatment F and the other mechanical units are assumed unchanged.)

The constant k may be determined by experiments. For our purpose we put $\alpha = \beta = 1$ and compare Equation 39 to Faraday's law in the Gaussian system. This gives $k = 1/c$ and from Equation 28 we get

$$\int \overline{E}\cdot\overline{ds} = -\frac{\beta}{a}\frac{d}{dt}\int \overline{H}\cdot\overline{da} \qquad \ldots.40$$

and the differential form is

$$\nabla \times \overline{E} = -\frac{\beta}{a}\frac{\partial\overline{H}}{\partial t} \qquad \ldots.41$$

Equation 40 may also be derived directly from Equation 33 by equating the mechanical and electric work involved in displacing a rigid, closed circuit in a static magnetic field (Slater and Frank, 1947, p. 210 f.).

Maxwell's equations in general form

The field around a linear current I is calculated from Equation 36

$$H = \frac{2}{a}\cdot\frac{I}{r} \qquad \ldots.42$$

The field lines are concentric circles around I.

The line-integral of H along one of the closed circular field lines is

$$\int_{\substack{\text{closed}\\ \text{curve}}} \overline{H}\cdot\overline{ds} = \frac{2}{a}I\int_0^{2\P}\frac{r\,d\theta}{r} = \frac{4\P}{a}\cdot I \qquad \ldots.43$$

The general form of Equation 43 is Ampère law

$$\int_{\substack{\text{closed}\\ \text{curve}}} \overline{H}\cdot\overline{ds} = \frac{4\P}{a}\Sigma\, I \qquad \ldots.44$$

where Σ I is the total encircled current. The differential form of Equation 44 is

$$\nabla \times \overline{H} = \frac{4\P}{a}\,\overline{J} \qquad \ldots.45$$

The Equations 41, 16, 25, and 45 with an added term are the basic equations for the combined, time-varying magnetic and electric fields:

$$\nabla \times \overline{E} = -\frac{\beta}{a}\frac{\partial\overline{H}}{\partial t} \qquad \ldots.41$$

$$\nabla \cdot \overline{E} = \frac{4\P}{\alpha}\rho \qquad \ldots.16$$

$$\nabla \cdot \overline{H} = 0 \qquad \ldots.25$$

and the incomplete equation

$$\nabla \times \overline{H} = \frac{4\P}{a} J \qquad \ldots .45$$

Of course, the fact that the constant *c* in Equation 28 is equal to the speed of light within the experimental error, inspired physicists to search for an electromagnetic theory of light. Maxwell saw that the electric and magnetic field vectors had to satisfy wave equations of the following form (Maxwell, 1954, Art. 784).

$$\nabla^2 \overline{E} - \frac{1}{c^2} \frac{\partial^2 \overline{E}}{\partial t^2} = f(x, y, z, t) \qquad \ldots .46$$

$$\nabla^2 \overline{H} - \frac{1}{c^2} \frac{\partial^2 \overline{H}}{\partial t^2} = g(x, y, z, t) \qquad \ldots .47$$

He could derive this from the four equations above if he added the term $\frac{\alpha}{a}\frac{\partial \overline{E}}{\partial t}$ to the right-hand side of Equation 45 giving

$$\nabla \times \overline{H} = \frac{4\P}{a} \overline{J} + \frac{\alpha}{a} \frac{\partial \overline{E}}{\partial t} \qquad \ldots .48$$

Applying ∇x and $\frac{\partial}{\partial t}$ to Equations 41 and 48 successively gives

$$\nabla \times (\nabla \times \overline{E}) + \frac{\beta}{a}\frac{\partial}{\partial t}(\nabla \times \overline{H}) = 0 \qquad \ldots .49$$

$$\frac{\partial}{\partial t}(\nabla \times \overline{E}) + \frac{\beta}{a}\frac{\partial^2 \overline{H}}{\partial t^2} = 0 \qquad \ldots .50$$

$$\nabla \times (\nabla \times \overline{H}) - \frac{\alpha}{a}\frac{\partial}{\partial t}(\nabla \times \overline{E}) = \frac{4\P}{a}(\nabla \times \overline{J}) \qquad \ldots .51$$

$$\frac{\partial}{\partial t}(\nabla \times \overline{H}) - \frac{\alpha}{a}\frac{\partial^2 \overline{E}}{\partial t^2} = \frac{4\P}{a}\frac{\partial \overline{J}}{\partial t} \qquad \ldots .52$$

Substituting Equation 50 into Equations 51 and 52 into Equation 49 and using the vector-operator relation

$$\nabla \times (\nabla \times \overline{A}) = \nabla(\nabla \cdot \overline{A}) - \nabla^2 \overline{A} \qquad \ldots .53$$

together with Equations 16 and 25 gives the wave equations

$$\nabla^2 \overline{H} - \frac{\alpha\beta}{a^2}\frac{\partial^2 \overline{H}}{\partial t^2} = \frac{4\P}{a}(\nabla \times \overline{J}) \qquad \ldots .54$$

$$\nabla^2 \overline{E} - \frac{\alpha\beta}{a^2}\frac{\partial^2 \overline{E}}{\partial t^2} = \frac{4\P}{\alpha}\nabla\rho + \frac{4\P\beta}{a^2}\frac{\partial \overline{J}}{\partial t} \qquad \ldots .55$$

where the constant in front of the time-derivatives of second order equals the reciprocal square of the speed of propagation in agreement with Equation 28.

Maxwell's modification of Equation 45 by adding the term $\frac{\alpha}{a}\frac{\partial \overline{E}}{\partial t}$ is quite understandable as he knew exactly what he was looking for. It is a piece of ingenious mathematical intuition and it led to the prediction and discovery of electromagnetic waves. The modification has no measurable effect on the description of experiments with static and slowly varying electromagnetic fields from which the Equations 41, 16, 25, and 45 were derived. Maxwell tried nevertheless after his successful accomplishment to explain and justify the modification in terms of the static electromagnetic concepts. This of course is impossible as it originates from an entirely different class of experiments. His attempt to explain $\frac{\partial \overline{E}}{\partial t}$ as "displacement current" is well known, and even in recent textbooks this notation is used, although it is clear that no current, i.e. transport of charge, is involved.

Lorenz (1867) suggested, two years after Maxwell, that the charge and current distributions in the Equations 15 and 44 should not be the instantaneous values, but the values r/c earlier where *r* is the distance from the observed point to the charge or current. His suggestion also leads to a solution of Maxwell's equations.

Lorenz had no difficulties with the interpretation of the modification. In his paper (Lorenz, 1867) he says:

"But as the laws of induced currents, generally admitted and based on experiment, did not lead to the expected result, the question was whether it was not possible so to modify the laws assumed that they would embrace both the experiments on which they rest and the phenomena which belong to the theory of light......... It is at once obvious that the equations, which are deduced in a purely empirical manner, are not necessarily the exact expression of the actual law; and it will always be permissible to add several members or to give the equations another form, always provided these changes acquire no perceptible influence on the results which are established by experiment."

This is the introduction, in 1867, of the correspondence principle, which has proven so successful in the theory of relativity and in quantum mechanics. It states that any new theory must approach the classical theory asymptotically when dealing with classical phenomena. Nobody today maintains that the quantum mechanical equations can be understood entirely in terms of classical mechanics and yet Maxwell's "displacement current" is still defended in textbooks.

Giorgi-pairs

The general form of Maxwell's equations independent of unit systems is

$$\nabla x \overline{E} + \frac{\beta}{a}\frac{\partial \overline{H}}{\partial t} = 0 \qquad \ldots .56$$

$$\nabla \cdot \overline{H} = 0 \qquad \ldots .57$$

$$\nabla \cdot \overline{E} = \frac{4\P}{\alpha}\rho \qquad \ldots .58$$

$$\nabla x \overline{H} - \frac{\alpha}{a}\frac{\partial \overline{E}}{\partial t} = \frac{4\P}{a}\overline{J} \qquad \ldots .59$$

If a new measure-ratio, $\overline{B}$, for magnetic field is defined as

$$\overline{B} = \frac{\beta}{a}\overline{H} \qquad \ldots .60$$

and a new measure-ratio for electric field

$$\overline{D} = \frac{\alpha}{a}\overline{E} \qquad \ldots .61$$

then Equations 56 to 59 become

$$\nabla x \overline{E} + \frac{\partial \overline{B}}{\partial t} = 0, \qquad \nabla \cdot \overline{B} = 0 \qquad \ldots .62, 63$$

$$\nabla x \overline{H} - \frac{\partial \overline{D}}{\partial t} = \frac{4\P}{a}\overline{J}, \qquad \nabla \cdot \overline{D} = \frac{4\P}{a}\rho \qquad \ldots .64, 65$$

This form of Maxwell's equations is independent of the constants α and β, but these are of course reintroduced by the Equations 60 and 61 and instead of the four equations (Equations 56 to 59) we now need the six equations (Equations 60 to 65).

The two relations (Equations 60 and 61) have been referred to as the Giorgi conditions (Stopes-Roe, 1969) and for purpose of the present discussion the two members of a Giorgi condition will be called a Giorgi-pair†. In Equation 60 $\overline{B}$ is the Giorgi-mate of $\overline{H}$ and the process of transforming one to the other is called giorgization.

In discussing the effects of giorgization it is often necessary to distinguish between the two members of a Giorgi-pair; therefore if a Giorgi-pair is given

$$G_2 = \frac{\beta}{a} \cdot G_1 \qquad \ldots .66$$

†In analogy to Maxwell's electrostatic and magnetic pairs, Treatise, Art. 621.

then G_1 is called the female member and G_2 the male member of the pair.† The same notation is used if the Giorgi-pair is related by

$$G_2 = \frac{\alpha}{a} G_1 \qquad \ldots .67$$

Giorgization is not limited to the field vectors only. The magnetic moment of a current loop is (Stratton, 1941) introduced as

$$\overline{i_m} = I \cdot A \cdot \overline{n} \qquad \ldots .68$$

where I is the current, A the area of the loop, and $\overline{n}$ a unit normal vector to A.

Others (Döring, 1948; and Bjerge, 1951) introduce the magnetic moment as

$$\overline{m_m} = \frac{\beta}{a} \cdot I \cdot A \cdot \overline{n} \qquad \ldots .69$$

Clearly $\overline{i_m}$ and $\overline{m_m}$ are a Giorgi-pair as

$$\overline{m_m} = \frac{\beta}{a}\,\overline{i_m} \qquad \ldots .70$$

Likewise magnetic poles, electric charges and all other electromagnetic measure-ratios may be giorgized.

An interesting feature of giorgization is that all formulas for force, torque, energy etc. contain both a male and female member of a giorgi-pair, side by side (both in numerator, or both in denominator).††

The torque on a magnetic dipole moment in a magnetic field is

$$\overline{T} = \overline{m_m} \; x \; \overline{H} = \overline{i_m} \; x \; \overline{B} \qquad \ldots .71$$

The force on an electric charge in an electric field is

$$\overline{F} = q \cdot \overline{E} = q^* \cdot \overline{D} \qquad \ldots .72$$

where

$$q = \frac{\alpha}{a} \cdot q^* \qquad \ldots .73$$

The work involved in moving a charge in an electric field is

$$W = \int q \cdot \overline{E} \cdot d\overline{r} = \int q^* \cdot \overline{D} \cdot d\overline{r} \qquad \ldots .74$$

†This notation has been suggested by G.V. Haines, Div. of Geomagnetism, Earth Physics Branch, EMR, Ottawa, Can.

††The Haines male-female rule.

and the energy associated with a current

$$P = V \cdot I = \int_{\text{circuit}} \bar{E} \cdot d\bar{s} \cdot I = \int_{\text{circuit}} \bar{D} \cdot d\bar{s} \cdot I^* \qquad \ldots .75$$

where

$$I = \frac{\alpha}{a} \cdot I^* \qquad \ldots .76$$

In view of this the force formula (Coulomb's law) looks like a monstrosity, but replacing one of the charges by its Giorgi-mate overcomes this problem

$$F = \frac{q\ q'}{\alpha\ r^2} = \frac{q^*\ q'}{a \cdot r^2} \qquad \ldots .77$$

From these considerations it is clear that by giorgizing the proper measure-ratios according to the Haines male-female rule it is possible to completely eliminate the constants α and β from all the electromagnetic formulas.

This elimination is of course only apparent as every quantity now is expressed by two measure-ratios related by one of the Giorgi conditions (Equation 66 or 67).

Giorgi-transformation of a unit system

Starting from one unit system characterized by the constants α, β, and a (the mechanical units are untouched) and consequently giorgizing all the measure-ratios will give two new unit systems characterized by α^*, $\beta^*=\beta$, a^* and $\alpha^{**}=\alpha$, β^{**}, a^{**}.

Coulomb's law for electric charges gives

$$F = \frac{q\ q'}{\alpha\ r^2} = \frac{\frac{\alpha}{a} q^* \cdot \frac{\alpha}{a} q^{*\prime}}{\alpha\ r^2} = \frac{q^*\ q^{*\prime}}{\frac{a^2}{\alpha}\ r^2} \qquad \ldots .78$$

which means that

$$\alpha^* = \frac{a^2}{\alpha} = c^2\beta \qquad \ldots .79$$

using Equation 28. Similarly we get for β^{**}

$$\beta^{**} = c^2\alpha \qquad \ldots .80$$

and thus

$$a^* = \frac{a^2}{\alpha} \qquad \ldots .81$$

and

$$a^{**} = \frac{a^2}{\beta} \qquad \ldots .82$$

By giorgizing the original system

$$\alpha, \beta, \underline{a} \qquad \ldots .83$$

we thus express some of the formulas in the System (1)

$$c^2\beta, \beta, \frac{a^2}{\alpha} \qquad \ldots .84$$

and other formulas in the System (2)

$$\alpha, c^2\alpha, \frac{a^2}{\beta} \qquad \ldots .85$$

All three systems are proper unit systems as

$$\frac{a^2}{\alpha\ \beta} = c^2 \qquad \text{Original system} \qquad \ldots .83a$$

$$\frac{a^4/\alpha^2}{c^2\beta \cdot \beta} = c^2 \qquad \text{System (1)} \qquad \ldots .84a$$

$$\frac{a^4/\beta^2}{\alpha \cdot c^2\alpha} = c^2 \qquad \text{System (2)} \qquad \ldots .85a$$

If α, β, and a are the characteristic constants of the unit system of Equations 56 to 59 then α^*, β^*, and a^* given by Equation 84 are the constants of the unit system of Equations 64 and 65, and α^{**}, β^{**}, and a^{**} given by Equation 85 are the constants of the unit system of Equations 62 and 63. As the measure-ratio for current in Equation 64 is unchanged from that of Equation 59 it means that current in System (1) is defined by the magnetostatic units (e.g. Equation 7) rather than by Equation 5 using a and not a^*.

The two new systems of course fulfil the conditions

$$\frac{\alpha^*}{a^*} = \frac{\beta^{**}}{a^{**}} = 1 \qquad \ldots .85b$$

so Maxwell's equations appear with no constants in front of the time derivatives.

Maxwell's equations in material bodies

The general form of Maxwell's equations is

$$\nabla x \overline{E} + \frac{\beta}{a}\frac{\partial \overline{H}}{\partial t} = 0 \quad ; \quad \nabla \cdot \overline{H} = 0 \quad \dots 56, 57$$

$$\nabla x \overline{H} - \frac{\alpha}{a}\frac{\partial \overline{E}}{\partial t} = \frac{4¶}{a}\overline{J}, \quad \nabla \cdot \overline{E} = \frac{4¶}{\alpha}\rho \dots 59, 58$$

H.A. Lorentz assumed the validity of these equations inside material bodies provided ρ and $\overline{J}$ included charges and currents from the atoms of the material bodies (Casimir, 1969).

ρ is thus taken as

$$\rho = \rho_e + \rho_p \qquad \dots 86$$

and $\overline{J}$ as

$$\overline{J} = \overline{J}_e + \overline{J}_{at} \qquad \dots 87$$

The division in external and material contributions is, as pointed out by Casimir, to a certain extent arbitrary; and ρ_p and $\overline{J}_{at}$ are statistical descriptions of the effect of the presence of matter.

For a small elementary volume, dv_n containing a large number of atoms in order to make average values statistically meaningful, the electric dipole moment per unit volume, $\overline{P}_n$ is introduced as

$$\overline{P_n} = \frac{1}{dv_n} \sum_{dv_n} \overline{m_e}^{\,i} \qquad \dots 88$$

where $\overline{m}_e^{\,i}$ is the i'th point-dipole inside dv_n. By a process of continuization (Knudsen, 1955) all the $\overline{P}_n$- values assigned to the individual volume elements, dv_n, are replaced by a continuous vector field $\overline{P}$ having the value $\overline{P} = \overline{P}_n$ at some point inside each volume element dv_n.

This polarization vector field $\overline{P}$ may be visualized as strings of dipoles starting with a negative charge and terminating with a positive. According to the Poisson-Thomson analysis (O'Rahilly, 1938, p. 37 ff) $\overline{P}$ is equivalent to a charge distribution ρ_p where

$$\rho_p = - \nabla \cdot \overline{P} \qquad \dots 89$$

A positive $\nabla \cdot \overline{P}$ means that more strings of dipoles are leaving the elementary volume than entering it, which in turn means that there are some negative ends of dipole strings inside the volume, hence the minus sign.

Equation 89 may be considered an "equation of continuity" following from the fact that the total charge of a collection of dipoles is zero.

The splitting into external and atomic contributions is arranged so that the external charges and currents satisfy the equation of continuity

$$\nabla \cdot \overline{J}_e + \frac{\partial \rho_e}{\partial t} = 0 \qquad \dots 90$$

and the same holds for the atomic charges and currents

$$\nabla \cdot \overline{J}_{at} + \frac{\partial \rho_p}{\partial t} = 0 \qquad \dots 90a$$

or

$$\nabla \cdot \left\{ \overline{J}_{at} - \frac{\partial}{\partial t}\overline{P} \right\} = 0 \qquad \dots 91$$

We may then write

$$\overline{J}_{at} = \frac{\partial \overline{P}}{\partial t} + \frac{a}{\beta} \nabla x \overline{M} \qquad \dots 92$$

or

$$\overline{J}_{at} = \overline{J}_p + \overline{J}_{amp} \qquad \dots 93$$

where $\overline{J}_p$ is the current due to changes in polarization and $\overline{J}_{amp}$ are the amperean neutral currents.

$\overline{M}$ is the magnetization vector derived as a mathematical continuization of the magnetic moment per unit volume analogous to $\overline{P}$.

The atoms contain small current loops originating in orbiting and spinning electrons and spinning nuclear particles. The actual description belongs to the quantum mechanics, but for our purpose it is sufficient to postulate an equivalent amperean current loop associated with every atom. The magnetic moment of the i'th current loop in an elementary volume dv_n, large enough to render average values statistically significant, is given by Equation 69

$$\overline{m_m}^{\,i} = \frac{\beta}{a} \cdot I_i \circ A_i \cdot \overline{n}_i \qquad \dots 94$$

where I_i is the current and A_i the area of the current loop.

The magnetic moment per unit volume is

$$\overline{M}_n = \frac{1}{dv_n} \sum_i \overline{m_m}^{\,i} \qquad \dots 95$$

and the continuization of all the $\overline{M}_n$'s is the vector field $\overline{M}$. $\overline{M}$, described by the equivalent current density distribution

$$\overline{J}_{amp} = \frac{a}{\beta} \nabla \times \overline{M} \qquad \ldots .96$$

and $\overline{P}$, described by the equivalent charge density distribution

$$\rho_p = - \nabla \cdot \overline{P} \qquad \ldots .97$$

are two mathematical models representing the effects of the presence of matter.

Equation 91 follows from

$$\nabla \cdot \overline{J}_{amp} = 0 \qquad \ldots .98$$

which means that the amperean currents are neutral.

Maxwell's Equations 56 to 59, using Equations 86, 87, 92, 93, and 89 become

$$\nabla x \overline{E} + \frac{\beta}{a} \frac{\partial \overline{H}}{\partial t} = 0, \quad \nabla \cdot \overline{H} = 0 \qquad \ldots .99, 100$$

$$\nabla x \overline{H} - \frac{\alpha}{a} \frac{\partial \overline{E}}{\partial t} = \frac{4\P}{a} \overline{J}_e + \frac{4\P}{a} \frac{\partial \overline{P}}{\partial t} + \frac{4\P}{\beta} \nabla x \overline{M} \qquad \ldots .101$$

$$\nabla \cdot \overline{E} = \frac{4\P}{\alpha} \rho_e - \frac{4\P}{\alpha} \nabla \cdot \overline{P} \qquad \ldots .102$$

Equations 99 to 102 are Maxwell's equations for magnetic and electric fields inside material bodies. The contributions from the atoms of the present matter are accounted for by the space-average vectors $\overline{P}$ and $\overline{M}$, electric and magnetic dipole moment per unit volume.† J_e and ρ_e are the external current and charge densities.

To solve the equations we must know something about $\overline{P}$ and $\overline{M}$ besides $\overline{J}_e$ and ρ_e.

The solution to the 'external' problem, $\overline{H}_e$ and $\overline{E}_e$, is given by

$$\nabla x \overline{E}_e + \frac{\beta}{a} \frac{\partial \overline{H}_e}{\partial t} = 0, \quad \nabla \cdot \overline{H}_e = 0 \qquad \ldots .103, 104$$

$$\nabla x \overline{H}_e - \frac{\alpha}{a} \frac{\partial \overline{E}_e}{\partial t} = \frac{4\P}{a} \overline{J}_e, \quad \nabla \cdot \overline{E}_e = \frac{4\P}{\alpha} \rho_e \qquad \ldots .105, 106$$

†A consequence of using P and M is that E and H now represent the space-average fields.

$\overline{H}_e$ and $\overline{E}_e$ may be interpreted as the fields we would have if the material bodies were removed. Subtracting Equations 103 to 106 from Equations 99 to 102 and introducing

$$\overline{H}_I = \overline{H} - \overline{H}_e \qquad \ldots .107$$

$$\overline{E}_I = \overline{E} - \overline{E}_e \qquad \ldots .108$$

gives

$$\nabla x \overline{E}_I + \frac{\beta}{a} \frac{\partial \overline{H}_I}{\partial t} = 0, \quad \nabla \cdot \overline{H}_I = 0 \qquad \ldots .109, 110$$

$$\nabla x \overline{H}_I - \frac{\alpha}{a} \frac{\partial \overline{E}_I}{\partial t} = \frac{4\P}{a} \frac{\partial \overline{P}}{\partial t} + \frac{4\P}{\beta} \nabla x \overline{M} \qquad \ldots .111$$

$$\nabla \cdot \overline{E}_I = - \frac{4\P}{\alpha} \nabla \cdot \overline{P} \qquad \ldots .112$$

where (H_I, E_I) is the solution to the 'internal' problem, i.e. the field contributions from $\overline{P}$ and $\overline{M}$. The total fields $\overline{E}$ and $\overline{H}$ are thus made up of the partial fields $\overline{E}_e + \overline{E}_I$ and $\overline{H}_e + \overline{H}_I$.

$\overline{P}$ and $\overline{M}$ are normally dependent on the field vectors; but whether the 'primary' cause is $(\overline{E}, \overline{H})$, $(\overline{E}_e, \overline{H}_e)$ or $(\overline{E}_I, \overline{H}_I)$ is impossible to say as all three sets of vectors are interdependent. If $\overline{P}$ and $\overline{M}$ have constant parts unaffected by $\overline{E}_e$ and $\overline{H}_e$ it is reasonable to assume $\overline{E}_I$ and $\overline{H}_I$ to be the causes†, but if $\overline{P}$ and $\overline{M}$ have parts dependent also on $\overline{E}_e$ and $\overline{H}_e$ these vectors as well (i.e. the total vectors $\overline{E}$ and $\overline{H}$) must be regarded as the causes.

The torque, $\overline{T}$, on a magnetic dipole, $\overline{m}_m$, in a field, $\overline{H}$, is

$$\overline{T} = \overline{m}_m \times \overline{H} \quad \text{or} \quad T = m_m \cdot H \cdot \sin\theta \ldots .113$$

where θ is the angle between the directions of $\overline{m}_m$ and $\overline{H}$. The work involved in changing θ from θ_1 to θ_2 is

$$W_{mag} = \int_{\theta_1}^{\theta_2} m_m \cdot H \cdot \sin\theta \, d\theta =$$

$$- m_m H(\cos\theta_2 - \cos\theta_1) \qquad \ldots .114$$

†For ferromagnetic materials H_I is not sufficient to explain the very strong spontaneous magnetization existing in the domains. The actual explanation is based on exchange integrals and belongs to quantum mechanics.

If $\theta_1 = 0$ is chosen then

$$W_{mag} = -m_m H(\cos\theta_2 - 1)\cdot \quad \ldots.115$$

is the potential energy of a magnetic dipole $\overline{m}_m$ in a field $\overline{H}$ where the direction of the dipole has the angle θ_2 to the field. The zero level of the potential energy is arbitrary so we may express

$$W_{mag} = -\overline{m}_m \cdot \overline{H} \quad \ldots.116$$

From this we have the potential energy of a volume element dv of a magnetized body

$$dW_{mag} = -\overline{M}\cdot\overline{H}\,dv \quad \ldots.116a$$

Normally there is also some mechanical energy associated with the magnetization so the total energy is

$$dW_{tot} = -\overline{M}\cdot\overline{H}\,dv + dW_{mech} \quad \ldots.117a$$

If the mechanical potential energy dW_{mech} is independent of the direction of $\overline{M}$ in the volume element then Equation 117a has a minimum when

$$dW_{mag} = -\overline{M}\cdot\overline{H}\,dv \quad \ldots.117$$

is minimum. The directional independence of dW_{mech} is the characteristic feature of the magnetically isotropic materials, in these materials $\overline{M}$ is thus parallel to $\overline{H}$ everywhere.

Anisotropic materials have dW_{mech} dependent on the orientation of $\overline{M}$ so that the total potential energy dW_{tot} is not necessarily minimum when $\overline{M}$ is parallel to $\overline{H}$, which means that in general $\overline{M}$ and $\overline{H}$ have different directions.

In some materials $\overline{M}$ is a linear function of $\overline{H}$. For isotropic materials

$$\overline{M} = k_m \cdot \overline{H} \quad \ldots.118$$

where k_m is the scalar proportionality factor between $\overline{H}$ and $\overline{M}$.† In the case of anisotropy we have

$$\overline{M} = \{k\}\cdot\overline{H} \quad \ldots.118a$$

†Mason & Weaver (1929) uses this notation with $\frac{4\P}{\beta} k_m = (1 - \frac{1}{\mu_r})$.

where (k) is a 3 X 3 matrix.

For the ferromagnetic materials, however, the relation between $\overline{M}$ and $\overline{H}$ is more involved than expressed by Equations 118 and 118a:

$$\overline{M} = \sum_{n=o}^{\infty} \begin{pmatrix} + \\ (-1)^{n-1} \end{pmatrix} m_n \overline{H}^n \quad \ldots.118b$$

where + is taken if H is decreasing and $(-1)^{n-1}$ if H is increasing. Furthermore the coefficients m_n are dependent on the immediately preceding maximum value of H. This is still the isotropic case where $\overline{M}$ is parallel to $\overline{H}$, so if anisotropy is introduced the relation between $\overline{M}$ and $\overline{H}$ becomes even more complicated.

Traditionally†, still another set of vector quantities is introduced in the treatment of magnetic and dielectric materials.

The basic Equations 101 and 102 are rearranged in the following manner

$$\nabla\times\left\{\overline{H} - \frac{4\P}{\beta}\overline{M}\right\} - \frac{\alpha}{a}\frac{\partial}{\partial t}\left\{\overline{E} + \frac{4\P}{\alpha}\overline{P}\right\} = \frac{4\P}{a}\overline{J}_e \quad \ldots.119$$

and

$$\nabla\cdot\left\{\overline{E} + \frac{4\P}{\alpha}\overline{P}\right\} = \frac{4\P}{\alpha}\rho_e \quad \ldots.120$$

Two new vectors are then introduced

$$\overline{K} = \overline{H} - \frac{4\P}{\beta}\overline{M} \quad \ldots.121$$

$$\overline{G} = \overline{E} + \frac{4\P}{\alpha}\overline{P} \quad \ldots.122$$

and inserted into Equations 99 to 102

$$\nabla\times\overline{G} + \frac{\beta}{a}\frac{\partial\overline{K}}{\partial t} = \frac{4\P}{\alpha}\nabla\times\overline{P} - \frac{4\P}{a}\frac{\partial\overline{M}}{\partial t} \quad \ldots.123$$

$$\nabla\cdot\overline{K} = -\frac{4\P}{\beta}\nabla\cdot\overline{M} \quad \ldots.124$$

$$\nabla\times\overline{K} - \frac{\alpha}{a}\frac{\partial\overline{G}}{\partial t} = \frac{4\P}{a}\overline{J}_e \quad \ldots.125$$

$$\nabla\cdot\overline{G} = \frac{4\P}{\alpha}\rho_e \quad \ldots.126$$

†Introduced by Maxwell.

There are two mathematical models of polarized media in existence. The Poisson-Thomson model replaces a polarized dielectric by an equivalent charge density distribution

$$\rho_p = -\nabla\cdot\overline{P} \qquad \ldots.97$$

and the amperen model replaces a magnetized medium by an equivalent current density distribution

$$\overline{J}_{amp} = \frac{a}{\beta}\,\nabla x M \qquad \ldots.96$$

Maxwell's equations for material bodies Equations 99 to 102 contain the terms $\nabla x\overline{M}$, $\nabla\cdot\overline{P}$ and $\frac{\partial\overline{P}}{\partial t}$ because in the derivation we used the amperean model for magnetic materials and the Poisson-Thomson model for dielectrics. In the Equations 123 to 126 we have the reverse situation as the terms $\nabla x\overline{P}$, $\nabla\cdot\overline{M}$ and $\frac{\partial\overline{M}}{\partial t}$ indicate the amperean model for dielectrics and the Poisson-Thomson model for magnetic materials.

The two models give the same field outside the polarized bodies, but in predicting the internal fields they differ. The Poisson-Thomson model is, however, often used to calculate the field around a magnetized body because of its analogy to similar problems already solved in electrostatics, but for calculating the internal magnetic fields this model fails to explain the experimental results and must be rejected. The self-induction of an air-cored coil increases when the coil is filled with magnetic material, indicating an increased internal field according to the amperean model. The $\overline{K}$-field is thus a hypothetical field based on a wrong model for magnetized materials.

Similarly the $\overline{G}$-field is a hypothetical field based on the amperean model for dielectrics. This model fails to explain experiments with the internal fields in dielectrics. The capacity of a condenser increases if the empty space between the plates is filled with a dielectric indicating a decreased field as predicted by the Poisson-Thomson model.

The question of $\overline{K}$-field vs. $\overline{H}$-field and $\overline{G}$-field vs. $\overline{E}$-field is closely related to the question of the existence of a free magnetic pole. In Equation 124 the term $-\nabla\cdot\overline{M}$ is equivalent to a magnetic charge density

$$m = -\nabla\cdot\overline{M} \qquad \ldots.97a$$

and in Equation 123 the terms $\frac{a}{\alpha}\nabla x\overline{P}$ and $\frac{\partial\overline{M}}{\partial t}$ are equivalent to a density of "magnetic current"

$$\overline{J}_m = \frac{\partial\overline{M}}{\partial t} - \frac{a}{\alpha}\,\nabla\times\overline{P} \qquad \ldots.96a$$

In the section on Giorgi-pairs it is shown that by introducing two units for electric field and two units for magnetic field it is possible to remove the constants in Maxwell's equations. To justify this, the MKSA-formulation simultaneously adopts both models for dielectrics and for magnetic materials, assigning different units to the resulting two electric and two magnetic fields. The $\overline{D}$, $\overline{E}$, $\overline{B}$ and $\overline{H}$-fields in the MKSA-formulation are thus related to the $\overline{G}$, $\overline{E}$, $\overline{H}$ and $\overline{K}$-fields in the notation used here by

$$\overline{D}_{MKSA} = \frac{\alpha}{a}\,\overline{G}$$

$$\overline{E}_{MKSA} = \overline{E}$$

$$\overline{B}_{MKSA} = \frac{\beta}{a}\,\overline{H}$$

$$\overline{H}_{MKSA} = \overline{K}$$

Static fields in material bodies

For static and slowly varying fields where the time-derivatives may be neglected the magnetic and electric fields can be separated. This leads to

$$\nabla x\overline{G} = \frac{4¶}{\alpha}\nabla x\overline{P}\ ; \qquad \nabla\cdot\overline{G} = \frac{4¶}{\alpha}\rho_e \qquad \ldots.127, 128$$

$$\nabla x\overline{K} = \frac{4¶}{a}\,\overline{J}_e\ ; \qquad \nabla\cdot\overline{K} = \frac{4¶}{\beta}\nabla\cdot\overline{M} \ldots 129, 130$$

Considering the magnetostatic part (Equations 129 and 130) we see that if the magnetization vector $\overline{M}$ is divergence-free then $\overline{K}$ is the solution to the external problem. An infinitely long cylinder magnetized uniformly along the axis is a case where $\nabla\overline{M} = 0$ and thus $\overline{K} = \overline{H}_e$. A toroid magnetized uniformly along its circular axis is another example of a body with divergence-less magnetization having $\overline{K} = \overline{H}_e$.

For *isotopic materials* $\overline{M}$ and $\overline{H}$ are collinear and proportional (Equation 118). From the definition (Equation 121) it then follows that $\overline{K}$ is proportional to and collinear with $\overline{H}$ which gives the following interpretation of $\overline{K}$.

A volume element dv, with magnetization $\overline{M}$ and the total field $\overline{H}$ inside a body of arbitrary geometrical shape, has a vector $\overline{K}$ equal to the external field H_e^c that applied axially to an infinite cylinder of the same material would produce the same magnetization $\overline{M}$ and the same total field $\overline{H}$.

For the volume element we have

$$\overline{H} = \overline{H}_e + \overline{H}_I = \overline{K} + \frac{4¶}{\beta}\overline{M} \qquad \ldots.131$$

and for the hypothetical cylinder

$$H = \overline{H}_e^c + \overline{H}_I^c = \overline{K} + \frac{4¶}{\beta}\overline{M} \qquad \ldots.132$$

H_I^c is the field from the distribution of magnetic moment per unit volume in the cylinder which may be calculated from Equations 110 and 111

$$\overline{H}_I^c = \frac{4\P}{\beta}\overline{M} \qquad \ldots .133$$

In the body of arbitrary shape the distribution of $\overline{M}$ in general gives a field H_I different from $\frac{4\P}{\beta}\overline{M}$, e.g. for uniformly magnetized sphere we have

$$\overline{H}_I = \frac{2}{3}\cdot\frac{4\P}{\beta}\overline{M} \qquad \ldots .134$$

and in the general case

$$\overline{H}_I = \frac{1}{\beta}\int_{Vol} \nabla' x[\overline{M} x \nabla\left(\frac{1}{r}\right)]dv \qquad \ldots .135$$

where dv is at (x,y,z), $\overline{H}_I$ at (x′,y′,z′), $\overline{M}$ at (x,y,z) and $\overline{r} = \{(x'-x),(y'-y),(z'-z)\}$. The operator ∇' works only on (x′,y′,z′), and ∇ only on (x,y,z).

Equations 132 and 133 give $H_e^c = \overline{K}$ and Equation 131 gives

$$\overline{K} = \overline{H}_e - \left\{\frac{4\P}{\beta}\overline{M} - \overline{H}_I\right\} \qquad \ldots .136$$

but as H_I from Equation 135 is dependent on the geometry it is seen that $\overline{K}$ is dependent on the shape of the body.

$$\overline{K}_D = \frac{4\P}{\beta}\overline{M} - \overline{H}_I$$

is normally referred to as the demagnetizing force.

From Equations 118 and 121 we may deduce

$$\overline{H} = \overline{K} + \frac{4\P}{\beta}\overline{M} = \overline{K} + \frac{4\P}{\beta}\cdot k_m \cdot \overline{H} \qquad \ldots .137$$

and from this

$$\overline{H} = \frac{1}{1-\frac{4\P}{\beta}k_m}\overline{K} \qquad \ldots .138$$

$$\overline{M} = \frac{k_m}{1-\frac{4\P}{\beta}k_m}\overline{K} \qquad \ldots .139$$

The factor $\kappa_m = 1/(1 - \frac{4\P}{\beta}k_m)$ is normally called the magnetic permeability and $\chi_m = k_m/(1-\frac{4\P}{\beta}k_m)$ the suscepti-ceptibility of the material.

κ_m and χ_m preassumes the introduction of the geometry-dependent, hypothetical vector K; which the paremeter k_m does not, Stratton (1941, p. 12) remarks that k_m would be a more logical way to describe the magnetic materials although he abstains from using this notation due to the traditionally adopted parameters κ_m and χ_m.

Anisotropic materials are normally investigated by cutting long cylinders out in the directions of the anisotropy axes and measuring κ_m or χ_m in these directions. From these measurements the matrix $\{k\}$ in Equation 118a may be constructed and used in further calculations.

The ferromagnetic materials present a special problem. An approximate solution is sometimes possible by assuming $\overline{M}$ to consist of a constant part $\overline{M}_0$ and a part $\overline{M}_H$ proportional to $\overline{H}$.

If Equation 118 and its analog for electric polarization

$$\overline{P} = k_e \cdot \overline{E} \qquad \ldots .140$$

are introduced in Maxwell's Equations 99 to 102 we get

$$\nabla x\overline{E} + \frac{\beta}{a}\frac{\partial\overline{H}}{\partial t} = 0\ ; \qquad \nabla\cdot\overline{H} = 0 \qquad \ldots .141, 142$$

$$\nabla x\overline{H} - \frac{\alpha}{a}\frac{\partial\overline{E}}{\partial t} = \frac{4\P}{a}\overline{J}_e + \frac{4\P}{a}\frac{\partial}{\partial t}(k_e\cdot\overline{E}) + \frac{4\P}{\beta}\nabla x(k_m\cdot\overline{H}) \qquad \ldots .143$$

$$\nabla\cdot\overline{E} = \frac{4\P}{\alpha}\rho_e - \frac{4\P}{\alpha}\nabla\cdot(k_e\cdot\overline{E}) \qquad \ldots .144$$

$$\nabla x\overline{E} + \frac{\beta}{a}\frac{\partial\overline{H}}{\partial t} = 0\ ; \qquad \nabla\cdot\overline{H} = 0 \qquad \ldots .145, 146$$

$$\nabla x\left\{(1-\frac{4\P}{\beta}k_m)\overline{H}\right\} - \frac{\alpha}{a}\frac{\partial}{\partial t}\left\{(1 + \frac{4\P}{\alpha}k_e)\,\overline{E}\right\} = \frac{4\P}{a}\overline{J}_e \qquad \ldots .147$$

$$\nabla\cdot\left\{(1 + \frac{4\P}{\alpha}k_e)\,\overline{E}\right\} = \frac{4\P}{\alpha}\rho_e \qquad \ldots .148$$

These equations assume *isotropy* so k_m and k_e are scalars; if we also assume *homogeneous* materials, i.e. k_m and k_e are constants inside the material bodies, we have for interior points

$$\nabla x \bar{E} + \frac{\beta}{a}\frac{\partial \bar{H}}{\partial t} = 0\,, \qquad \nabla \cdot \bar{H} = 0 \qquad \ldots\ldots 149, 150$$

$$\nabla x \bar{H} - \frac{(1 + \frac{4\pi}{\alpha} k_e)}{(1 - \frac{4\pi}{\beta} k_m)} \cdot \frac{\alpha}{a}\frac{\partial \bar{E}}{\partial t} = \frac{1}{(1 - \frac{4\pi}{\beta} k_m)} \cdot \frac{4\pi}{a} \bar{J}_e \qquad \ldots\ldots 151$$

$$\nabla \cdot \bar{E} = \frac{1}{(1 + \frac{4\pi}{\alpha} k_e)} \frac{4\pi}{\alpha} \rho_e \qquad \ldots\ldots 152$$

By introducing three new constants $\alpha' = (1 + \frac{4\pi}{\alpha} k_e)\alpha$, $\beta' = (1 - \frac{4\pi}{\beta} k_m)\beta$ and $a' = (1 - \frac{4\pi}{\beta} k_m)\, a$ the equations are given the form

$$\nabla x \bar{E} + \frac{\beta'}{a'}\frac{\partial \bar{H}}{\partial t} = 0, \qquad \nabla \cdot \bar{H} = 0 \qquad \ldots\ldots 153, 154$$

$$\nabla x \bar{H} - \frac{\alpha'}{a'}\frac{\partial \bar{E}}{\partial t} = \frac{4\pi}{a'} \bar{J}_e \qquad \ldots\ldots 155$$

$$\nabla \cdot \bar{E} = \frac{4\pi}{\alpha'} \rho_e \qquad \ldots\ldots 156$$

so that a change of constants formally accounts for the presence of material bodies.

The effective speed of propagation is thus changed from c to c'.

$$c' = \frac{a'}{\sqrt{\alpha' \cdot \beta'}} = c\sqrt{\frac{(1 - \frac{4\pi}{\beta} k_m)}{(1 + \frac{4\pi}{\alpha} k_e)}} \qquad \ldots\ldots 157$$

as a result of a complicated interaction between the electromagnetic wave and the atoms of the present matter.

Unit systems

On the preceding pages it is shown that a unit system is completely determined by six basic "units" including the constants δ, α, γ, a, and β in this concept. Two sets of mechanical units are in common use, the cm, the sec., the gram, and $\delta = 1$ called the c.g.s.-system, and the metre, the kilogram and the sec. also with $\delta = 1$ called the MKS-system. The reason for including δ in the analysis is that other unit systems having $\delta \neq 1$ exist and so this constant is necessary to make the equations fully general.

Specifying c.g.s. or MKS and tacitly assuming $\delta = 1$ then determines the unit for force in the two systems. If a measure-ratio for force is $F_{c.g.s.}$ in the c.g.s.-system and F_{MKS} in the MKS-system we have

$$F_{MKS} = 10^{-5} \cdot \bar{F}_{c.g.s.} \qquad \ldots\ldots 158$$

and the speed of light in the two systems is

$$c_{MKS} = 10^{-2} \cdot c_{c.g.s.} \qquad \ldots\ldots 159$$

The electromagnetic units are then specified by choosing two of the three constants α, β, and a. The values of these constants are characteristic for the unit system and once they have been stated no need really exists for any names of the units. This is the reason for the common practice of stating E.M.U. or E.S.U. after measure-ratios when using the equations of the absolute electromagnetic system of the electrostatic system.

Table I gives the values of the constants in the three c.g.s.-systems, in the Practical system, in the MKS-system and in the three versions of the giorgized, rationalized MKSA-system, which is the SI presently being adopted internationally.

The constant c is the measure-ratio for the speed of light using the mechanical units of each particular system.

The absolute electromagnetic system is referred to as E.M.U. and so is the magnetostatic part of the Gaussian system. Maxwell's electrostatic system and the electrostatic part of the Gaussian system are called E.S.U.

The absolute electromagnetic system is in a sense the basic unit system as both the practical system and the MKS-system are derived from it.

The practical system has $\alpha = (\frac{10}{c})^2$ and $a = 10$ which make the unit of current equal to the technical unit, ampere. (1 E.M.U. = 10 amperes). The practical unit for potential difference then becomes 10 times bigger than the E.M.U. for potential (1 E.M.U. = 10^{-1} pra-volts). In the practical system Equation 40 is

$$V = \int \bar{E} \cdot \bar{ds} = -\frac{1}{10}\frac{d}{dt}\int \bar{H} \cdot \bar{da} \qquad \ldots\ldots 160$$

Table I Characteristic constants of electromagnetic unit systems

Unit System		α	β	a	Mechanical units (δ=1)
Gaussian		1	1	c	c.g.s.
Absolute Electromagnetic		$\frac{1}{c^2}$	1	1	c.g.s.
Maxwell's Electrostatic		1	$\frac{1}{c^2}$	1	c.g.s.
Practical (ampere)		$\frac{100}{c^2}$	1	10	c.g.s.
MKS (volt, ampere)		$\frac{10^7}{c^2}$	10^{-7}	1	MKS
Giorgized, Rationalized MKSA-System (Giorgi-Syst. or SI)	Basic	$\frac{10^7}{c^2}$	$(4\P)^2 \cdot 10^{-7}$	4¶	MKS
	Subsyst. (1)	$(4\P c)^2 \cdot 10^{-7}$	$(4\P)^2 \circ 10^{-7}$	$(4\P c)^2 \cdot 10^{-7}$	
	Subsyst. (2)	$\frac{10^7}{c^2}$	10^7	10^7	

substituting N for $\int \overline{H} \cdot \overline{da}$ and introducing volts by dividing both sides by 10^7 gives (1 volt = 10^8 E.M.U.):

$$V' = \frac{V}{10^7} = -10^{-8}\frac{dN}{dt} \quad \ldots .161$$

which is the familiar form of Faraday's law in the practical system. As $\beta = 1$ the magnetostatic measure-ratios are in E.M.U., V is in pra-volts and V′ in volts. Because the volt is not the "natural" practical unit for potential difference it became necessary to state explicitly "V′ in volts" and to include the conversion factor 10^7.

This conversion factor of 10^7 between volts and pra-volts was considered impractical so a search for a new unit system based on the volt and the ampere began.

Because of a confusion between conversion factors and constants of nature in the equations the idea came up that by choosing a "proper" unit system any factor in the equations could be eliminated.

In E.M.U. Maxwell's Equations 56 to 59 are

$$\nabla x\overline{E} + \frac{\partial \overline{H}}{\partial t} = 0, \qquad \nabla \cdot \overline{H} = 0 \qquad \ldots .162, 163$$

$$\nabla x\overline{H} - \frac{1}{c^2}\frac{\partial \overline{E}}{\partial t} = 4\P \cdot \overline{J}, \qquad \nabla \cdot \overline{E} = 4\P c^2 \rho \ldots .164, 165$$

and in the Gaussian system

$$\nabla x\overline{E} + \frac{1}{c}\frac{\partial \overline{H}}{\partial t} = 0, \qquad \nabla \cdot \overline{H} = 0 \qquad \ldots .166, 167$$

$$\nabla x\overline{H} - \frac{1}{c}\frac{\partial \overline{E}}{\partial t} = \frac{4\P}{c} \cdot \overline{J}, \qquad \nabla \cdot \overline{E} = 4\P \circ \rho \ldots .168, 169$$

The $\frac{1}{c}$ and $\frac{1}{c^2}$ constants and especially the 4¶-factors annoyed people. Oliver Heaviside considered the 4¶'s "irrational" and suggested the equations "rationalized" by simply removing the 4¶-factors. Now, the only way to remove the 4¶'s is to choose a = 4¶ and multiply the two other constants α and β by 4¶. Obviously the units then change by factors like $\sqrt{4\P}$ or $1/\sqrt{4\P}$ and the 4¶'s will emerge in other equations, but this did not seem to bother Heaviside. In fact very few people seem to realize what "rationalization" is, and even in recent literature the confusion exists (Avčin, 1961, pp. 5 and 10).

The choice of the values of α, β, and a is, of course, a matter of taste, and it has as such been discussed violently. No real physical arguments can be put forward in favour of one choice over the other; the only thing to be said is that there is no reason for choosing a set of values that will give grossly impractical constants in the equations, or complicated conversion factors to other unit systems.

The statement that, for instance, $\alpha = 1$, $\beta = 1$, and $a = c$ cause 4¶-factors to appear where "they have nothing to do with the geometry" is purely emotional; who can, anyway, say where 4¶-factors really "ought to" be? Maybe Newton's law for mass-attraction really "ought to" be

$$F = \frac{\gamma}{4\P} \frac{MM'}{r^2} \qquad \ldots.170$$

and Newton's second law

$$F = \frac{1}{\sqrt{4\P}} \overline{M} \frac{d^2r}{dt^2} \qquad \ldots.171$$

by rationalizing the mass unit?

However, a new unit system was desired and the conditions it had to satisfy were:

1) 1 Volt = 10^8 E.M.U. should be the unit for potential difference.
2) 1 ampere = 10^{-1} E.M.U. should be the unit for current.
3) No constants in Maxwell's equations.

As will be shown later 3) is contradictory to 1) and 2), but by the technique called giorgization the constants have been removed from Maxwell's equations and put into the Giorgi-conditions. It thus appears as if 3) is fulfilled; but it is not, and the manipulation has caused great confusion in electromagnetic theory.

Logometric formulas

To analyze the problem of changing the E.M.U. to a system having volt and ampere as the units for potential difference and current, we have to investigate the effects of changing also the mechanical units including the constant δ.

In the following the unmarked symbols stand for measure-ratios in the absolute electromagnetic system defined by gram, second, centimetre, $\delta = 1$, $a = 1$, $\beta = 1$, and thus $\alpha = \frac{1}{c^2}$. The symbols marked with a star stand for measure-ratios in some new unit system whose basic units for length, time, and mass and characteristic constants δ^*, α^*, β^*, and a^* we are going to evaluate.

The general form of Newton's second law is

$$F = \delta M \frac{d^2L}{dT^2} \qquad \ldots.172$$

and changing to the starred system we get

$$\frac{F}{F^*} = \frac{\delta}{\delta^*} \circ \frac{M}{M^*} \cdot \frac{L}{L^*} \cdot \frac{T^{*2}}{T^2} \qquad \ldots.172a$$

Coulomb's law for electric charges

$$F = \frac{Q \quad Q'}{\alpha \cdot L^2} \qquad \ldots.173$$

and

$$\frac{F}{F^*} = \left(\frac{Q}{Q^*}\right)^2 \cdot \frac{\alpha^*}{\alpha} \cdot \left(\frac{L^*}{L}\right)^2. \qquad \ldots.173a$$

Electric potential is

$$V = \frac{Q}{\alpha L} \qquad \ldots.174$$

and

$$\frac{V}{V^*} = \frac{Q}{Q^*} \cdot \frac{\alpha^*}{\alpha} \cdot \frac{L^*}{L} \qquad \ldots.174a$$

Electric current

$$I = \frac{dQ}{dT} \qquad \ldots.175$$

and

$$\frac{I}{I^*} = \frac{Q}{Q^*} \cdot \frac{T^*}{T} \qquad \ldots.175a$$

The formulas (Equations 172a to 175a) are the so-called logometric formulas introduced by O'Rahilly, except for a small change in notation†. These logometric formulas, of course, perform much the same function as dimensional equations, and except for the inclusion of the constants δ and α etc. the logometric expressions are identical to the dimensions.

The reason for including δ, α, and other characteristic constants is that we are completely free to choose and change these independently of the basic units for length, mass, and time. The logometric formulas are thus more general than the dimensional equations.

The four equations (172a to 175a) contain the following nine unknown transformation-ratios

$$\frac{M}{M^*};\ \frac{L}{L^*};\ \frac{T}{T^*};\ \frac{\delta}{\delta^*};\ \frac{F}{F^*};\ \frac{\alpha}{\alpha^*};\ \frac{Q}{Q^*};\ \frac{V}{V^*} \text{ and } \frac{I}{I^*} \qquad \ldots.176$$

†O'Rahilly uses $[L] = \frac{L}{L^*}$, $[\alpha] = \frac{\alpha}{\alpha^*}$ etc. to symbolize the transformation-ratios between the equivalent measure-ratios in the unstarred and the starred systems. To avoid introducing new symbols, the transformation ratios are used directly here.

which means that we are free to choose five of the transformation-ratios.

The aim of this investigation is to find out how the volt and the ampere can be incorporated in a unit system, so the first two choices are

$$\frac{I}{I^*} = 10^{-1} \quad \ldots.177$$

and

$$\frac{V}{V^*} = 10^8 \quad \ldots.178$$

Then we choose

$$\frac{\delta}{\delta^*} = 1 \quad \ldots.179$$

to keep Newton's second law free from constants and

$$\frac{T}{T^*} = 1 \quad \ldots.180$$

so that the time-unit is unchanged.

If we, as the last choice, put M/M* = 1 then by solving the Equations 172a to 175a we get $L/L^* = 10^7$ which is a rather crastic change. More acceptable is

$$\frac{L}{L^*} = 10^2 \quad \ldots.181$$

which means that the unit for length is changed from cm to metre. By solving the equations we get

$$\frac{M}{M^*} = 10^3 \quad \ldots.182$$

so that the unit for mass in system (*) is the kilogram instead of the gram: a very acceptable change.

The transformation-ratio for α is then

$$\frac{\alpha}{\alpha^*} = 10^{-11} \quad \ldots.183$$

or

$$\alpha^* = 10^{11}\ \alpha = \frac{10^{11}}{c^2} \quad \ldots.184$$

The definition of a speed is

$$v = \frac{dL}{dT} \quad \ldots.185$$

so

$$\frac{v}{v^*} = \frac{L}{L^*} \cdot \frac{T^*}{T} \quad \ldots.186$$

and

$$\frac{c}{c^*} = \frac{L}{L^*} = 10^2 \quad \ldots.187$$

so we get

$$\alpha^* = \frac{10^{11}}{10^4 c^{*2}} = \frac{10^7}{c^{*2}} \quad \ldots.188$$

which is the value of the constant α in the MKS-system and in the basic Giorgi-system.

MKS-systems

In the preceding section it has been shown that using kg, metre, second, and $\delta = 1$ as the mechanical units it is possible to construct a unit system with the volt and the ampere as units for potential difference and for current if we at the same time take

$$\alpha = \frac{10^7}{c^2} \quad \ldots.189$$

We are still free to choose one of the constants a or β, and it would have been tempting to take $\beta = 10^{-1}$ so the magnetostatic unit for field strength would have been equal to the E.M.U. for magnetic field. The constant a would then have been $a = 10^3$ using Equation 28.

However, the historical fact is that a was chosen as

$$a = 1 \quad \ldots.190$$

and thus

$$\beta = 10^{-7} \quad \ldots.191$$

These values characterize the unrationalized MKS-system, and Maxwell's equations in this system are

$$\nabla \times \overline{E} + 10^{-7}\frac{\partial \overline{H}}{\partial t} = 0, \qquad \nabla \cdot \overline{H} = 0 \quad \ldots.192, 193$$

$$\nabla \mathrm{x} \overline{H} - \frac{10^7}{c^2}\frac{\partial \overline{E}}{\partial t} = 4\P \cdot \overline{J}, \quad \nabla \cdot \overline{E} = \frac{4\P \cdot c^2}{10^7} \cdot \rho \quad \ldots .194, 195$$

By giorgization, as shown earlier, it is possible to get rid of the constants in front of the two time derivatives. But to cancel the $4\P$'s we have to choose $a = 4\P$ additionally. This means that $\beta = (4\P)^2 \cdot 10^{-7}$ as we can not change α if we still want the volt and the ampere.

The magnetostatic units then become badly "rationalized" so the conversion factors to E.M.U.'s include $4\P$-factors.

The rationalized MKS-system thus has the following characteristic constants

$$\alpha = \frac{10^7}{c^2}, \; a = 4\P, \text{ and } \beta = (4\P)^2 \cdot 10^{-7} \quad \ldots .196$$

and Maxwell's equations are

$$\nabla \mathrm{x} \overline{E} + 4\P \cdot 10^{-7} \cdot \frac{\partial \overline{H}}{\partial t} = 0 \, ; \quad \nabla \cdot H = 0 \quad \ldots .197, 198$$

$$\nabla \mathrm{x} \overline{H} - \frac{10^7}{4\P \cdot c^2} \cdot \frac{\partial \overline{E}}{\partial t} = \overline{J} \, ; \quad \nabla \cdot \overline{E} = \frac{4\P \cdot c^2}{10^7} \rho \quad \ldots .199, 200$$

In the giorgized, rationalized MKS-system the equations are

$$\nabla \mathrm{x} \overline{E} + \frac{\partial B}{\partial t} = 0 \, ; \qquad \nabla \cdot \overline{B} = 0 \quad \ldots .201, 202$$

$$\nabla \mathrm{x} \overline{H} - \frac{\partial D}{\partial t} = \overline{J} \, ; \qquad \nabla \cdot \overline{D} = \rho \quad \ldots .203, 204$$

together with the Giorgi-conditions

$$\overline{B} = 4\P \cdot 10^{-7} \cdot H \quad \ldots .205$$

$$\overline{D} = \frac{10^7}{4\P \cdot c^2} \cdot \overline{E} \quad \ldots .206$$

As shown in the section on Giorgi-transformation of a unit system, the giorgization means that we use two different unit systems in Equations 201, 202 and 203, 204, and that 205 and 206 are the necessary transformation rules between these systems.

Referring to Table I, Equations 201, 202 are in Giorgi-subsystem (2) and Equations 203, 204 are in subsystem (1). The basic system is used when potential difference and current simultaneously enter the equations, e.g. in Kirchhoff's equations involving resistance, self-induction and capacitance in AC-circuits.

The two transformation rules Equations 205 and 206 are, by a misinterpreted analogy to the Equation 138 and the corresponding equation for dielectrics, stated to be the relations between 'magnetic induction' and 'magnetizing force', 'electric induction', and 'polarizing force', respectively, in free space, the two constants are concealed behind the symbols

$$\mu_0 = 4\P \cdot 10^{-7} \quad \ldots .207$$

and

$$\epsilon_0 = \frac{10^7}{4\P \cdot c^2} \quad \ldots .208$$

where μ_0 is called the "permeability of free space" and ϵ_0 the "permittivity of free space".

The original aim of all these manipulations was to include the volt and the ampere in an absolute unit system; but this has long been lost in the discussions about "rationalization" and a "fourth unit".

Heaviside's suggestion of rationalization has truly been disastrous; we are now stuck with μ_0 and ϵ_0, two sets of electromagnetic units belonging to three unit systems, and a stubborn confusion about magnetic and dielectric materials.

Life was simpler when the only conversion factors to remember were

$$1 \text{ volt} = 10^8 \text{ E.M.U.}$$

$$1 \text{ ampere} = 10^{-1} \text{ E.M.U.}$$

and only one magnetic and one electric field existed.

Conclusions

It is possible to develop an electromagnetic theory independent of any unit system, and this formulation is extremely useful in distinguishing the features inherent in the theory from the features imposed by the different existing unit systems.

Inherent in the theory is thus the necessity of constants in the equations including the occasional occurrence of $4\P$-factors. One system in particular, the MKSA-system or the SI, contains a duality of dimensionally different units for the physical phenomena.

The problems in the MKSA-system can be traced back to Maxwell's "displacement current" which caused him unnecessarily to introduce two different electric field concepts, to

Heaviside's "removal" of 4¶-factors from Maxwell's equations, and to the simultaneous adoption of two conflicting models in the theory of polarizable media. The "brute force" removal of constants from Maxwell's equations by giorgization has resulted in two sets of units with the conversion factors μ_0 and ϵ_0, which, of course, have nothing to do with permeability or permittivity.

Once this is realized all the ambiguity about B- or H-units, about I·A or μ_0·I·A being the "real" magnetic moment, etc. disappears, and we are free to use which one we please, as long as the formulas are consistent.

The magnetostatic E.M.U.'s can then be converted to SI-units using the following power-of-ten conversion factors.

Magnetic field strength:

$$1 \text{ oersted} = 10^{-4} \text{ tesla}$$

Magnetization intensity:

$$1 \frac{\text{E.M.U. of mag. moment}}{\text{cm}^3} = 10^3 \frac{\text{amp}}{\text{m}}$$

In this way we are, of course, avoiding the "rationalized" SI-units and leaving it to the user in SI to rationalize or to work his problem out in a consistent set of unrationalized SI-equations.

Acknowledgments

This work was carried out at the Division of Geomagnetism, Earth Physics Branch, Department of Energy, Mines and Resources, Ottawa, Ont., Canada under a postdoctorate fellowship granted by the National Research Council of Canada.

The author is greatly indebted to Prof. Alfred O'Rahilly who, in his book, among other things pointed out the need for a third constant in electromagnetism, the existence of two units for magnetic field in the MKSA-formulation, and who reintroduced reality into dimensions by stating them as ratios between measures of the same quantity in different unit systems.

Special thanks are given to G.V. Haines for his contributions to the analysis of Giorgi-pairs, to E.I. Loomer and to P.H. Serson for many clarifying discussions and for encouragement during the work; also E. Irving is thanked for discussions on the units in paleomagnetism.

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Appendix I – Units in the MKSA-subsystems and application of the Haines rule

Because of the existence of two sets of units in the MKSA-system it is necessary to state the names of the units together with the measure-ratios in order to avoid inconsistency. However, once this is done it is always possible to go back and forth between the subsystems by giorgization of the measure-ratios observing the Haines male-female rule.

The MKSA-notation is used

$$\epsilon_0 = \frac{\alpha}{a} \qquad \ldots.\text{A1.1}$$

$$\mu_0 = \frac{\beta}{a} \qquad \ldots.\text{A1.2}$$

where α, β, and *a* belong to the basic MKSA-system (see Table I). The symbols used in the following two tables are not completely consistent with international practice nor with the earlier notation but no misunderstandings should arise from this.

The force on an electric charge Q in the field E is:

$$\overline{F} = \overset{\text{♂}}{Q} \cdot \overset{\text{♀}}{\overline{E}} = \overset{\text{♀}}{Q^*} \cdot \overset{\text{♂}}{\overline{D}} \qquad \ldots.\text{A1.3}$$

Power is:

$$P = \overset{\text{♀}}{V} \cdot \overset{\text{♂}}{I} = \overset{\text{♂}}{V^*} \cdot \overset{\text{♀}}{I^*} \qquad \ldots.\text{A1.4}$$

Work:

$$W = \overset{\text{♀}}{V} \cdot \overset{\text{♂}}{Q} = \overset{\text{♂}}{V^*} \cdot \overset{\text{♀}}{Q^*} \qquad \ldots.\text{A1.5}$$

Torque on an electric dipole in the field E

$$\overline{T} = \overset{\text{♂}}{\overline{m}_e} \times \overset{\text{♀}}{\overline{E}} = \overset{\text{♀}}{\overline{m}_e^*} \times \overset{\text{♂}}{\overline{D}} \qquad \ldots.\text{A1.6}$$

Force between two electric charges (Coulomb's law):

$$F = \frac{\overset{\text{♂}}{Q} \cdot \overset{\text{♂}}{Q'}}{a \cdot \epsilon_0 \cdot r^2} = \frac{\epsilon_0 \cdot \overset{\text{♀}}{Q^*} \cdot \overset{\text{♀}}{Q^{*\prime}}}{a \cdot r^2} = \frac{\overset{\text{♂}}{Q} \cdot \overset{\text{♀}}{Q^{*\prime}}}{a \cdot r^2} \qquad \ldots.\text{A1.7}$$

Table II. Electrostatic part of the MKSA-system

Name	Symbol, Name of unit, gender	Symbol, Name of unit, gender	Giorgization rule $\epsilon_0 \left[\frac{\text{Coulomb}}{\text{volt·m}}\right]$
	Basic system	Sub-system (1)	
Charge	Q Coulomb, ♂	Q* volt·m, ♀	$Q = \epsilon_0 \cdot Q^*$
Electric field intensity	E volt/m, ♀	D Coulomb/m², ♂	$D = \epsilon_0 \cdot E$
Electric flux	Ψ* volt·m, ♀	Ψ Coulomb, ♂	$\Psi = \epsilon_0 \cdot \Psi^*$
Potential difference	V volt, ♀	V* Coulomb/m, ♂	$V^* = \epsilon_0 \cdot V$
Polarization	P Coulomb/m², ♂	P* volt/m, ♀	$P = \epsilon_0 \cdot P^*$
Electric dipole-moment	$\overline{m}_e$ Coulomb·m, ♂	$\overline{m}_e^*$ volt·m² ♀	$\overline{m}_e = \epsilon_0 \cdot \overline{m}_e^*$
Current	I ampere $= \frac{\text{Coulomb}}{\text{sec}}$, ♂	I* $\frac{\text{volt·m}}{\text{sec}}$, ♀	$I = \epsilon_0 \cdot I^*$
Time-derivative of flux	$\frac{d\Psi^*}{dt}$ $\frac{\text{volt·m}}{\text{sec}}$, ♀	$\frac{d\Psi}{dt}$ $\frac{\text{Coulomb}}{\text{sec}}$ = amp, ♂	$\frac{d\Psi}{dt} = \epsilon_0, \frac{d\Psi^*}{dt}$

Table III. Magnetostatic part of the MKSA-system

	Symbol, Name of unit, gender	Symbol, Name of unit, gender	Giorgization rule $\mu_0 \left[\frac{\text{Webers}}{\text{amp}\cdot\text{m}}\right]$
	Basic system	Sub. system	
Magnetic pole strength	m Webers, ♂	m* † amp•m, ♀	$m = \mu_0 \cdot m^*$
Magnetic field strength	$\overline{H}$ amp/m, ♀	$\overline{B}$ Webers/m^2, ♂ = tesla	$\overline{B} = \mu_0 \cdot \overline{H}$
Magnetic flux	Φ^* amp•m, ♀	Φ Webers, ♂	$\Phi = \mu_0 \cdot \Phi^*$
Magnetization	$\overline{M}$ Webers/m^2, ♂	$\overline{J}$ †† amp/m, ♀	$\overline{M} = \mu_0 \cdot \overline{J}$
Magnetic dipole-moment	$\overline{m}_m$ Webers•m, ♂	$\overline{m}^*_m$ amp•m^2, ♀	$\overline{m}_m = \mu_0 \cdot \overline{m}^*_m$
Time-derivative of flux	$\frac{d\Phi^*}{dt}$ amp•m/sec, ♀	$\frac{d\Phi}{dt}$ † $\frac{\text{Webers}}{\text{sec}}$ = volt, ♂	$\frac{d\Phi}{dt} = \mu_0 \cdot \frac{d\Phi^*}{dt}$

† Volt and ampere change gender according to whether they are members of a magnetic or an electric Giorgi-pair.

†† Not to be confused with the symbol for current density.

If we want to avoid the constant ϵ_0 in the formula, Haines rule has to be observed.

The force on a magnetic pole in a magnetic field

$$\overline{F} = \overset{♂}{m} \cdot \overset{♀}{\overline{H}} = \overset{♀}{m^*} \cdot \overset{♂}{\overline{B}} \qquad \ldots . \text{A1.8}$$

Torque on a magnetic dipole:

$$\overline{T} = \overset{♂}{\overline{m}_m} \times \overset{♀}{\overline{H}} = \overset{♀}{\overline{m}^*_m} \times \overset{♂}{\overline{B}} \qquad \ldots . \text{A1.9}$$

Force between poles:

$$F = \frac{\overset{♂\,♂}{mm'}}{a \cdot \mu_0 \cdot r^2} = \frac{\mu_0}{a} \cdot \frac{\overset{♀\,♀}{m^* m^{*\prime}}}{r^2} = \frac{\overset{♂\,♀}{m\, m^{*\prime}}}{a \cdot r^2} \qquad \ldots . \text{A1.10}$$

Again the constant μ_0 disappears if the Haines rule is followed.

In magnetic materials the following general formula is given (Equation 121):

$$\overline{H} = \overline{K} + \frac{4\P}{\beta}\overline{M} \qquad \ldots . \text{A1.11}$$

if all the terms are giorgized we get

$$\overline{B} = \mu_0 \overline{K} + \overline{M} \qquad \ldots . \text{A1.12}$$

as

$$\mu_0 = \frac{\beta}{a} \text{ and } a = 4\P$$

By giorgizing $\overline{M}$ according to Table III then Equation A1.12 becomes

$$\overline{B} = \mu_0 (\overline{K} + \overline{J}) \qquad \ldots . \text{A1.13}$$

In Equation A1.11 the three measure-ratios $\overline{H}$, $\overline{K}$, and $\overline{M}$ all belong to the basic system of Table III. Equation A1.12 thus contains measure-ratios belonging to two different unit systems as $\overline{B}$ is the magnetic field strength in the subsystem. The same is the case in Equation A1.13, $\overline{B}$ and $\overline{J}$ belong to the subsystem in Table III and $\overline{K}$ to the basic system. The factor $\frac{4\P}{\beta}$ is thus eliminated in the two MKSA-versions of Equation A1.11 by mixing the units; but this elimination is, of course, only apparent as it must have been included earlier in the definition of $\overline{M}$ or of $\overline{B}$.

Maxwell's equations in the giorgized, rationalized MKSA-system are (Equations 201 to 204)

$$\nabla x\overline{E} + \frac{\partial \overline{B}}{\partial t} = 0 \; ; \qquad \nabla \cdot \overline{B} = 0 \qquad \ldots. 201, 202$$

$$\nabla x\overline{H} - \frac{\partial \overline{D}}{\partial t} = \overline{J} \; ; \qquad \nabla \cdot \overline{D} = \rho \qquad \ldots. 203, 204$$

The unit for $\nabla x\overline{E}$ is [volt/m^2] so $\overline{E}$ belongs to the basic system of Table II. $\frac{\partial \overline{B}}{\partial t}$ is in [$\frac{\text{Webers}}{\text{m}^2 \cdot \text{sec}}$] or [tesla/sec] so this term belongs to subsystem (2) of Table III.

$\nabla x\overline{H}$ is in [ampere/m^2] belonging to the basic system and $\frac{\partial \overline{D}}{\partial t}$ is in [$\frac{\text{Coulomb}}{\text{m}^2 \cdot \text{sec}}$] belonging to subsystem (1).

Now, subsystem (1) and the magnetic part of the basic system form a consistent unit system, and subsystem (2) together with the electric part of the basic system also constitute a unit system. These are the two systems referred to as (1) and (2) in Table I, and they are both different from the basic MKSA-system. Maxwell's Equations 201 to 204 are thus given in two different unit systems and the conversion factors from these systems to the basic MKSA-system are

$$\overline{B} = \mu_0 \cdot \overline{H}$$

and

$$\overline{D} = \epsilon_0 \cdot \overline{E}$$

Appendix 2 – Units, dimensions and logometric expressions

A unit is a name attached to a measure-ratio to state which unit system it belongs to. Units like volt, ampere, coulomb, etc. and compounded units such as volt•sec, ampere•m, etc. are used to ensure that the formulas we use are correct or that the measure-ratios are in the same unit system.

The equation

$$\nabla x\overline{E} + \frac{\partial \overline{B}}{\partial t} = 0 \qquad \ldots. A2.1$$

is stated in a unit system having volt/m as the unit for electric field strength $\overline{E}$ and tesla as the unit for magnetic field strength. If the measure ratio for magnetic field strength has been given in ampere/m we would immediately recognize either the formula in Equation A2.1 as wrong or the measure-ratio as belonging to another unit system. We may then use the corresponding formula in the other system

$$\nabla x\overline{E} + 4¶ \cdot 10^{-7} \frac{\partial \overline{H}}{\partial t} = 0 \qquad \ldots. A2.2$$

or, which amounts to the same thing, convert the measure-ratio for magnetic field from ampere/m to tesla by using the conversion factor μ_0

$$\overline{B} = \mu_0 \, \overline{H} \qquad \ldots. A2.3$$

where

$$\mu_0 = 4¶ \cdot 10^{-7} \left[\frac{\text{Webers}}{\text{amp}\cdot\text{m}}\right] \qquad \ldots. A2.4$$

The unit attached to μ_0 is of the same kind as the unit metres/inch attached to the conversion factor between inches and metres

$$L\left[\text{metres}\right] = 0.0254\left[\frac{\text{metres}}{\text{inch}}\right] \cdot L^* \left[\text{inches}\right] , \qquad \ldots. A2.5$$

i.e. it is a mnemotechnical help to remember whether it is the measure-ratio in inches or in metres we have to multiply by the conversion factor.

Also, if we change the unit of a measure-ratio to a subunit, i.e. from volts to millivolts, microvolts, kilovolts etc., then the units are useful to derive the conversion factors. If $\overline{E}$ is the measure-ratio for electric field in volts/m and $\overline{E}^*$ is the measure-ratio for the same field in microvolts/cm we have

$$\overline{E}\left[\frac{\text{volts}}{\text{m}}\right] = \overline{E}^* \left[\frac{\text{microvolts}}{\text{cm}}\right] \cdot k\left[\frac{\text{cm}}{\text{m}} \cdot \frac{\text{volts}}{\text{microvolts}}\right] \ldots. A2.6$$

where k is the conversion factor. It is easily derived that

$$k = 10^2 \cdot 10^{-6} = 10^{-4} \qquad \ldots. A2.7$$

and thus

$$\overline{E}^* = 10^4 \cdot \overline{E} \qquad \ldots. A2.8$$

Dimensions are the units stated in terms of the chosen basic units, but ignoring the basic constants. In the c.g.s. systems the basic units are cm, gram, and second normally represented by L, M, and T and the basic constants are $\delta = 1$, $\beta = 1$ or $\frac{1}{c^2}$, and $\alpha = \frac{1}{c^2}$ or 1.

The dimension of force is

$$[F] = \left[\frac{ML}{T^2}\right] \qquad \ldots. A2.9$$

which states how the measure-ratios for force change if we change the units, and thus the measure-ratios, for mass, length, and time. Comparing to Equation 172a we see that tacitly we have limited ourselves to $\delta = 1$, unchanged, as this constant is not included in Equation A2.9. It is well known that the three-unit dimensions lead to contradictions in the electromagnetic part of the C.G.S.-systems.

The MKSA-systems have four basic units - metre, kilogram, second, and ampere, symbolized by L, M, T, and I, and the two constants $\delta = 1$ and $a = 4¶$. The use of four units in the dimensions removes the double-dimensions in electromagnetism, but as δ and a are not included we are restricted to unit systems having $\delta = 1$ and $a = 4¶$. The change from the unrationalized MKS-system to the rationalized MKSA-system cannot be analyzed by dimensions as the rationalization is equivalent to a change from $a = 1$ to $a = 4¶$. Neither can the change from the absolute electromagnetic system to the unrationalized MKS-system be analyzed by dimensions as it involves a change of the constant α from $\alpha = \frac{1}{c^2}$ to $\alpha = \frac{10^7}{c^2}$ and of β from $\beta = 1$ to $\beta = 10^{-7}$.

Logometric formulas are the fully general expressions including all six basic units and constants. The logometric expression for force is (Equation 172a)

$$\frac{F}{F^*} = \frac{\delta}{\delta^*} \cdot \frac{M}{M^*} \cdot \frac{L}{L^*} \cdot \left(\frac{T^*}{T}\right)^2 \qquad \ldots. A2.10$$

where F, δ, M, etc. are measure-ratios and constants in system (1) and F*, δ*, M*, etc. belong to system (2). Using O'Rahilly's notation $[F] = F/F^*$, $[\delta] = \delta/\delta^*$, etc., we have

$$[F] = \left[\frac{\delta ML}{T^2}\right] \qquad \ldots. A2.11$$

and it is seen that the logometric formula reduces to the dimension of force if $[\delta] = \delta/\delta^* = 1$.

The expression "logometric" is also O'Rahilly's. According to him it is derived from Greek "logos" = ratio and "metron" = measure and means "ratio between measures".

As an example of the use of logometric formulas we will derive the logometric expressions for the constants α and β in the MKSA-system.

The MKSA-system is defined by metre, kilogram, second, $\delta = 1$, ampere, and $a = 4¶$. The logometric expression for force is Equation A2.11

$$[F] = \left[\frac{\delta ML}{T^2}\right] \qquad \ldots. A2.11$$

The definition of the ampere is based on Equation 8

$$d^2F = \frac{\beta}{a^2} \frac{I_1\, ds_1 \times (I_2 ds_2 \times r)}{r^2} \qquad \ldots. 8$$

which gives the following logometric formula

$$[F] = \left[\frac{\beta}{a^2} \cdot I^2\right] \qquad \ldots. A2.12$$

From Coulomb's law for two electric charges we get

$$[F] = \left[\frac{Q^2}{\alpha L^2}\right] \qquad \ldots. A2.13$$

and from Equation 5

$$[I] = \left[\frac{Q}{T}\right] \qquad \ldots. A2.14$$

Combining Equations A2.13 and A2.14 and equating [F] to [F] in Equations A2.11 and A2.12 we have

$$[F] = \left[\frac{\delta ML}{T^2}\right] = \left[\frac{\beta}{a^2} \cdot I^2\right] = \left[\frac{I^2T^2}{\alpha\, L^2}\right] \qquad \ldots. A2.15$$

From this we may derive

$$\left[\frac{a^2}{\alpha\,\beta}\right] = \left[\frac{L}{T}\right]^2 \qquad \ldots. A2.16$$

which is the logometric expression corresponding to Equation 28. Further we have

$$[\alpha] = \left[\frac{I^2 \cdot T^4}{\delta \cdot M \cdot L^3}\right] \qquad \ldots. A2.17$$

and

$$[\beta] = \left[\frac{\delta \cdot a^2 M \cdot L}{I^2 \cdot T^2}\right] \qquad \text{.... A2.18}$$

From the section on Giorgi-pairs we have

$$\epsilon_0 = \frac{\alpha}{a} \qquad \text{.... A2.19}$$

and

$$\mu_0 = \frac{\beta}{a} \qquad \text{.... A2.20}$$

The logometric expressions are then

$$\epsilon_0 \left[\frac{I^2 \cdot T^4}{\delta \cdot a \cdot M \cdot L^3}\right] \qquad \text{.... A2.21}$$

and

$$\mu_0 \left[\frac{\delta \cdot a \cdot M \cdot L}{I^2 \cdot T^2}\right] \qquad \text{.... A2.22}$$

If we choose $\delta = \delta/\delta^* = 1$ and $a = a/a^* = 1$ we get the four-unit dimensions

$$\epsilon_0 \left[\frac{I^2\ T^4}{M \cdot L^3}\right] \qquad \text{.... A2.23}$$

and

$$\mu_0 \left[\frac{M \cdot L}{I^2 \cdot T^2}\right] \qquad \text{.... A2.24}$$

If we replace M by kg, L by metre, T by second, and I by ampere and use the relations

$$\frac{\text{kg} \cdot \text{metre}}{\text{sec}^2} = \text{newton}$$

$$\frac{\text{Newton} \cdot \text{metre}}{\text{Coulomb}} = \text{volt} \qquad \text{.... A2.25}$$

$$\text{Volt} \cdot \text{sec} = \text{weber}$$

$$\text{ampere} \cdot \text{sec} = \text{coulomb}$$

we get the units for μ_0 and ϵ_0

$$\mu_0 \left[\frac{\text{Webers}}{\text{amp} \cdot \text{m}}\right] \qquad \text{.... A2.26}$$

$$\epsilon_0 \left[\frac{\text{Coulomb}}{\text{volt} \cdot \text{m}}\right] \qquad \text{.... A2.27}$$