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Abstract. Measurements of the geomagnetic components with the proton magnetometer require a coil system for producting a homogeneous bias field in a direction which can be determined accurately. The theoretical advantages of various symmetrical arrangements of coils are reviewed, and the precision of mechanical construction necessary to achieve the theoretical performance is estimated. It is concluded that where a Helmholtz pair would be too large, the most practical choice is a system of four circular coaxial coils of equal diameter.

Résumé. Les mesures des composantes géomagnétiques effectuées à l'aide d'un magnétomètre à protons exigent l'utilisation d'un jeu de bobines qui produit un champ homogène dévié dans une direction qui peut être déterminée avec précision. L'auteur étudie les avantages théoriques de diverses dispositions symétriques des bobines et il évalue l'exactitude mécanique nécessaire pour atteindre le rendement théorique. Il conclut que lorsqu'une paire de bobines Helmholtz occuperait un espace trop considérable, le choix le plus pratique serait un jeu de quatre bobines circulaires coaxiales de diamètre égal.

Introduction

The invention of the proton precession magnetometer in 1954 provided for the first time a convenient method of measuring the geomagnetic total intensity with an accuracy of 1 nT. Soon, several methods were developed to adapt the proton and other precession-type magnetometers for the absolute measurement of the components of the geomagnetic field (for reviews see Wienert, 1970; Stuart, 1972). Most of these methods require the addition to the geomagnetic field of an artificial magnetic field, of similar order of magnitude, in a known direction. A precession magnetometer measures the magnitude of the resultant, from which the desired component is deduced.

A basic difficulty is that the magnitude of the resultant field must be highly uniform for satisfactory operation of the precession magnetometer. Using free nuclear precession, a difference of a few nT across the sample will cause the precessing nuclei to fall out of phase in a second or so, and the signal decays to the noise level before an accurate measurement of frequency can be made. Other types of precession magnetometers are less susceptible to gradients, but all suffer some degradation of performance in non-uniform fields.

In the following discussion it is assumed that the free proton precession technique is used, with a sample which would fit inside a sphere 10 cm in diameter. Since bias fields as large as 40,000 nT may be required (Bobrov and Trofimov, 1968; Wienert, 1970), we adopt the criterion that within this sphere the components of the bias field must vary by no more than 1 part in 10^4 of the central axial field.

The classical way to produce a uniform magnetic field is by means of a Helmholtz coil system. The above criterion requires a Helmholtz pair about 1 m in diameter. In determining the azimuth of the geomagnetic field, it is necessary to invert the coil system, rotating it about a horizontal axis, and this is very awkward with large coils. Moreover, the external field at a large distance x from a coil system of radius a is approximately $(a/x)^3$ x the field in the centre, so that if the dimensions of the coil system can be reduced, the disturbing effect on other instruments in the same building can be greatly diminished. This paper reviews coil systems more compact than the Helmholtz arrangement, and estimates the accuracy of construction necessary to achieve the theoretical homogeneity.

Magnetic field of a circular current

Consider the scalar magnetic potential V of a current i flowing in a circle of radius a. We take the origin on the axis of the loop at a distance d from the plane of the loop (Figure 1). At a point with polar coordinates (r, θ) where $r^2 < d^2 + a^2$, the appropriate solution of Laplace's equation is

$$V = -\sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \qquad \text{..... 1}$$

where P_n are Legendre polynomials. The axial and radial components of the field can be written immediately

$$H_x = -\frac{\partial V}{\partial x} = \sum_{n=1}^{\infty} n A_n r^{n-1} P_{n-1}(\cos\theta) \qquad \text{..... 2}$$

$$H_y = -\frac{\partial V}{\partial y} = -\sum_{n=1}^{\infty} \sin\theta\, A_n r^{n-1} P'_{n-1}(\cos\theta) \qquad \text{. 3}$$

where $$P'_n(\cos\theta) = \frac{\partial}{\partial\cos\theta} P_n(\cos\theta) \qquad \text{..... 4}$$

Defining $$B_n = (n+1) A_{n+1}/A_1 \qquad \text{..... 5}$$

and writing r, P_n and P'_n in terms of $x = r\cos\theta$ and $y = r\sin\theta$

$$H_x = A_1 \left[1 + \sum_{n=1}^{\infty} B_n r^n P_n \cos\theta \right]$$

$$= A_1 \left[1 + B_1 x + \tfrac{1}{2}B_2(2x^2 - y^2) + \tfrac{1}{2}B_3(2x^2 - 3y^2)x + \tfrac{1}{8}B_4(8x^4 - 24x^2y^2 + 3y^4) + + \right] \qquad \text{. 6}$$

$$H_y = -A_1 \sum_{n=1}^{\infty} \sin\theta \frac{B_n}{n+1} r^n P'_n(\cos\theta)$$

$$= -A_1 \left[\tfrac{1}{2}B_1 y + B_2 xy + \tfrac{3}{8}B_3(4x^2 - y^2)y + \tfrac{1}{2}B_4(4x^2 - 3y^2)xy + + \right] \quad \ldots 7$$

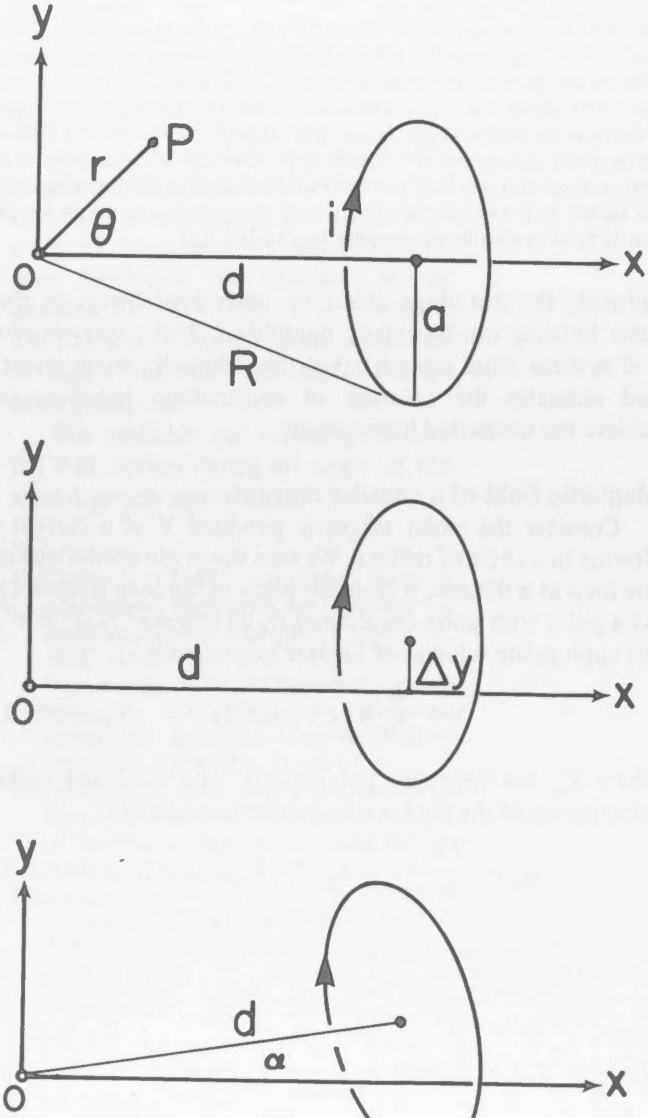

Figure 1. Top: circular current loop and coordinate system.
Centre: current loop displaced in y-direction.
Bottom: current loop tilted about origin.

It is easily verified that Equations 6 and 7 satisfy Laplace's equation in the form

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{H_y}{y} = 0 \quad \ldots 8$$

On the axis, Hy = 0 and

$$H_x = A_1 \left[1 + B_1 x + B_2 x^2 + B_3 x^3 + B_4 x^4 + + \right]. \; 9$$

A straightforward although tedious way to find the coefficients A_1 and B_n is to compare Equation 9 with the expression found in textbooks for the induction on the axis of a circular current loop:

$$H = \tfrac{1}{2}\mu_0 i a^2 \left[a^2 + (d-x)^2 \right]^{-3/2} \quad \ldots 10$$

where H is in tesla, i is in amperes, and lengths are in metres. Writing R^2 for d^2+a^2, and expanding by the binomial theorem

$$H = \tfrac{1}{2}\mu_0 i a^2 R^{-3} \left[1 - x(2d-x)R^{-2} \right]^{-3/2} \quad \ldots 11$$

we find that

$$A_1 = \tfrac{1}{2}\mu_0 i a^2 R^{-3}$$
$$B_1 = 3dR^{-2}$$
$$B_2 = \tfrac{3}{2}(4d^2 - a^2)R^{-4}$$
$$B_3 = \tfrac{5}{2}(4d^2 - 3a^2)dR^{-6}$$
$$B_4 = \tfrac{15}{8}(8d^4 - 12d^2a^2 + a^4)R^{-8} \quad \ldots 12$$

Combinations of current loops

For combinations of coaxial pairs of similar loops spaced symmetrically with respect to the plane $x = 0$, the coefficients B_1, B_3, B_5 etc. will cancel, since they are odd functions of d. With a single pair of loops, there is one free parameter, d/a. By choosing $a = 2d$, B_2 can be made to vanish, and the series for the error in Equations 6 and 7 begin with the fourth power of x and y. Near the origin, the first non-zero term provides a good estimate of the inhomogeneity.

Maxwell (1873), Neumann (1884) and Fanselau (1929) showed that with four loops one can make the fourth-order term vanish as well. Braunbek (1934) pointed out that four loops carrying the same current provide three free parameters, d_1/a_1, d_2/a_2, and a_1/a_2, and showed that it is possible to make B_2, B_4 and B_6 vanish simultaneously. The first non-zero error terms are then of the eighth order.

If one allows unequal currents (or different numbers of turns) an additional parameter is available, and one would hope to be able to cancel the eighth-order term with four loops, but Sauter and Sauter (1944) have shown that no real solution to this problem exists. With four free parameters,

however, a great variety of four-loop systems with eighth-order errors are possible. A selection is included in Table I.

The purpose of Table I is to show the overall length and diameter of various arrangements of circular loops required to produce a field uniform to 1 part in 10^4 within a sphere 10 cm in diameter. The calculations are based on the first non-zero term in the axial component (except for the system of Everett and Osemeikhian, 1966, where the full series is used). This procedure is believed to provide a safe estimate of the homogeneity for all the examples shown: the error on the axis is over-estimated. Also, at any point on the sphere the radial component, and the vectorial sum of the radial and axial errors, are no greater than the error on the axis.

Table I
Combinations of Circular Currents with Errors $< 10^{-4}$ Within a Sphere 10 cm in Diameter

System	Number of loops	Order of error	Characteristic	Overall dimensions (cm) length	Overall dimensions (cm) diameter
Ampere	1	2	–	0	1220
Helmholtz	2	4	–	52	104
Maxwell	3	6	equal R	24.6	37.5
Barker	3	6	equal a	34.2	45.0
Fanselau	4	6	equal i	33.9	37.0
Braunbek	4	8	equal i	27.5	32.5
McKeehan	4	8	equal R	26.6	33.4
Sauter	4	8	equal d	20.7	47.6
Sauter	4	8	equal a	27.9	29.7
Garrett	6	12	equal a	23.9	20.0
Garrett	8	16	equal a	21.2	15.4
Garrett	10	20	equal a	20.6	13.5
Braunbek	4	8	equal i	27.5	32.5
McKeehan	6	12	equal i	22.3	22.5
Garrett	8	16	equal i	19.0	17.5
Garrett	10	20	equal i	17.6	15.4
Everett	100	2	equal i,R	15.0	15.0

Garrett (1967) gives convenient tabulations of the parameters of most of the systems of Table I, with characteristics of the region of uniformity.

The most compact system of Table I is the spherical coil of Everett and Osemeikhian (1966). However, its error on a sphere of 7.5 cm diameter is 10^{-5}, whereas the error of the 10-loop systems would be $(0.75)^{20} \times 10^{-4} = 3 \times 10^{-6}$. Thus the mean inhomogeneity within the proton sample would be less with a 10-loop or even a 6-loop system.

Precision of construction

A subject rarely discussed in the literature of coil systems is the precision of construction necessary to realize the theoretical homogeneity of ideal systems. The only systematic treatment known to the author (Blednov and Rotshteyn, 1972) is unfortunately based on an incorrect version of Equation 7. Here we investigate various distortions one at a time, and calculate the tolerances in dimensions which must be maintained in order to avoid inhomogeneities greater than 10^{-4} on the sphere where the maximum inhomogeneity of the ideal system is 10^{-4}.

The tolerances were calculated by partial differentiation of Equations 6 and 7. The easiest case is when a loop is moved along the axis of symmetry through a distance Δd from its ideal position. The resulting change in field is

$$\Delta H = \frac{\partial H}{\partial d} \Delta d \quad \ldots\ldots 13$$

Since

$$\frac{\partial H}{\partial d} = -\frac{\partial H}{\partial x} \quad \ldots\ldots 14$$

$$\Delta H_x = -A_1 \Delta d \left[B_1 + 2B_2 x + \tfrac{3}{2} B_3 (2x^2 - y^2) + 2B_4 (2x^2 - 3y^2) x + + \right] \quad \ldots\ldots 15$$

$$\Delta H_y = A_1 \Delta d \left[B_2 y + 3B_3 xy + \tfrac{3}{2} B_4 (4x^2 - y^2) y + + \right] \ldots 16$$

In these series, and in similar series for other deformations, the second-order terms are comparable in magnitude to the first, and must be included in the calculation. The Helmholtz case is unique in that the coefficient B_2 is zero for each loop individually, making it relatively insensitive to displacements of one loop as far as homogeneity is concerned. However, the diameters of the two loops must be equal to 1 part in 1,000, since here the first-order term does not vanish.

Probably the deformation most likely to occur with the usual method of construction is that the loops are parallel but not coaxial. If one loop is moved so that its axis is a distance Δy from the axis of the others

$$\Delta H_x = -\frac{\partial H_x}{\partial y} \Delta y = A_1 \Delta y \left[B_2 y + 3B_3 xy + \tfrac{3}{2} B_4 (4x^2 - y^2) y + + \right] \quad \ldots\ldots 17$$

$$\Delta H_y = -\frac{\partial H_y}{\partial y} \Delta y = A_1 \Delta y \left[\tfrac{1}{2} B_1 + B_2 x + \tfrac{3}{8} B_3 (4x^2 - 3y^2) + \tfrac{1}{2} B_4 (4x^2 - 9y^2) x + + \right] \quad \ldots\ldots 18$$

It will be noticed that this deformation changes the direction of the central field. First-order terms vanish in the Helmholtz case.

Finally, the effect of tilting one loop, so that its axis still passes through the origin but at an angle α to the axes of the other loops, was investigated by rotating x and y coordinates. Remembering that y is always positive

$$\frac{\partial x}{\partial \alpha} = \pm\, y \qquad\qquad \frac{\partial y}{\partial \alpha} = \pm\, x$$

$$\Delta H_x = \pm\, \alpha \left[\frac{\partial H_x}{\partial x}\, y \pm \frac{\partial H_x}{\partial y}\, x \right]$$

$$\Delta H_y = \pm\, \alpha \left[\frac{\partial H_y}{\partial x}\, y \pm \frac{\partial H_y}{\partial y}\, x \right]$$

with the signs depending on the quadrant of the field point.

The effect on the homogeneity of departures of the loops from true circles was not calculated. Knowing that circular loops can be replaced by square ones with little loss of homogeneity, such effects are assumed to be small provided that some degree of symmetry is maintained.

Table II gives two sets of tolerances. The first tolerances refer to *asymmetrical* deformations, when one loop of a system is varied. The tolerances in brackets apply to *symmetrical* distortions, when both loops of a pair are varied by the same amount. In the coaxial cases, symmetry about the mid-plane is preserved. In the Δy case, the two loops of a pair are moved in opposite directions. In the last case, both loops are tilted in the same sense about the origin. With symmetrical deformations, odd powers of x and y vanish.

Table II
Dimensions and Tolerances for Homogeneity of 10^{-4} on Sphere of Radius r

	Helmholtz	Braunbek	Sauter
r/a_1	.0965	.307	.337
a_1	1.0000 ± 8 (±141)	1.0000 ± 4 (± 11)	1.0000 ± 5 (± 8)
a_2		.7639 ± 11 (± 8)	1.0000 ± 6 (± 7)
d_1	.5000 ± 50 (± 28)	.2780 ± 3 (± 3)	.2432 ± 3 (± 4)
d_2		.8457 ± 4 (± 11)	.9407 ± 5 (± 32)
i_1	1.0000 ± 17	1.0000 ± 9 (± 17)	1.0000 ± 11 (± 17)
i_2		1.0000 ± 8 (± 17)	2.2604 ± 15 (± 38)
Δy_1	.0000 ± 99 (± 56)	.0000 ± 6 (± 7)	.0000 ± 7 (± 8)
Δy_2		.0000 ± 8 (± 22)	.0000 ± 10 (± 63)
α_1	0 ± 6′	0 ± 3′ (± 4′)	0 ± 3′ (± 4′)
α_2		0 ± 3′ (± 4′)	0 ± 2′ (± 4′)

Note: The first tolerances quoted refer to *asymmetrical* deformations; those in brackets refer to *symmetrical* deformations. See text for explanation.

Table II does not include the effects of deformation on the magnitude and direction of the field at the origin. The most troublesome case, as far as rigidity of the system is concerned, is likely to be the Δy deformation. Moving one loop in its own plane by .0003 a will produce a radial field at the origin of 10^{-4} H_x in the Helmholtz case. The four-loop systems are about half as sensitive to the displacement of one loop, since it contributes a smaller fraction of the central field.

Coils with finite cross-section

Practical coil systems usually employ coils of many turns rather than single loops, and the cross-section of the winding is much larger than the tolerance of the dimensions discussed above.

When the required number of turns is not too large, the simplest procedure is to wind them in a single cylindrical layer with the same radius a and mean distance d as the loop of the prototype. In the Helmholtz case, for example, one then has an array of identical Helmholtz pairs distributed along the x-axis. The axial field of the pair located at $d + l_i, -d + l_i$ will be

$$H = 2A_1 \left[1 + B_4(x - l_i)^4 + B_6(x - l_i)^6 + + \right]$$

Adding the field of the corresponding pair at $d - l_i, -d - l_i$, the odd powers of x cancel

$$H = 4A_1 \left[1 + B_4(x^4 + 6l_i^2x^2 + l_i^4) + + \right]$$

Averaged over $l_i = 0$ to $l_i = l$, the first error term becomes

$$B_4(x^4 + 2l^2x^2 + \tfrac{1}{5}\, l^4)$$

If l is one tenth of the useful radius x, the inhomogeneity is increased by only 2 per cent over that of the prototype.

When more turns are necessary than can be accommodated in a single layer, the breadth and depth of the winding cross-section provide additional parameters which can sometimes be used to advantage. Maxwell (1873, Sect. 713) showed that the Helmholtz loops can be replaced by coils of rectangular cross-section without significant effect on the homogeneity if the ratio of breadth to depth is $(31/36)^{1/2}$ = 0.9280. However, Maxwell's approach has not been very successful with higher order systems; for example, there is no analogous solution for the Braunbek arrangement (McKeehan, 1936). More recently, Garrett (1967) has pointed out that in general any single loop of radius a_o can be replaced by a coil of square cross-section of mean radius a_c without significantly affecting the homogeneity if

$$a_c^2 = a_o^2 - \frac{1}{12}\, D^2$$

where D is the depth of the square cross-section.

Obviously any of the prototype coil systems can be expanded into systems of solenoids with equivalent orders of homogeneity. Thick solenoids are difficult to wind accurately, and do not appear to have any advantages in the present application.

Practical considerations

In most instrument workshops it is difficult to make accurately circular coils larger than 40 cm in diameter, and square coils are often used in the larger sizes. The Helmholtz loops can be replaced by escribed square loops with no significant loss of uniformity (Fanselau, 1956). The appropriate separation is 0.5445 time the length of the side of the square. However, Lee-Whiting (1957) has shown that at least five square coils are required to match the uniformity of the four-coil systems with circular loops. For diameters less than 40 cm, it is probably easier to make circular coils.

When a Helmholtz system would be too large, the best choice for accurate construction is the system of Sauter and Sauter (1944) with four coils of equal diameter. Its only disadvantage relative to the Braunbek arrangement is that the number of turns on the outer coils must be 2.2604 ±38 times the number on the inner coils. This ratio can be approximated by pairs of integers such as 52/23 or 113/50. The currents can be adjusted by resistors in parallel with the coils. In fact, it is a simple matter to trim the system empirically by varying a resistor connected across one or two of the coils and observing the decay of the proton precession signal. Often errors in the geometry can be compensated in this way.

Table II shows that for a coil system 30 cm in diameter tolerances of the order of .05 mm must be maintained. It is not difficult to achieve this precision in the radius of the coil forms, but it is difficult to assemble the system with the required accuracy, especially in the centring of the coils. The best way would seem to be machining the complete form from a single cylindrical tube. As Stuart (1972) points out, the most serious practical problem is in winding uniformly multi-layer coils.

Everett and Osemeikhian (1966) estimated that their spherical coils were constructed with a precision of .05 mm. However, with a sphere 17 cm in diameter they found that the proton signal decayed more quickly than expected, and in their final design they used spheres having diameters of 21 cm and 25 cm, which did not appreciably increase the rate of decay.

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