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DISCUSSION AND INTERPRETATION OF THE MATHEMATICS GOVERNING RADIAL FLOW IN A SINGLE HORIZONTAL ROCK FRACTURE

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ABSTRACT

From the viewpoint of practical rock engineering, radial flow must be considered one of the most important aspects of groundwater hydrology. All standard field testing - including pump tests, packer tests, flow into drainage chambers and grouting - must be analysed based on radial flow concepts.

Two distinct flow systems are encountered. These are:

(a) Conductivity tests in soil, where the medium can be idealized as a continuum and a statistical Darcy approach assumed valid.

(b) Conductivity tests in fractured rock where the test section will include only one or a few discrete fluid conduits, where a statistical interpretation is seldom valid.

This report deals only with the latter. The fracture flow formulations discussed are all based on the basic parallel plate model.

The aim is to compare the various derivations, assumptions, etc. using consistent nomenclature so that any substantive variance in the results could be analysed. This is done for both laminar and turbulent flow regimes.

RESUME

Du point de vue de l'étude pratique des roches, l'écoulement radial doit être consideré comme l'un des plus importants aspects de l'hydrologie des eaux souterraines. Tous les essais standard sur le ter‥ain - y compris les essais de production à la pompe, les essais de packer, l'écoulement dans les chambres de drainage et le fonçage des puits par cimentation - doivent être analysés selon les principes de l'écoulement radial.

Deux systèmes distincts d'écoulement sont rencontrés. Ils sont:

(a) Les essais de conductivité dans un sol, où le milieu peut être considéré comme continu et une approche statistique de Darcy est supposée valide.

(b) Les essais de conductivité dans la roche fracturée, où la section sous épreuve ne comprend qu'une ou quelques passages discontinus de fluide et où une interprétation statistique est rarement valide.

Ce rapport ne se concerne qu'avec le deuxième cas. Les formulations de l'écoulement dans les fractures discutées sont toutes établies d'après le modèle de base des plaques parallèles.

Le but est de comparer les diverses dérivations, suppositions, etc. utilisant une nomenclature consistante de sorte qu'une variation réelle dans les résultats peut être analysée. Ceci est accompli pour les régimes d'écoulement laminaire et turbulent.

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A report prepared for

Energy, Mines and Resources, Canada
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Division of Seismology and Geothermal Studies,
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Any opinions expressed in this report are those of the authors and the Earth Physics Branch takes no responsibility and neither does it endorse the findings.

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NOMENCLATURE

Q	Flow rate	L^3/T
e	Fracture Aperture	L
h,H	Fluid Head	L
γ	Unit Weight of Fluid	F/L^3
g	Gravitational Acceleration	L/T^2
ρ_m	Mass Density	FT^2/L^4
μ	Dynamic Viscosity	$F\text{-}T/L^2$
ν	Kinetic Viscosity	L^2/T
L	Flow Length	L
r,R	Radial Flow Length	L
v	Fluid Velocity	L/T
R_e	Reynold's Number	—
D_h	Hydraulic Diameter	L
k	Fracture Wall Roughness (Absolute)	L
k/D_h	Relative Roughness	—
P	Pressure	M/L^2

CHAPTER I

INTRODUCTION

From the viewpoint of practical rock engineering, radial flow must be considered one of the most important aspects of groundwater hydrology. All standard field testing - including pump tests, packer tests, flow into drainage chambers and grouting - must be analysed based on radial flow concepts.

In geotechnical engineering, two separate and distinct problems are encountered. These are:

(a) Conductivity tests in soil, where the medium can be idealized as a continuum and a statistical Darcy approach assumed valid.

(b) Conductivity tests in fractured rock where the test section will include only one or a few discrete fluid conduits, where a statistical interpretation is seldom valid.

The following report deals only with the latter. The fracture flow formulations discussed are all based on the basic parallel plate model. The basic laws for streamline (laminar) flow between parallel plates can be derived from the Navier-Stokes' equation. It is easily shown that

$$Q \propto e^3$$

where Q = flowrate

e = aperture between the plates

In radial flow further complications arise, however, as the effects of inertia, kinetic energy and turbulence tend to be more influential.

The main contributors to the analytical analyses of radial flow are Baker (1955), Maini (1971), Iwai (1976) and Rissler (1978). At first glance, due largely to varying conventions, nomenclature, etc. these various authors' results appear radically different.

The purpose of the present report is to compare the various derivations, assumptions, etc. using consistent nomenclature so that any substantive variance in the results could be analysed. This is done for both laminar and turbulent flow regimes.

The various formulations are briefly reviewed and critically compared through the remainder of this report. The detailed mathematical formulations for each author are presented in the Appendices at the end.

CHAPTER II

DISCUSSION OF EXISTING RADIAL FRACTURE FLOW FORMULATIONS

II.1 Generalities

All of the following discussions deal with radial flow in a horizontal fracture as outlined in Figure 1. Inclined fractures can also easily be incorporated (Rissler 1978).

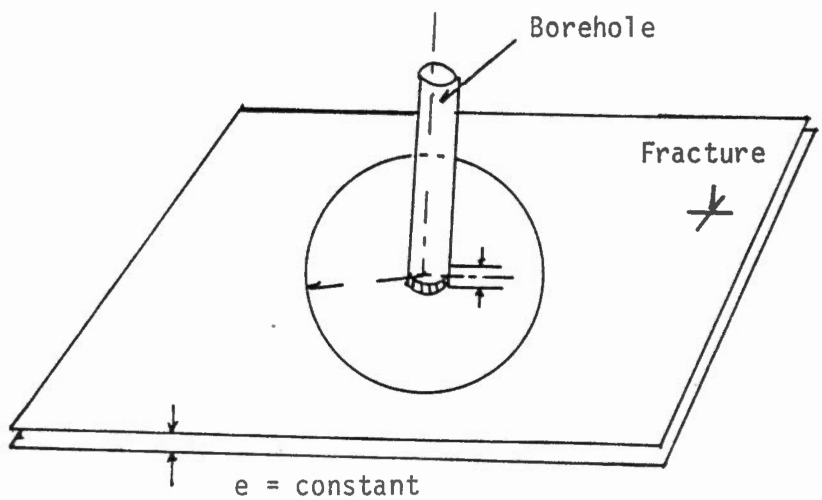


Fig. 1

II.2 Baker (1955)

This author presents one of the earliest comprehensive discussions of radial fracture flow. His assumptions are:

a) fracture is of uniform size

b) fracture is restricted to a horizontal plane

c) fracture aperture is very small compared to its width.

Baker then presents the following general equation for steady state flow in self-consistent units;

$$P_f = \frac{L \rho V_m^2}{T} \phi \left(\frac{\eta}{V_m T \rho}\right) \tag{2-1}$$

Baker's terms are defined and translated into units consistent with this report below:

where:
- P_f = fluid pressure in psi = $h \cdot \gamma$
- ρ = density in pcf = γ
- γ = unit weight = $g \cdot \rho_m$
- ρ_m = mass density
- η = dynamic viscosity = $\mu = \nu \cdot \rho_m$
- ν = kinematic viscosity
- T = fracture aperture = e

This previous equation may then be rewritten as:

$$h_f \cdot \gamma = \frac{L \cdot \gamma \cdot V_m^2}{e} \phi \left(\frac{\mu}{V_m \cdot e \cdot \gamma}\right) \tag{2-2}$$

It should be noted that this approach is applicable to one-dimensional (i.e. linear) flow (not radially symmetrical). The author then states that for laminar flow

$$\phi \left(\frac{\mu}{V_m \cdot e \cdot \gamma}\right) = k \left(\frac{\mu}{V_m \cdot e \cdot \gamma}\right) \quad \text{where } k = 12 \tag{2-3}$$

Substituting into (2.2) and rearranging the terms leads to

$$\frac{h_{fs}}{L} = \frac{12 \cdot \nu}{g \cdot e^2} \cdot V_m \tag{2-4}$$

or $$Q = \frac{g \cdot e^3}{12 \cdot \nu} \cdot \frac{h_{fs}}{L} \qquad (2\text{-}5)$$

Equations (2.4) and (2.5) can, of course, be simply derived from the Navier Stokes' equation for one-dimensional flow between smooth parallel plates (Ref. Louis, 1969).

For convergent radial flow, Baker gives the following:

$$dp = \frac{12\ \rho\ Q}{2\pi r\ T^3}\ dr \qquad (2\text{-}6)$$

However, it can be shown that this should be (Ref. Appendix B):

$$dp = \frac{12\ \eta\ Q}{2\pi r\ T^3}\ dr \qquad (2\text{-}7)$$

Translating units again gives:

$$dh = \frac{12\ \nu\ Q}{2\pi r\ g\ e^3}\ dr \qquad (2\text{-}8)$$

Integrating equation (2.8) between an outer radius R_1 and an inner radius R_2 gives:

$$h_{fs} = \frac{6\ \nu\ Q}{\pi \cdot g \cdot e^3}\ \ell n\ R_1/R_2 \qquad (2\text{-}9)$$

This last equation describes the frictional (viscous) head loss in the fracture. However, for radial flow into a well, kinetic energy losses may also be important. Calculation of these losses requires a knowledge of the velocity distribution in the fissure. Baker assumes that the error will be small if one assumes that the distribution curve is flat; i.e. at any given radius the velocity is constant at all points between the fissure faces and is equal to the mean flow velocity. For streamline flow, Baker gives the following expressions:

$$v = \frac{P_{fs}}{\eta\ L}\left[\frac{T^2}{8} - \frac{t^2}{2}\right] \tag{2-10}$$

where v represents the velocity at a distance t from the axial plane of the fissure. Translating terms once more gives:

$$v = \frac{h_{fs} \cdot g}{\nu \cdot L}\left(\frac{e^2}{8} - \frac{z^2}{2}\right) \tag{2-11}$$

Then the mean square of velocities between the fissure faces is

$$(v^2)_m = \frac{h^2{}_{fs} \cdot g^2 \cdot e^4}{120 \cdot \nu^2 \cdot L^2} \tag{2-12}$$

Baker then gives the pressure drop due to kinetic energy as

$$P_{KS} = \frac{1}{2}\rho\ (v^2)_m \tag{2-13}$$

However, for an incompressible fluid this should be (refer to Appendix B),

$$P_{KS} = \frac{(v^2)_m}{2 \cdot g} = \frac{h^2{}_{fs} \cdot g \cdot e^4}{240 \cdot \nu^2 \cdot L^2} \tag{2-14}$$

Or, using equation (2.4),

$$P_{KS} = \frac{3\ V^2_m}{5\ g} \tag{2-15}$$

Note that this last equation has been derived based on one-dimensional flow conditions. Baker then substitutes for V^2_m from radial flow to get (see Appendix B):

$$P_{KS} = \frac{3\ Q^2}{20 \cdot g \cdot \pi^2 \cdot e^2} \left(\frac{1}{R_2{}^2} - \frac{1}{R_1{}^2} \right) \tag{2-16}$$

Therefore, the total head drop for streamline radial flow may be expressed as

$$h_s = \frac{6 \cdot \nu \cdot Q}{\pi \cdot g \cdot e^3} \ln \left(R_1/R_2 \right) + \frac{3\ Q^2}{20 \cdot g \cdot \pi^2 \cdot e^2} \left(\frac{1}{R_2{}^2} - \frac{1}{R_1{}^2} \right) \tag{2-17}$$

For the case of radial turbulent flow, Baker follows Miessback's law and assumes (see Appendix B):

$$P_{ft} \propto V_m{}^n \tag{2-18}$$

such that

$$\phi \left(\frac{\eta}{V_m\ T\ \rho} \right) = k \left(\frac{\eta}{V_m\ T\ \rho} \right)^{2-n} \tag{2-19}$$

where k and n are experimental constants.

From pipe flow analogy, Baker assumes n = 2 for the fully turbulent case, giving for one-dimensional flow

$$V_m{}^2 = \frac{e}{k} \cdot \frac{h_{ft}}{L} \tag{2-20}$$

For radial turbulent flow Baker gives:

$$d\,P_{ft} = \frac{K \cdot \rho \cdot Q^2}{4\,\pi^2\,r^2\,T^3}\,dr \qquad (2\text{-}21)$$

Translating terms and integrating between R_1 and R_2 then gives:

$$h_{ft} = \frac{K \cdot Q^2}{4\,\pi^2 \cdot e^3}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \qquad (2\text{-}22)$$

For the calculation of turbulent kinetic energy losses Baker uses:

$$V_m^{\,2} = \frac{Q^2}{4\,\pi^2\,r^2\,e^2} \qquad (2\text{-}23)$$

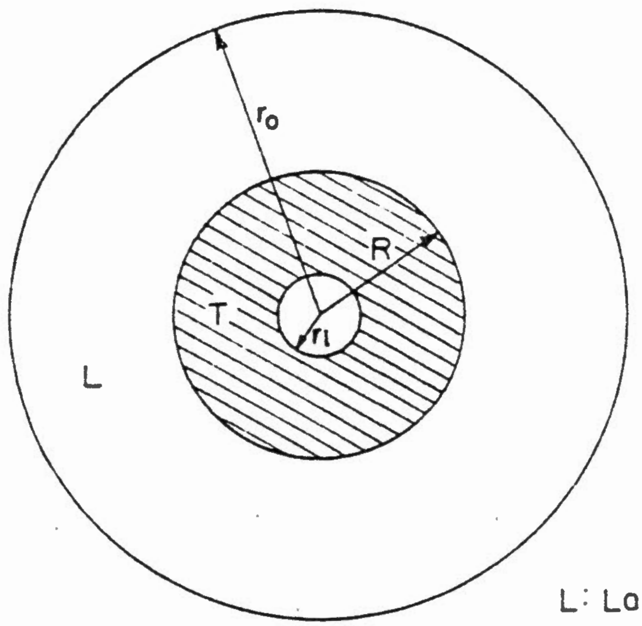
Substituting into equation (2-15) leads to:

$$h_{Kt} = \frac{Q^2\,\rho_m}{8 \cdot g \cdot \pi^2 \cdot e^2}\left(\frac{1}{R_2{}^2} - \frac{1}{R_1{}^2}\right) \qquad (2\text{-}24)$$

and, therefore, the total head drop for radial turbulent flow is given by:

$$h_T = \frac{K\,Q^2}{4\,\pi^2\,e^3}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + \frac{Q^2}{8\,g\,\pi^2\,e^2}\left(\frac{1}{R_2{}^2} - \frac{1}{R_1{}^2}\right) \qquad (2\text{-}25)$$

Baker further realized that although flow through a fracture into a well at low velocities will be streamlined, at higher velocities the flow will become turbulent at some radius 'R' as shown in Figure 2.



L: Laminar Region
T: Turbulent Region

Figure 2 Laminar and turbulent regions in radial flow (after Baker, 1955).

Under these conditions, the head drop will be given by

$$h_T = \frac{6\ \nu\ Q}{\pi\ g\ e^3}\ \ell n\ R_1/R + \frac{3\ Q^2}{20 \cdot g \cdot \pi^2 \cdot e^2}\left(\frac{1}{R^2} - \frac{1}{R_1{}^2}\right)$$

$$+ \frac{K\ Q^2}{4\ \pi^2\ e^3}\left(\frac{1}{R_2} - \frac{1}{R}\right) + \frac{Q^2}{8 \cdot g \cdot \pi^2 \cdot e^2}\left(\frac{1}{R_2{}^2} - \frac{1}{R^2}\right) \qquad (2\text{-}26)$$

II.3 Maini (1970)

Maini's radial flow formulation is fundamentally the same as that of Baker. The author begins his formulation from equation (2-11). Then using (2-12) and (2-14) he arrives at (2-15).

For radial flow conditions Maini gives (see Appendix C):

$$v = K_J\ \frac{\delta p}{\delta r} \qquad (2\text{-}27)$$

and for flow into a fissure

$$Q = 2\pi r \cdot e \cdot \bar{v} \qquad (2\text{-}28)$$

Now substituting into equation (2-15) leads to equation (2-16) giving the kinetic energy head loss. Then substituting (2-28) into (2-27) and integrating gives

$$h = \frac{Q\ \ell n\ (r_1/r_0)}{2\pi\ K_J\ e} \qquad (2\text{-}29)$$

expressing the viscous head loss. Hence Maini's final relation between flow and energy loss for streamline radial flow is

$$h = \frac{Q \, \ell n \, (r_1/r_0)}{2\pi \, K_J \, e} + \frac{3 \, Q^2}{20 \, g \, \pi^2 \, e^2} \left(\frac{1}{r_1{}^2} - \frac{1}{r_0{}^2} \right)$$

or $$h = \frac{6 \, \nu \, Q}{\pi \cdot g \cdot e^3} \, \ell n \, (r_1/r_0) + \frac{3 \, Q^2}{20 \, g \, \pi^2 \, e^2} \left(\frac{1}{r_1{}^2} - \frac{1}{r_0{}^2} \right) \qquad (2\text{-}30)$$

or $$h = A \, Q + B \, Q^2$$

For the case of turbulent radial flow, Maini also bases his development on Miessback's law:

$$(V_m)^n = C \, \frac{\delta P}{\delta r} \qquad (2\text{-}31)$$

where n = degree of non-linearity $1 < n < 2$

C = constant - depends on ν and the medium and is experimentally determined in the field

Assuming V_m to be the mean velocity in the fracture, Maini substitutes (2-28) into (2-31) to get

$$\left(\frac{Q}{2\pi r e}\right)^n = C \, \frac{\delta h}{\delta r} \qquad (2\text{-}32)$$

Integrating the last equation between r_0 and r_1 and h_0 and h_1 then gives

$$Q^n = C \, (2\pi e)^n \left[\frac{r_1{}^{n-1} \cdot r_0{}^{n-1}}{r_1{}^{n-1} - r_0{}^{n-1}} \right] (h_0 - h_1)(1 - n) \qquad (2\text{-}33)$$

or for the fully turbulent case where n=2

$$Q^2 = C \cdot 4 \cdot \pi^2 \, e^2 \left[\frac{r_1 \cdot r_0}{r_1 - r_0} \right] (h_1 - h_0) \qquad (2\text{-}34)$$

Maini further notes that equation (2-33) can be rewritten as

$$E \cdot Q^n = (h_0 - h_1)$$

or $\quad n \log Q + \log E = \log (h_0 - h_1)$ (2-35)

Consequently, for non-linear flow, a diagram $\log (h_0 - h_1)$ versus $\log Q$ is represented by a straight line, such that the slope of the line is given by n while the intercept on the h-axis is equal to E.

where $E \equiv$ non-linear permeability function

II.4 Iwai (1976)

Iwai took a somewhat different, and for his radial flow development much more fundamental approach in which he derived everything from the Navier Stokes' equations.

For linear viscous flow of an incompressible fluid, the Navier Stokes' equation is:

$$\frac{D\bar{v}}{Dt} = \bar{f} - \frac{1}{\rho} \nabla P + \nu(\nabla^2 \bar{v}) \tag{2-36}$$

Assuming that:

a) the flow is governed only by mechanical and thermal energy present within the system

b) the flow is isothermal.

c) the flow is Newtonian and homogeneous

d) Stokes' equation is valid.

The equation of continuity is:

$$\nabla \cdot \bar{v} = 0 \qquad (2\text{-}37)$$

Those two last equations represent a system of four equations and four unknowns. Solving for the appropriate boundary conditions (refer to Appendix D), Iwai derives the following expression for linear streamline flow:

$$Q = \frac{g \cdot e^3}{12\ \nu} (- \nabla h) \qquad (2\text{-}38)$$

For fractures with smooth walls in an axisymmetric coordinate system, Iwai gives the Navier Stokes' equation in cylindrical coordinates as:

$$\begin{cases} \left[\dfrac{D\ V_r}{Dt} + \dfrac{V_\theta^{\ 2}}{r} \right] = f_r - \dfrac{1}{\rho} \dfrac{\delta P}{\delta r} + \nu \left[\nabla^2\ V_r - \dfrac{V_r}{r^2} - \dfrac{2}{r^2} \dfrac{\delta V_\theta}{\delta \theta} \right] \\ \left[\dfrac{D\ V_\theta}{Dt} + \dfrac{V_r\ V_\theta}{r} \right] = f_\theta - \dfrac{1}{\rho} \dfrac{1}{r} \dfrac{\delta P}{\delta \theta} + \nu \left[\nabla^2\ V_\theta + \dfrac{2}{r^2} \dfrac{\delta V_r}{\delta \theta} - \dfrac{V_\theta}{r^2} \right] \\ \dfrac{D\ V_z}{Dt} = f_z - \dfrac{1}{\rho} \dfrac{\delta P}{\delta z} + \nu\ \nabla^2\ V_z \end{cases} \qquad (2\text{-}39)$$

where:

$$\frac{D}{Dt} = \frac{\delta}{\delta t} + V_r \frac{\delta}{\delta r} + \frac{V_\theta}{r} \frac{\delta}{\delta \theta} + V_z \frac{\delta}{\delta z}$$

$$\nabla^2 = \frac{\delta}{\delta r^2} + \frac{1}{r} \frac{\delta}{\delta r} + \frac{1}{r^2} \frac{\delta^2}{\delta \theta^2} + \frac{\delta^2}{\delta z^2}$$

The equation of continuity for an incompressible fluid in this system is

$$\frac{1}{r}\frac{\delta}{\delta r}(r\ V_r) + \frac{1}{r}\frac{\delta V_\theta}{\delta \theta} + \frac{\delta V_z}{\delta z} = 0 \tag{2-40}$$

Assuming that for axisymmetric steady state flow conditions the two last equations may be rewritten as

$$\begin{cases} V_r \dfrac{\delta V_r}{\delta r} = -\ g\dfrac{\delta h}{\delta r} + \nu\left[\dfrac{\delta^2 V_r}{\delta r^2} + \dfrac{1}{r}\dfrac{\delta V_r}{\delta r} + \dfrac{\delta^2 V_r}{\delta z^2} - \dfrac{V_r}{r^2}\right] & (2\text{-}41) \\ 0 = -\ g\dfrac{\delta h}{\delta z} & \end{cases}$$

$$\frac{\delta}{\delta r}(r\ V_r) = 0 \tag{2-42}$$

Equation (2-41) contains a nonlinear term which is dependent on the variation of V_r in the r-direction, one of the important characteristics of radial flow.

Iwai then assumes that the inertial term in that equation can be ignored. Then, recognizing that

$$V_r = \frac{F(z)}{r} \tag{2-43}$$

equation (2-41) becomes

$$0 = -\ g\frac{\delta h}{\delta r} + \nu \cdot \frac{1}{r} \cdot \frac{d^2\ F(z)}{dz^2} \tag{2-44}$$

However, h is independent of z, and hence

$$\frac{d^2 F(z)}{dz^2} = \text{constant}$$

therefore, $$F(z) = \frac{1}{2} C_3 z^2 + C_4 z + C_5 \qquad (2\text{-}45)$$

Solving this equation for the appropriate boundary conditions results in

$$F(z) = - \frac{\gamma}{2\mu} \frac{(h_i - h_o)}{\ell n\, r_o/r_i} (z^2 - (e/2)^2) \qquad (2\text{-}46)$$

Substituting into equation (2-43) gives

$$V_r = - \frac{\gamma}{2\mu r} \frac{(h_i - h_o)}{\ell n\, r_o/r_i} \left[z^2 - (e/2)^2\right] \qquad (2\text{-}47)$$

Integrating between z = e/2 and z = -e/2, the average velocity can be found

$$V_r = \frac{\gamma e^2}{12\mu r} \frac{(h_i - h_o)}{\ell n\, r_o/r_i} \qquad (2\text{-}48)$$

The flow rate into the fissure is then

$$Q = 2\pi r \cdot e \cdot V_r = \frac{\pi \gamma e^3}{6\mu} \frac{(h_i - h_o)}{\ell n\, r_o/r_i} \qquad (2\text{-}49)$$

Hence when the inertial force is negligible, analogy with Darcy's approach gives

$$K_f = \frac{\gamma e^2}{12 \mu} = \frac{g \cdot e^2}{12 \nu} \qquad (2\text{-}50)$$

Where both the velocity and its gradient are significant, the effect of the inertia term must therefore be considered.

$$A = B + C \tag{2-51}$$

Since the term $(V_r \; \delta V_r/\delta r)$ is always negative and approaches zero in the limiting case, Iwai assumes that B and C are always opposite in sign. Then:

a) For divergent flow, B is positive, therefore $(\delta h/\delta r)$ is negative and C must be negative. Therefore $|C|>|B|$.

b) For convergent flow, B is negative, $(\delta h/\delta r)$ is positive and therefore C is positive. Hence $|C|<|B|$.

Therefore Iwai concludes that:

a) For divergent flow one gets an apparent increase in permeability.

b) For convergent flow one gets an apparent decrease in permeability.

In drawing the above conclusions, Iwai ignores the possibility of changes in effective stress and subsequent fracture deformations causing these effects. This will be discussed further in this report.

Iwai states that since variations in velocity and velocity gradient are greatest at the inner boundary one can avoid these erroneous values of K due to inertia effects by measuring $(\delta h/\delta r)$ at an appropriate distance from the inner boundary.

The author then determined the upper limit of applicability of Darcy's Law such that inertial effects could be neglected. This was accomplished by assuming that, as a first approximation, V_r and $(\delta V_r/\delta r)$ can be determined from Darcy using equation (2-47). From this he derives that

$$V_r \frac{\delta V_r}{\delta r} = \left[\frac{\gamma}{2\mu} \; \frac{\Delta h}{\ell n \; r_o/r_i}\right]^2 \left(-\frac{1}{r^3}\right) \left[(e/2)^2 - z^2\right]^2 \tag{2-52}$$

Then taking the average over the fracture aperture as

$$[V_r \frac{\delta V_r}{\delta r}]_a = \frac{1}{e} \int_{-b/2}^{b/2} V_r \frac{\delta V_r}{\delta r} \, d z \qquad (2\text{-}53)$$

One obtains

$$[V_r \frac{\delta V_r}{\delta r}]_a = [\frac{\gamma}{2\mu} \frac{\Delta h}{\ell n \; r_o/r_i}]^2 (-\frac{1}{\gamma^3}) [\frac{e^4}{30}] \qquad (2\text{-}54)$$

Using equation (2-47), the head gradient is then given by

$$- g \frac{\delta h}{\delta r} = \frac{g}{r} \frac{\Delta h}{\ell n \; r_o/r_i} \qquad (2\text{-}55)$$

Iwai then determines the ratios between the inertia and head gradient terms as

$$\eta_i = \frac{| V_r \frac{\delta V_r}{\delta r} |}{| - g \frac{\delta h}{\delta r} |} = \frac{\rho Q e}{20 \mu \pi r_i^2} \qquad (2\text{-}56)$$

Using this ratio he determines the effects of inertia on radial flow as discussed later in this report.

Iwai also briefly discusses the two dimensionless parameters:

Re - Reynolds' Number

λ - friction factor

where

$$Re = \frac{V D_h}{\nu} \qquad (2\text{-}57)$$

$$\lambda = \frac{D_h \ (- \nabla h)}{V^2_{r,i}/2g} \tag{2-58}$$

where $D_h = 2e$

For linear flow these factors are given by Louis (1969). For radial flow Iwai defines the Reynolds' Number as

$$Re = \frac{V \ (2b)}{\nu} = \frac{\rho \ Q}{2 \ \mu \ R} \tag{2-59}$$

The Reynolds' Number is the ratio of viscous to inertial forces. If viscous forces predominate over inertial, i.e. where $R = r_i$ in figure 3, then we may take

$$\phi = 96/Re \tag{2-60}$$

This last equation applies for laminar flow, independent of geometry. Baker suggests from his results that this may be applicable to critical Reynolds' Numbers of 4000 to 8000. This is very high relative to linear flow values.

Finally, Iwai quotes Maini (1971) for turbulence, and for correction factors for kinetic energy (see equations (2-30) and (2-33)).

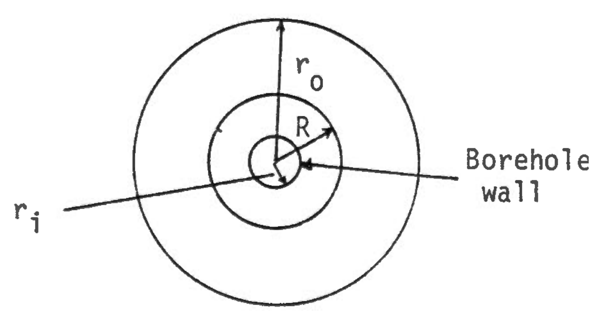


Figure 3

II.5 Rissler (1978)

2.5.1 Introduction

Rissler begins his flow formulation in exactly the same manner as Iwai. Starting from the basic Navier Stokes' equation and integrating for parallel plate boundaries, he shows that $\bar{v} \propto I$ (refer to equations (2.36) and (2.38)).

Following this, Rissler develops the governing flow laws for one-dimensional flow in a fissure in some detail. He bases his development on the basic law for energy losses in pipes of any cross-section given by

$$I = \lambda \cdot \frac{1}{D_h} \cdot \frac{v^2}{2g} \qquad (2\text{-}61)$$

where
- λ = friction coefficient
- D_h = hydraulic diameter
- $\frac{v^2}{2g}$ = kinetic energy relative to the unit weight

For a fissure of aperture e

$$D_h = 2e \qquad (2\text{-}62)$$

and Reynolds' number is: $$Re = \frac{D_h \cdot \bar{v}}{\nu} \qquad (2\text{-}63)$$

Hence λ can be calculated

$$\lambda = \frac{96}{Re} \qquad (2\text{-}64)$$

This equation is valid for parallel walls and relative roughnesses below

0.032. The relative roughness term that defines the roughness of the plate (fracture walls) is an important parameter in fracture flow. The terms are as defined below.

k = absolute roughness

D_h = hydraulic diameter

The remaining flow laws have been determined experimentally on artificial fractures (Ref. Louis, 1969) (see Appendix E).

For one-dimensional flow with non-parallel walls ($k/D_h > 0.032$)

$$\lambda = \frac{96}{Re} [1 + 8.8 (k/D_h)^{1.5}] \qquad (2\text{-}65)$$

For turbulent flow:

a) hydraulically smooth ($k/D_h = 0$)

$$\lambda = 0.316 \, Re^{-\frac{1}{4}} \qquad (2\text{-}66)$$

b) completely rough

(i) $k/Dh < 0.032$

$$\frac{1}{\sqrt{\lambda}} = - 2 \log \frac{k/D_h}{3.7} \qquad (2\text{-}67)$$

(ii) $k/D_h > 0.032$

$$\frac{1}{\sqrt{\lambda}} = - 2 \log \frac{k/D_h}{1.9} \qquad (2\text{-}68)$$

For the laminar-turbulent transition for $k/D_h < 0.032$, the law proposed by Colebrook-White is used

$$\lambda = 0.316\ Re^{-1/4}$$

Louis (1969) gives the critical Reynold's number for the laminar-turbulent transition for parallel flow and relative roughness less than 0.0168 as

$$Re_K = 2300 \qquad (2\text{-}69)$$

For relative roughness greater than 0.0168

$$Re_K = F(k/D_h) \qquad (2\text{-}70)$$

On the λ - Re diagram (refer to Figure 4) Rissler approximated this relationship as a straight line. Using the equation of this line as well as (2-67) and (2-68), he developed the following relations between Re and k/D_h (refer to Appendix E):

a) For $0.0168 \leq k/D_h \leq 0.032$

$$\log Re_K = \frac{1}{1.76} \log \left[142{,}000 \left(\log \frac{3.7}{k/D_h}\right)^2\right] \qquad (2\text{-}71)$$

b) For $k/D_h > 0.032$

$$\log Re_K = \frac{1}{1.76} \log \left[142{,}000 \left(\log \frac{1.9}{k/D_h}\right)^2\right] \qquad (2\text{-}72)$$

c) For the transition from hydraulically smooth (2-66) to completely rough (2-67) gives

$$Re_{K_1} = 2.552 \left[\log \frac{3.7}{k/D_h}\right]^8 \qquad (2\text{-}73)$$

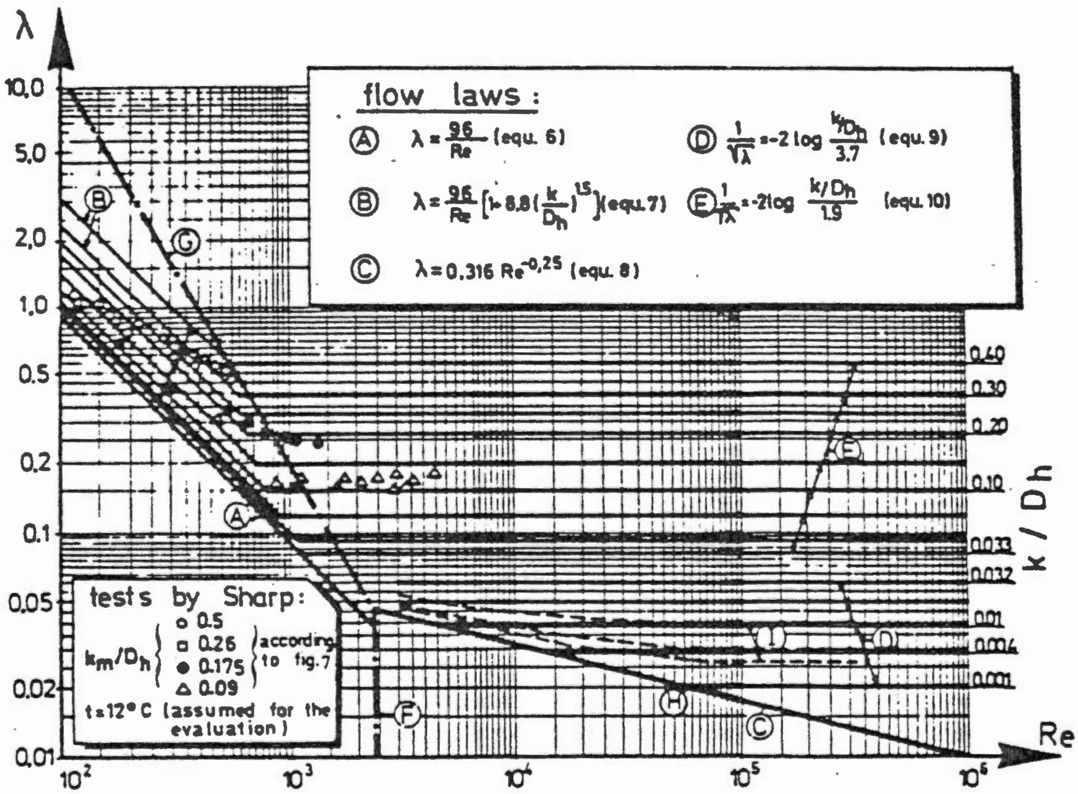


Fig. 4 Laws for one-dimensional flow in a fissure

(After Rissler 1978)

Wittke and Louis (1969) showed that the velocity profile for laminar divergent radial flow varies only slightly from that corresponding to one-dimensional flow conditions (see figure 5). They concluded that the flow laws applicable to one-dimensional flow can also be used as a good approxi-

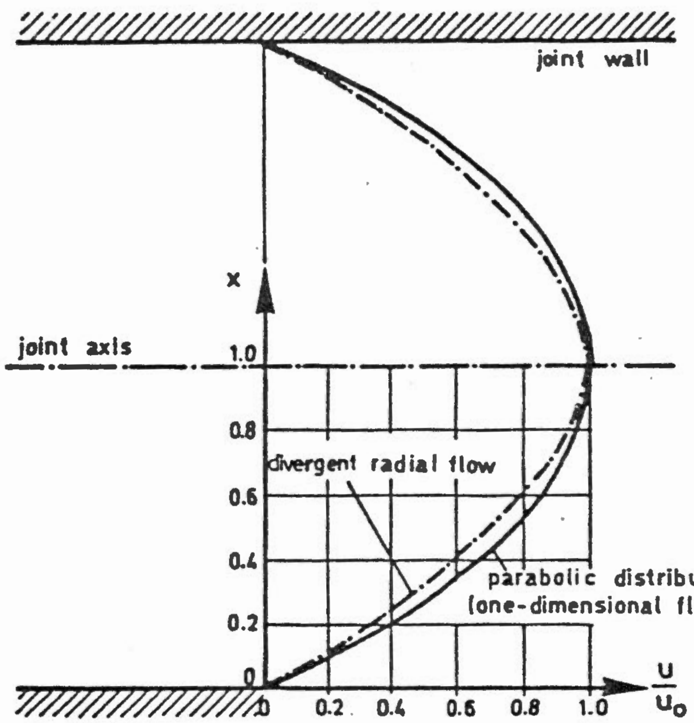


Figure 5 Distribution of the flow velocities for divergent radial flow and for one-dimensional flow in a fissure. Laminar flow conditions

(After Rissler 1978)

mation for divergent radial flow and can be handled using potential theory.

Rissler also discussed the field implications.

A typical water test set-up is shown schematically in figure 6. Knowledge of the energy head of water at the joint entrance is required for

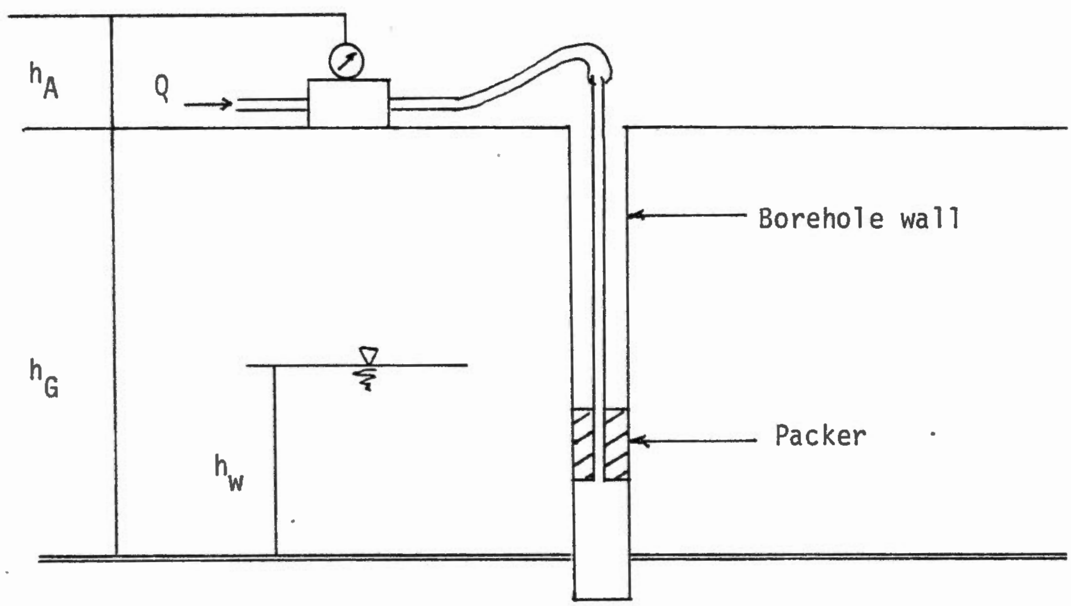


Figure 6 (After Rissler 1978)

reliable evaluation of water pressure tests. If the pressure is measured at the wellhead, the energy losses between the gauge and joint entrance must be calculated.

If gravity is the only mass force acting, then the total energy head of a fluid element can be computed using Bernoulli's equation, i.e.

$$H = z + \frac{P}{\gamma} + \frac{\bar{v}^2}{2g} \qquad (2\text{-}74)$$

where
z = geometric head
P = static pressure
γ = fluid density
$\bar{v}$ = mean flow velocity
g = acceleration of gravity

The total head is then composed of two parts:

(i) piezometric head h

(ii) velocity head $\bar{v}^2/2g$.

Since in most practical rock engineering situations $\bar{v}$ is very small, the equations can be simplified, and

$$H = z + \frac{P}{\gamma} = h \qquad (2\text{-}75)$$

Referring to figure 6

$$H = h_A + h_G - h_W \qquad (2\text{-}76)$$

The total energy loss between the gauge and the fracture consists of the following terms:

h_1: losses due to bends in tubing

h_2: friction losses in lines

h_3: losses due to enlargement in cross-section below packer

h_4: losses due to bending and contracting at entrance from borehole to fracture

The energy head acting at the fracture is consequently

$$H_o = h_A + h_G - h_W - h_1 - h_2 - h_3 - h_4 \qquad (2\text{-}77)$$

Rissler calculates expressions for h_1 through h_4 and presents these graphically. (Refer to Appendix E.)

2.5.2 Steady Radial Symmetrical Flow in Horizontal Joints

The conditions for this development are limited to nonparallel, laminar flow in a fissure. The friction law of Poiseuille for parallel flow is however also included.

Rissler considers a circular cut of radius r (see figure 1) and defines the continuity condition as

$$Q = \bar{v} \cdot F = \bar{v} \cdot 2\pi r \cdot e \qquad (2\text{-}78)$$

giving

$$\bar{v} = \frac{Q}{2\pi r \cdot e} \qquad (2\text{-}79)$$

Introducing the flow law of Louis (refer to equation (2-65)) into (2-61) and replacing I by $(- \frac{dH}{dr})$,

$$\bar{v} = - \frac{g \quad e^2}{12 \cdot \nu[1 + 8.8 \; (k/D_h)^{1.5}]} \cdot \frac{dH}{dr} \qquad (2\text{-}80)$$

Equating the right-hand sides of both equations leads to:

$$\frac{dH}{dr} = - \frac{12 \; \nu[1 + 8.8 \; (k/D_h)^{1.5}]}{g \cdot e^2} \cdot \frac{Q}{2\pi r \cdot e} \qquad (2\text{-}81)$$

Integrating between appropriate boundary conditions gives

$$H = H_o - \frac{6\,\nu[\,1 + 8.8\,(k/D_h)^{1.5}]}{g \cdot e^3} \cdot \frac{Q}{\pi}\,\ell n\,\frac{r}{r_o} \tag{2-82}$$

This last equation describes the energy head 'H' in the fracture as $f\,(H_o, r_o, g, \nu, e$ and $k/D_h)$ which is constant for one test. H decreases with increasing r. Generally it is sufficient to introduce H = 0 at r = R as a boundary condition if R is very large. Then (2-82) gives a linear relation between the energy head at the fracture entrance (H_o) and the flow rate (Q).

$$H_o = Q \cdot \frac{6\,\nu}{g\,\pi\,e^3}\,[\,1 + 8.8\,(k/D_h)^{1.5}\,]\,\ell n\,R/r_o \tag{2-83}$$

Equation (2-83) contains the measurable values ν and r_o along with the parameters e and k/D_h which are decisive to the fracture conductivity.

2.5.3 Turbulent Flow Near the Borehole - Smooth and completely rough with nonparallel walls

The following discussion is valid for $k/D_h = 0$ and $k/D_h \geqslant 0.0168$. For $0 < k/D_h < 0{,}0168$ further considerations are necessary as will be discussed later.

a) Extent of the turbulent zone

From the continuity equation (2-79) one can easily see that the mean flow velocity decreases as r increases. Since

$$Re = \frac{\bar{v}\,D_h}{\nu} \tag{2-84}$$

if the flow adjacent to the borehole wall is turbulent, the Reynolds' number decreases with increasing r until it reaches a critical value r_K at which a change from turbulent ($r < r_K$) to laminar ($r > r_K$) flow occurs (see figure 7).

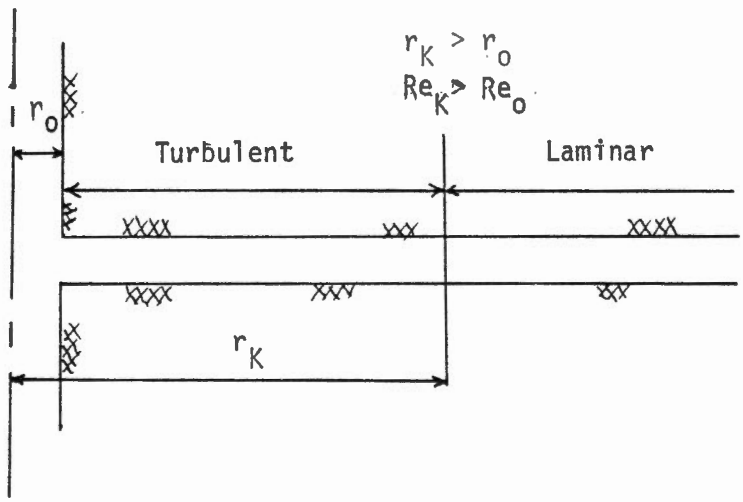


Figure 7 (After Rissler 1978)

To calculate r_K substitute

$$\left\{ \begin{array}{l} Re_K \rightarrow Re \\ \frac{Q}{2\pi re} \rightarrow \bar{v} \\ r_K \rightarrow r \end{array} \right.$$

in equation (2-80) to obtain

$$r_K = \frac{Q}{\pi \cdot \nu \cdot Re_K} \qquad (2\text{-}85)$$

Now substituting for Re_K from (2-69) through (2-72)

- for $k/D_h < 0.0168$

$$r_K = \frac{Q}{\pi \cdot \nu \cdot 2300} \tag{2-86}$$

- for $0.0168 \leqslant k/D_h \leqslant 0.032$

$$r_K = \frac{Q}{\pi \cdot \nu} [142{,}000 \ (\log \frac{3.7}{k/D_h})^2]^{-\frac{1}{1.76}} \tag{2-87}$$

- for $k/D_h > 0.032$

$$r_K = \frac{Q}{\nu \cdot \pi} [142{,}000 \ (\log \frac{1.9}{k/D_h})^2]^{-\frac{1}{1.76}} \tag{2-88}$$

Hence the extent of r_K for a given Q and ν is not constant but depends on k/D_h.

2.5.4 Distribution of energy head in section of turbulent flow and transition conditions

a) For $k/D_h > 0.032$

The general relation for energy losses in radially symmetrical flow is

$$-\frac{dH}{dr} = \lambda \cdot \frac{1}{D_h} \cdot \frac{\bar{v}^2}{2g} \tag{2-89}$$

Using (2-79) and (2.68), this equation becomes:

$$\frac{dH}{dr} = -\frac{Q^2}{64 \ e^3 \ g \ \pi^2 \ (\log 1.9/k_{/D_h})^2} \cdot \frac{1}{r^2} \tag{2-90}$$

Integrating for the appropriate boundary conditions then gives

$$H = H_o - \frac{Q^2}{64\ e^3\ g\ \pi^2\ (\log \frac{1.9}{k/D_h})^2} (\frac{1}{r_o} - \frac{1}{r}) \qquad (2\text{-}91)$$

Hence

$$H = F(r)$$

Rissler now extends this concept into the laminar zone. Hence referring to figure 7, he first determines the energy head $H = H_K$ at the outer boundary of the turbulent zone ($r = r_K$) using equations (2-88) and (2-91). This is then applied as the inner boundary condition for the laminar flow area.

Hence the energy head at the turbulent boundary is obtained first by substituting equations (2-88) into (2-91), ($r_K \rightarrow r$),

$$H_K = H_o - \frac{Q^2}{64\ e^3\ g\ \pi^2\ (\log \frac{1.9}{k/D_h})^2} [\frac{1}{r_o} - \frac{\nu\pi}{Q} [142{,}000\ (\log \frac{1.9}{k/D_h})^2]^{\frac{1}{1.76}} \qquad (2\text{-}92)$$

The function $H = F(r)$ for laminar flow was derived previously. Substituting the right-hand side of equation (2-92) for the energy head H_o in (2-82) then gives

$$H = F(r) = H_o - \frac{Q^2}{64\ e^3\ g\ \pi^2\ (\log \frac{1.9}{k/D_h})^2} \left\{ \frac{1}{r_o} - \frac{\nu \cdot \pi}{Q} [142{,}000\ (\log \frac{1.9}{k/D_h})^2]\ \frac{1}{1.76} \right\}$$

$$- \frac{6 \cdot \nu \cdot Q}{\pi \cdot g \cdot e^3} [\ 1 + 8.8\ (k/D_h)^{1.5}]$$

$$\cdot \ \ln \frac{r}{\frac{Q}{\nu \cdot \pi} [142{,}000 \ (\log \frac{1.9}{k/D_h})^2]^{\frac{1}{1.76}}} \qquad (2\text{-}93)$$

If this equation is to be considered as a function $H_o = F(Q)$ then constant numerical values must be assigned to H and r. Therefore,

$$r = R \quad \text{at} \quad H = 0$$

and,

$$H_o = \frac{Q^2}{64 \ e^3 \ g \ \pi^2 \ (\log \frac{1.9}{k/D_h})^2} \left\{ \frac{1}{r_o} - \frac{\nu \cdot \pi}{Q} [142{,}000 \ (\log \frac{1.9}{k/D_h})^2]^{\frac{1}{1.76}} \right\} +$$

$$\frac{6 \ \nu \ Q}{g \ \pi \ e^3} [\ 1 + 8.8 \ (k/D_h)^{1.5}] \cdot \left\{ \ln \frac{R \cdot \nu \cdot \pi}{Q} \cdot [142{,}000 \ (\log \frac{1.9}{k/D_h})^2]^{\frac{1}{1.76}} \right\} \qquad (2\text{-}94)$$

Analogous considerations can be made for completely rough ($0.0168 \leqslant k/D_h \leqslant 0.032$) and for smooth ($k/D_h = 0$) conditions. The corresponding flow laws must of course be introduced and result in

- for $0.0168 \leqslant k/D_h \leqslant 0.032$

$$H_o = \frac{Q^2}{64 \ e^3 \ g \ \pi^2 \ (\log \frac{3.7}{k/D_h})^2} \left\{ \frac{1}{r_o} - \right.$$

$$\frac{\nu \cdot \pi}{Q} [142{,}000 \ (\log \frac{3.7}{k/D_h})^2]^{\frac{1}{1.76}} \Big\} +$$

$$\frac{6 \ \nu \ Q}{g \ \pi \ e^3} [1 + 8.8 \ (k/D_h)^{1.5}] \ \cdot$$

$$\ell n \left\{ \frac{R \ \nu \ \pi}{Q} [142{,}000 \ (\log \frac{3.7}{k/D_h})^2]^{\frac{1}{1.76}} \right\} \qquad (2\text{-}95)$$

- for $k/D_h = 0$

$$H_o = 0.0263 \sqrt[4]{\frac{\pi \ \nu}{Q}} \cdot \frac{Q^2}{g \cdot \pi^2 \cdot e^3} [r_o^{-3/4} - (\frac{Q}{\nu \cdot \pi \cdot 2300})^{-3/4}]$$

$$+ \frac{6 \ \nu \ Q}{g \ \pi \ e^3} \cdot \ell n \frac{2300 \ \nu \ \pi \ R}{Q} \qquad (2\text{-}96)$$

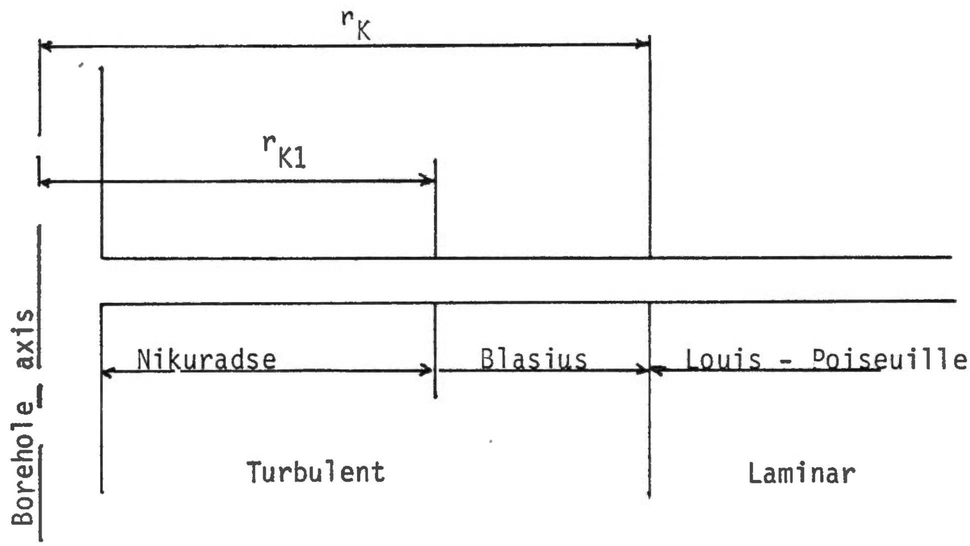
Equations (2-94) through (2-96) represent, for the given roughness range, a relationship between the data H_o and Q resulting from the test and e and k/D_h decisive for the fracture permeability.

2.5.5 Turbulent flow near the borehole - transition zone

The previous development has assumed that there is an abrupt change from hydraulically smooth to rough (turbulent) flow at a certain Re = $F(k/D_h)$. However, under certain conditions the change may be gradual in which case there will be two flow changes:

i.e. turbulence (Nikuradse) → transition (Blasius)
→ laminar (Louis, Poiseuille)

as shown in figure 8.



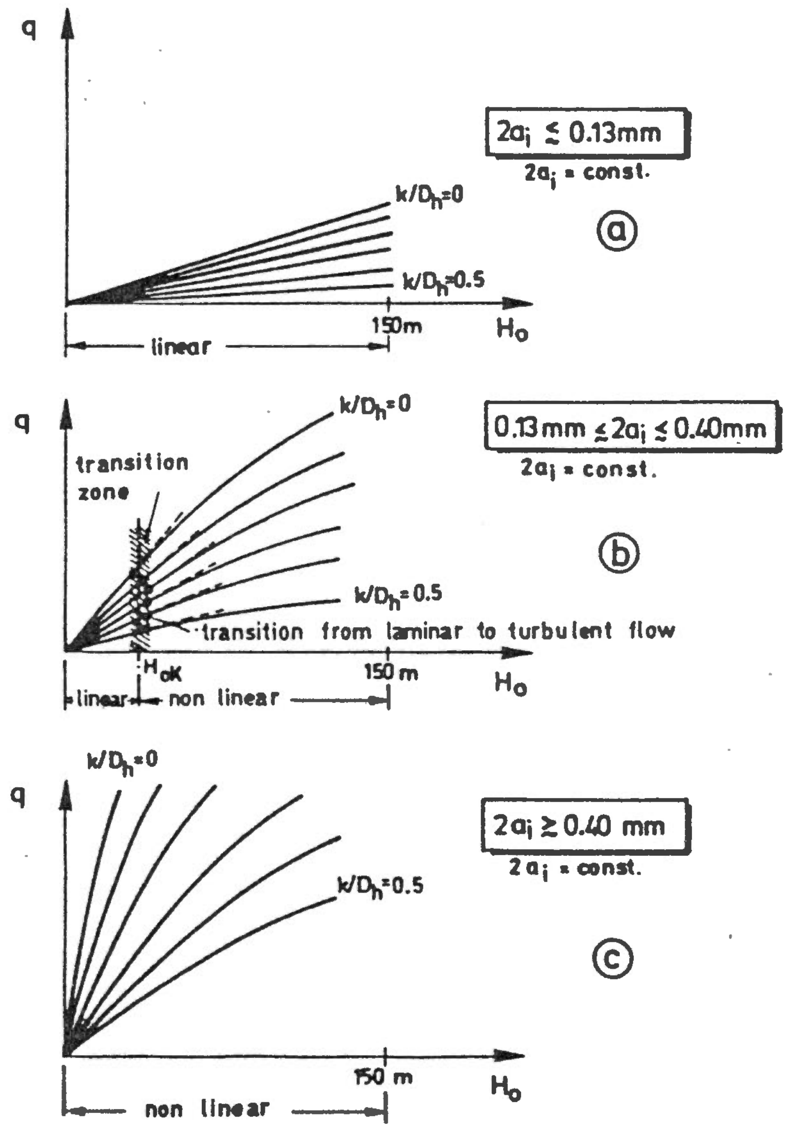
(After Rissler 1978)

Figure 8

The radius for the transition from completely rough to hydraulically smooth (i.e. r_{K_1} in figure 7) can be immediately derived from equations (2-84) through (2-88), i.e.

$$r_{K_1} = \frac{Q}{\pi \nu} \cdot \frac{1}{2.552 \left(\log \frac{3.7}{k/D_h}\right)^8} \qquad (2\text{-}97)$$

Figure 9 shows the characteristic curves calculated by Rissler for $Q = f(H_o)$. He found that for apertures $0.13 \leq e \leq 0.4$ mm a numerical evaluation of the theory showed that - for certain apertures - the laminar-turbulent change is nearly independent of (k/D_h) at a nearly constant energy head H_o. Rissler then uses this fact to determine the fracture aperture from the critical energy head without knowing (k/D_h). However, in order to do this it is first necessary to represent the relationship found between H_{o_K} and e using equations (2-83) and either (2-94), (2-95) or (2-96).



Characteristic curves $q = f(H_o)$

(After Rissler 1978)

Figure 9

H_{o_K} is the critical energy head at which laminar fracture flow becomes turbulent directly at the borehole wall. Using a numerical evaluation of these equations Rissler derives

$$H_{o_K} = \frac{6\ \nu^2\ r_o}{g\ e^3} \cdot Re_K\ (k/D_h)\ [\ 1 + 8.8\ (k/D_h)^{1.5}]\ \ln \frac{R}{r_o} \qquad (2\text{-}98)$$

where the flowrate Q has been replaced by the critical Reynolds' number $Re_K = f\ (k/D_h)$.

Solving equation (2-98) for e gives

$$e = \sqrt[3]{\frac{6 \cdot \nu^2 \cdot r_o}{g \cdot H_{o_K}} \cdot Re_K\ (k/D_h) \cdot [1 + 8.8\ (k/D_h)^{1.5}] \cdot \ln \frac{R}{r_o}} \qquad (2\text{-}99)$$

Introducing then the following values

$$B = \frac{6}{g \cdot H_{o_K}} \cdot Re_K\ (k/D_h) \cdot [1 + 8.8\ (k/D_h)^{1.5}]$$

$$= \frac{6}{g \cdot H_{o_K}} \cdot f\ (k/D_h) \qquad (2\text{-}100)$$

$$C = \nu^2 \cdot r_o \cdot \ln \left(\frac{R}{r_o} \right) \qquad (2\text{-}101)$$

Then $$e = \sqrt[3]{B \cdot C} \qquad (2\text{-}102)$$

Rissler evaluated the influence of (k/D_h) on the relation between H_{o_K} and e and came up with a possible 8 percent error.

For practical purposes, Rissler produced several nomographs which will be discussed in the following section of this report.

CHAPTER III

COMPARISON OF RADIAL FLOW FORMULATIONS

The previous section of this report discussed the basic developments of each of the various authors dealing with radial flow conditions. In this section the various approaches are compared and analyzed.

3.1 Laminar Flow

When dealing with laminar radial flow, the basic flow relation for frictional losses must be examined prior to considering the correction factors for kinetic energy and inertia.

All derivations considered previously were based on the following basic laws.

a) the Navier-Stokes' equation

b) the continuity equation

In addition, the following assumptions were made:

a) the fracture is of uniform size

b) the fracture is restricted to the horizontal plane (although this can be easily extended)

c) the fracture aperture is very small with respect to its width

d) the flow is governed by the mechanical and thermal energy in the system

e) the flow is isothermal

f) the flow is Newtonian and homogeneous

g) the Navier-Stokes' equation is valid.*

Based on the previous assumptions and theories, and neglecting kinetic

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* Although Baker (1955) does not state that his general flow equation is derived from Navier-Stokes' law, it is apparent from the form of the expression that it must be.

energy and inertial terms, the authors all derive the following expression for the viscous head loss in a fracture.

$$\Delta h = \frac{6 \cdot \nu \cdot Q}{\pi \cdot g \cdot e^3} \ln \frac{R_1}{R_0} \tag{3-1}$$

Baker (1955) and Maini (1971) both derive correction factors for kinetic energy losses. This requires an assumption that the velocity distribution curve in the fracture is flat, i.e. at any given radius the velocity is constant at all points between the fissure faces and is equal to the mean flow velocity. Then using Bernoulli's equation plus the equation for the mean square velocity between the fissure faces (derived from Navier-Stokes) for linear flow they derive

$$P_{KS} = \frac{3\ V_m^2}{5g.} \tag{3-2}$$

Substituting V_m^2 for radial flow, (assuming that this substitution is valid), gives

$$P_{KS} = \frac{3\ Q^2}{20 \cdot g \cdot \pi^2 \cdot e^2} \left(\frac{1}{R_2{}^2} - \frac{1}{R_1{}^2} \right) \tag{3-3}$$

Hence the relation between flow and energy loss for streamline divergent radial flow is given by

$$\Delta h = \frac{6\ \nu\ Q}{\pi\ g\ e^3} \ln \frac{r_1}{r_0} + \frac{3\ Q^2}{20 \cdot g \cdot \pi^2 \cdot e^2} \left(\frac{1}{r_1{}^2} - \frac{1}{r_0{}^2} \right) \tag{3-4}$$

Note that this equation is for radial divergent flow while equation (2-17) was for radial convergent flow.

Maini notes that equation (3-3) corresponds to a small correction term; and for most practical field problems, can easily be ignored. He notes, however, that if nonlinear flow is observed this term should definitely be evaluated before assuming turbulence.

If one evaluates (3-3) more closely, it can easily be seen that it may be rewritten as

$$P_{KS} = 0{,}00155 \frac{Q^2}{e^2} \left(\frac{1}{R_2{}^2} - \frac{1}{R_1{}^2} \right) \qquad (3\text{-}5)$$

in metric units.

For most practical cases, the following assumption can be made:

$$R_2 >> R_1$$

and, therefore,

$$P_{KS} \approx \frac{0{,}155\ Q^2}{e^2\ R_2{}^2} \qquad (3\text{-}6)$$

Since $Q \alpha e^3$, then as e decreases, Q will decrease much more rapidly. Hence the influence of the kinetic energy term will largely vary as follows:

$$P_{KS} \ \alpha \ \frac{1}{R_2{}^2} \qquad (3\text{-}7)$$

Hence for most practical cases where fracture apertures are very small, the influence of the kinetic energy term will be small and as R_2 increases, will become negligible.

Iwai (1977) begins his development from the basic Navier-Stokes' equations written in cylindrical coordinates, and shows that for axi-symmetric steady state flow it contains a nonlinear inertia term which is dependent on the velocity gradient.

He first calculates the viscous head loss as if the inertia term is negligible. He then calculates the inertia term by assuming that, as a first approximation, V_r can be taken from the derivation based on Navier-Stokes' equation. Finally, by integrating, he obtains the average over the fracture aperture:

$$\left[V_r \frac{\delta V_r}{\delta r} \right]_a = \left[\frac{\gamma}{2\mu} \frac{\Delta h}{\ell n\, r_o/r_i} \right]^2 \left(- \frac{1}{r^3} \right) \left[\frac{e^4}{30} \right] \qquad (3\text{-}8)$$

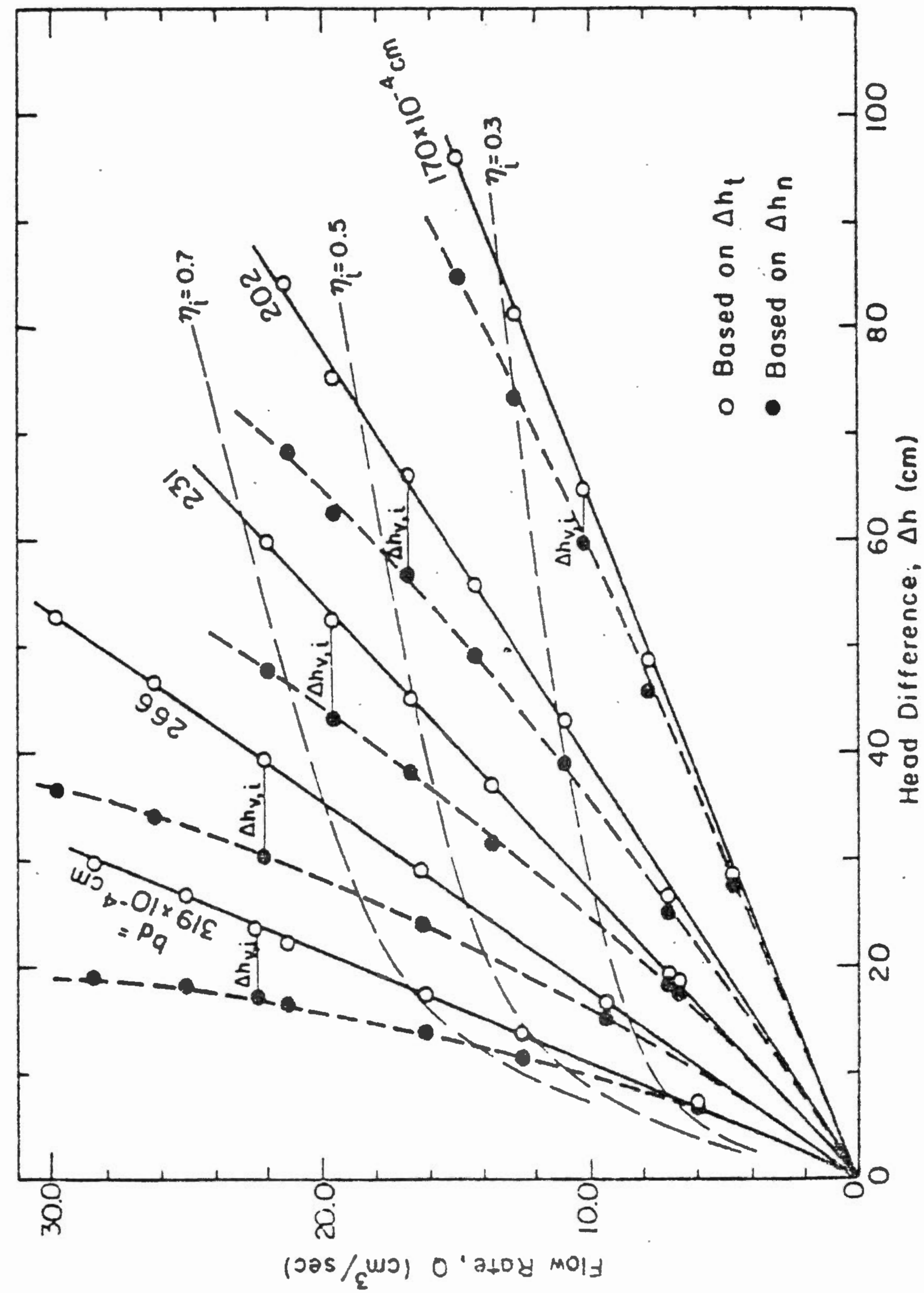
Iwai then takes the ratio of this inertia term to the head gradient and gets

$$\eta_i = \frac{\left| V_r \frac{\delta V_r}{\delta r} \right|}{\left| -g \frac{\delta h}{\delta r} \right|} = \frac{\rho\, Q\, e}{20\, \mu\, \pi\, r_i^{\,2}} \qquad (3\text{-}9)$$

which is shown plotted in figure 10. This graph is used to determine the limit of applicability of Darcy's assumption. On figure 10 Iwai also shows two sets of plots, one based on Δh_n and the second on Δh_t. The difference, $\Delta h_{v,i}$, is:

$$\Delta h_{v,i} = \frac{3\, Q^2}{20\, \pi^2\, e^2\, g} \frac{1}{r_i^{\,2}} \qquad (3\text{-}10)$$

Equation (3-10) is, of course, the same as equation (3-6). As can be easily



Flow Rate, Q (cm³/sec)
30.0
20.0
10.0
0
Head Difference, Δh (cm)
0
20
40
60
80
100
bd = 319x10-4 cm
266
231
202
170x10-4 cm
η_i = 0.7
η_i = 0.5
η_i = 0.3
Δh_v,i
o Based on Δh_t
● Based on Δh_n

Figure 10

seen on figure 10, for a constant flowrate and increasing head difference (and hence decreasing aperture) $\Delta h_{v,i}$ appears to increase slightly. More importantly, however, $\Delta h_{v,i}$ increases significantly for increasing flow-rate for a set fracture aperture.

Iwai notes that if Darcy's Law is valid the pressure should decrease linearly with the log of the radius. He shows two cases, one showing perfect linearity at the lower injection pressures but departing from linearity at higher pressures, while the second case for a larger aperture shows a rather curved profile except at the lowest injection pressures. The author states that when η_i becomes significant the velocity head around the injection hole begins to increase and the piezometric head therefore decreases. He concludes that the line $\eta_i = 0.5$ in figure 10 roughly borders the region where constant permeability can be assumed.

From equation (3-9),

$$\eta_i = \frac{\rho\, Q\, e}{20\, \mu\, \pi\, r_i^2} = \frac{Q\, e}{20\, \nu\, \pi\, r_i^2} \tag{3-11}$$

taking $\nu = 10^{-6}$

$$\eta_i = 1.6 \times 10^4 \frac{Q \cdot e}{r_i^2} \tag{3-12}$$

Hence for a given Q and e

$$\eta_i \propto \frac{1}{r_i^2} \tag{3-13}$$

as was concluded in equation (3-7) for the kinetic energy term. Hence for practical rock engineering purposes the effects of both the kinetic energy and inertial losses can be avoided by reading in an observation well which is sufficiently remote from the injection well. However, if nonlinear effects are seen, then the effect of kinetic energy and inertia should be evaluated prior to assuming turbulence.

Rissler (1978) takes a different approach in developing the radial flow laws. He based his development on the general law for losses in pipes of any cross-section. Introducing then Louis' law for one-dimensional flow with nonparallel walls he derives the flow law

$$\lambda = \frac{96}{Re} \left[1 + 8.8 \ (k/D_h)^{1.5} \right] \tag{3-14}$$

Rissler then, based on work by Wittke and Louis, where they showed that the velocity profile for laminar divergent radial flow varies only slightly from that for a corresponding one-dimensional flow - concluded that one-dimensional flow laws can be used. Ignoring kinetic energy and inertia effects he derived

$$H = H_o - \frac{6 \ \nu \left[1 + 8.8 \ (k/D_h)^{1.5} \right]}{g \cdot e^3} \cdot \frac{Q}{\pi} \ \ell n \ \frac{r}{r_o}$$

which for the case of smooth parallel walls ($k/D_h = 0$) reduces to

$$\Delta h = \frac{6 \ \nu \ Q}{g \cdot \pi \cdot e^3} \ \ell n \ \frac{r}{r_o} \tag{3-15}$$

where $\Delta h = H_o - H$.

Hence equations (3-15) and (3-1) are identical, proving that the approximation using one-dimensional flow laws appears valid.

3.2 Turbulent Flow

Baker (1955), Maini (1971) and Rissler (1978) all derive formulations for turbulent radial flow, while Iwai (1977) does not. Iwai does, however, recommend extension of the flow tests to the turbulent region to confirm if any of the existing flow equations are applicable.

Baker and Maini both base their turbulent flow derivations on Miessbach's law. Baker derives terms for both the frictional head loss and the kinetic energy loss in obtaining the following equation for the total pressure drop:

$$h_T = \frac{k\ Q^2}{4\ \pi^2\ e^3}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + \frac{Q^2}{8 \cdot g \cdot \pi^2\ e^2}\left(\frac{1}{R_2{}^2} - \frac{1}{R_1{}^2}\right) \qquad (3\text{-}16)$$

Maini, working from the same basis derives the following for the case of fully turbulent radial flow:

$$Q^2 = C \cdot 4 \cdot \pi^2\ e^2 \left[\frac{r_1 \cdot r_0}{r_1 - r_0}\right](h_1 - h_0) \qquad (3\text{-}17)$$

Maini ignores the kinetic energy term for turbulent flow.

Baker's equation changed to divergent flow conditions can be written as ($R_2 = r_0$)

$$\Delta h_T = \frac{k\ Q^2}{4\ \pi^2\ e^2}\left(\frac{1}{R_1} - \frac{1}{R_0}\right)$$

or

$$Q^2 = \frac{4\ \pi^2\ e^2}{k}\left(\frac{R_1\ R_0}{R_1 - R_0}\right)\Delta h_T \qquad (3\text{-}18)$$

Equating equations (3-17) and (3-18) gives

$$C \cdot 4\,\pi^2\,e^3\,\left(\frac{r_1 \cdot r_0}{r_1 - r_0}\right)\,\Delta h = \frac{4\,\pi^2\,e^2}{k}\,\left(\frac{R_1\,R_0}{R_1 - R_0}\right)\,\Delta h_T$$

or

$$C \cdot k = e \qquad (3\text{-}19)$$

Baker further realized that convergent flow through a fissure into a well, at low velocity will be streamlined. At higher velocities the flow will change from streamline to turbulent at some intermediate radius. Under these conditions the pressure drop will be given by

$$P = \frac{6\,\nu\,Q}{\pi\,g\,e^3}\,\ell n\,R_1/R + \frac{3\,Q^2}{20\,g\,\pi^2\,e^2}\,\left(\frac{1}{R^2} - \frac{1}{R_1{}^2}\right)$$

$$+ \frac{k\,Q^2}{4\,\pi^2\,e^3}\,\left(\frac{1}{R_2} - \frac{1}{R}\right) + \frac{Q^2}{8\,g\,\pi^2\,e^2}\,\left(\frac{1}{R_2{}^2} - \frac{1}{R^2}\right) \qquad (3\text{-}20)$$

This equation assumes that the flow will change instantly from streamline to fully turbulent,

Rissler (1978) notes that water pressure tests may be linear or nonlinear. Both overproportional and underproportional relationships have been observed where:

1) overproportional relations are generally due to expansion or cracking of the fracture

2) underproportional relations are generally due to turbulence.

In the literature, researchers have generally attempted to describe these curves by parabolic formulae such as

$$P = A\,Q + B\,Q^2 \qquad (3\text{-}21)$$

where P = the pressure in the borehole at the point of entrance

Q = flow rate

A & B = coefficients (with differing dimensions)

The coefficients A and B are empirically determined from tests, but usable conclusions about the causes of these relations (i.e. aperture, spacing, etc.) have not been made.

Rissler develops formulae for the energy head distribution for turbulent and transitional conditions. He derives all of these laws based on the general relation for energy losses in radially symmetrical flow given below:

$$-\frac{dH}{dr} = \lambda \cdot \frac{1}{D_h} \cdot \frac{\bar{v}^2}{2g} \tag{3-22}$$

Then using the continuity equation for flow from a borehole into a fracture and the appropriate frictional coefficient, λ, the basic differential equation can be set up. For the case of turbulent flow with completely rough conditions ($k/D_h > 0.032$) the following formulae can be derived

$$H = H_o - \frac{Q^2}{64 \cdot e^3 \cdot g \cdot \pi^2 \left(\log \frac{1.9}{k/D_h}\right)^2} \left(\frac{1}{r_o} - \frac{1}{r}\right) \tag{3-23}$$

describing the head loss in the turbulent flow section.

This equation is applicable to fully turbulent flow conditions and should therefore be comparable to Baker and Maini's formulations given in equations (3-16) and (3-17) (for $n = 2$, i.e. fully turbulent). Assuming that these various formulations describe the same phenomenon, it should then be possible to investigate Baker's empirical constant k as well as

Maini's constant 'C'. Hence, omitting the kinetic energy terms,

$$\frac{k}{4} = [64g(\log \frac{1.9}{k/D_h})^2]^{-1}$$

where $R_1 = r$, $R_2 = r_o$

and
$$k = \frac{1}{16\ g\ (\log \frac{1.9}{k/D_h})^2} \tag{3-24}$$

Similarly

$$\frac{1}{C \cdot 4} = \frac{1}{64\ g\ e\ (\log \frac{1.9}{k/D_h})^2}$$

$$C = 16 \cdot g \cdot e\ (\log \frac{1.9}{k/D_h})^2 \tag{3-25}$$

From equations (3-24) and (3-25) one concludes again that

$$C \cdot k = e$$

Hence,

Baker's constant $k = f\ (1/(k/D_h))$

Maini's constant $C = f\ (e, k/D_h)$ (3-26)

These two constants have been calculated and are plotted on figures 11 and 12.

For the case of full turbulence, but with $k/D_h \leq 0.032$, the friction

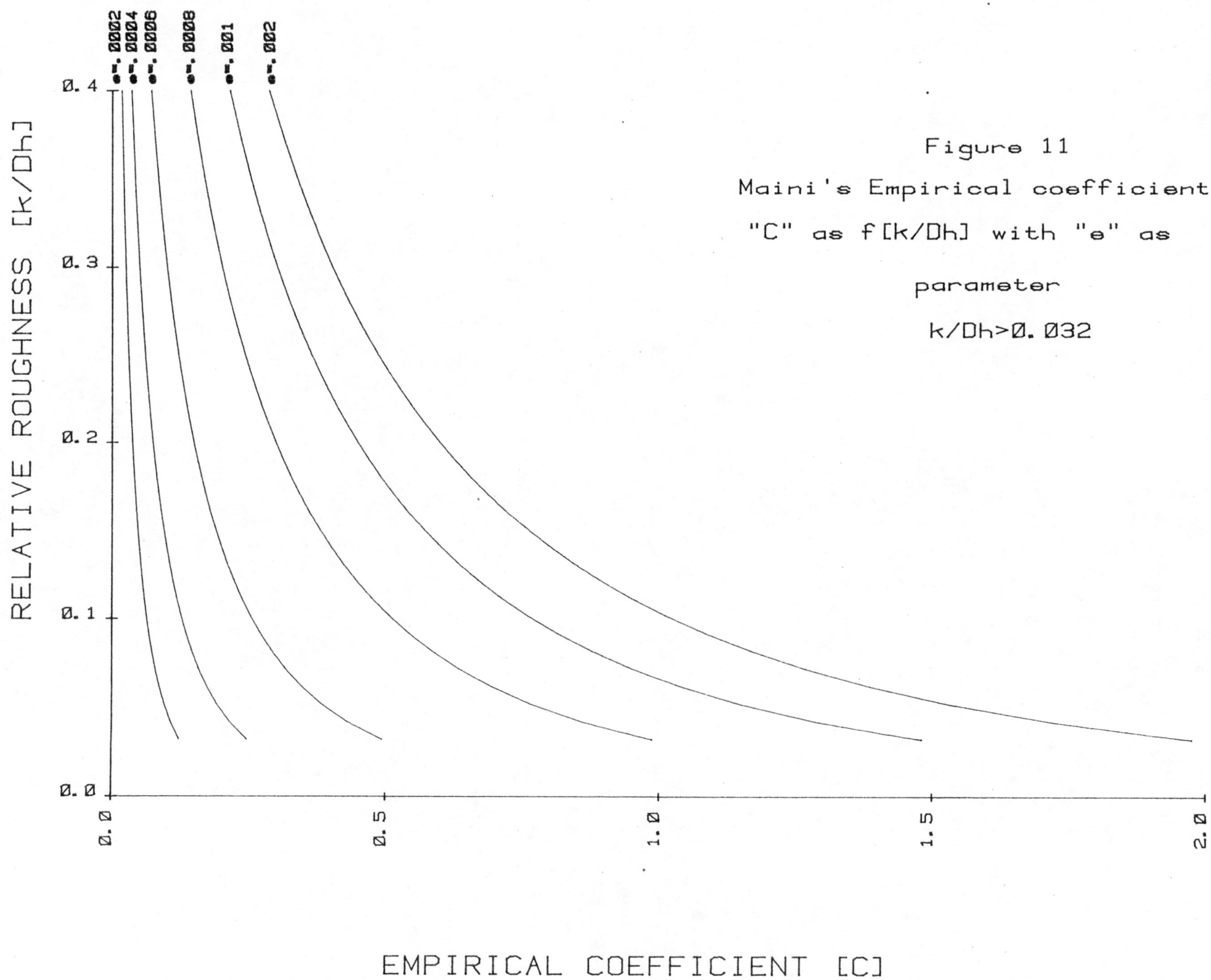
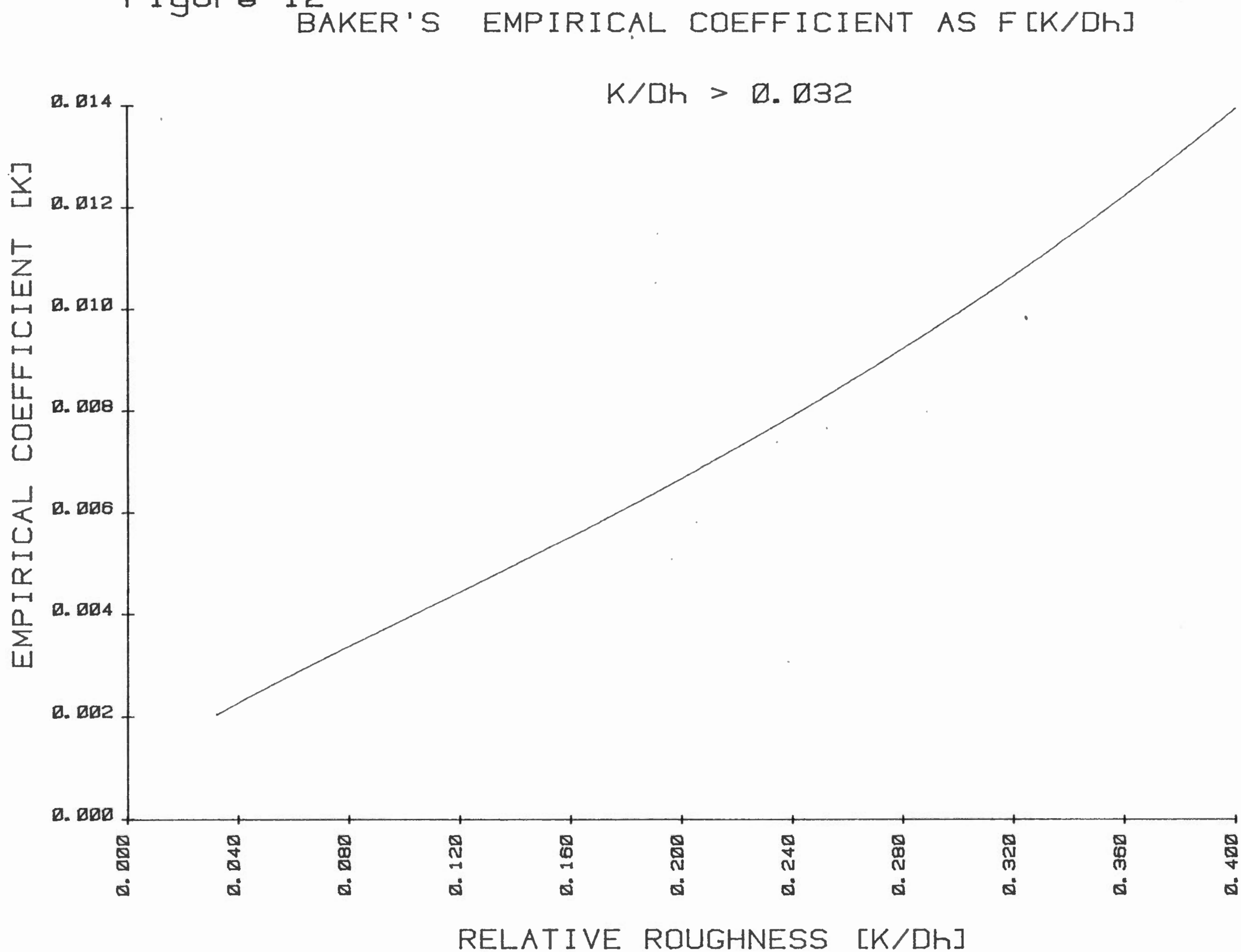


Figure 11

Maini's Empirical coefficient "C" as f[k/Dh] with "e" as parameter

k/Dh>0.032

Figure 12


BAKER'S EMPIRICAL COEFFICIENT AS F[K/Dh]
K/Dh > 0.032
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000
EMPIRICAL COEFFICIENT [K]
0.000
0.040
0.080
0.120
0.160
0.200
0.240
0.280
0.320
0.360
0.400
RELATIVE ROUGHNESS [K/Dh]

factor is given by equation (2-67)

$$\frac{1}{\sqrt{\lambda}} = - 2 \log \frac{k/D_h}{3.7} .$$

Hence C and K will vary from equations (3-24) and (3-25) only by a slight numerical factor. These values are shown graphically on figures 13 and 14. It should be immediately apparent from figures 10 through 14 that Baker and Maini's empirical constants both tend to increase the turbulent head loss with increasing roughness as would be expected. Furthermore, figures 10 and 12 show the very marked dependence of Maini's constant C on fracture aperture.

The previous comparison of turbulent flow laws ignored the kinetic energy losses since for most problems these can be considered negligible. The reason for this is that the kinetic energy head loss terms developed by Baker and Maini may be represented as

$$\Delta h_K \propto \frac{1}{R^2} \tag{3-27}$$

Hence as R increases away from the borehole, Δh_K goes to zero.

Rissler's formulation is based on equation (3-22), the general law for head losses in conduits of any cross-section, which includes a kinetic energy term, $(\bar{v}^2/2g)$. However, if the assumption that this term is negligible is really true, then the direct comparison of equations (3-16), (3-17) and (3-27) is still valid.

Rissler derives the extent of the turbulent zone for radially divergent

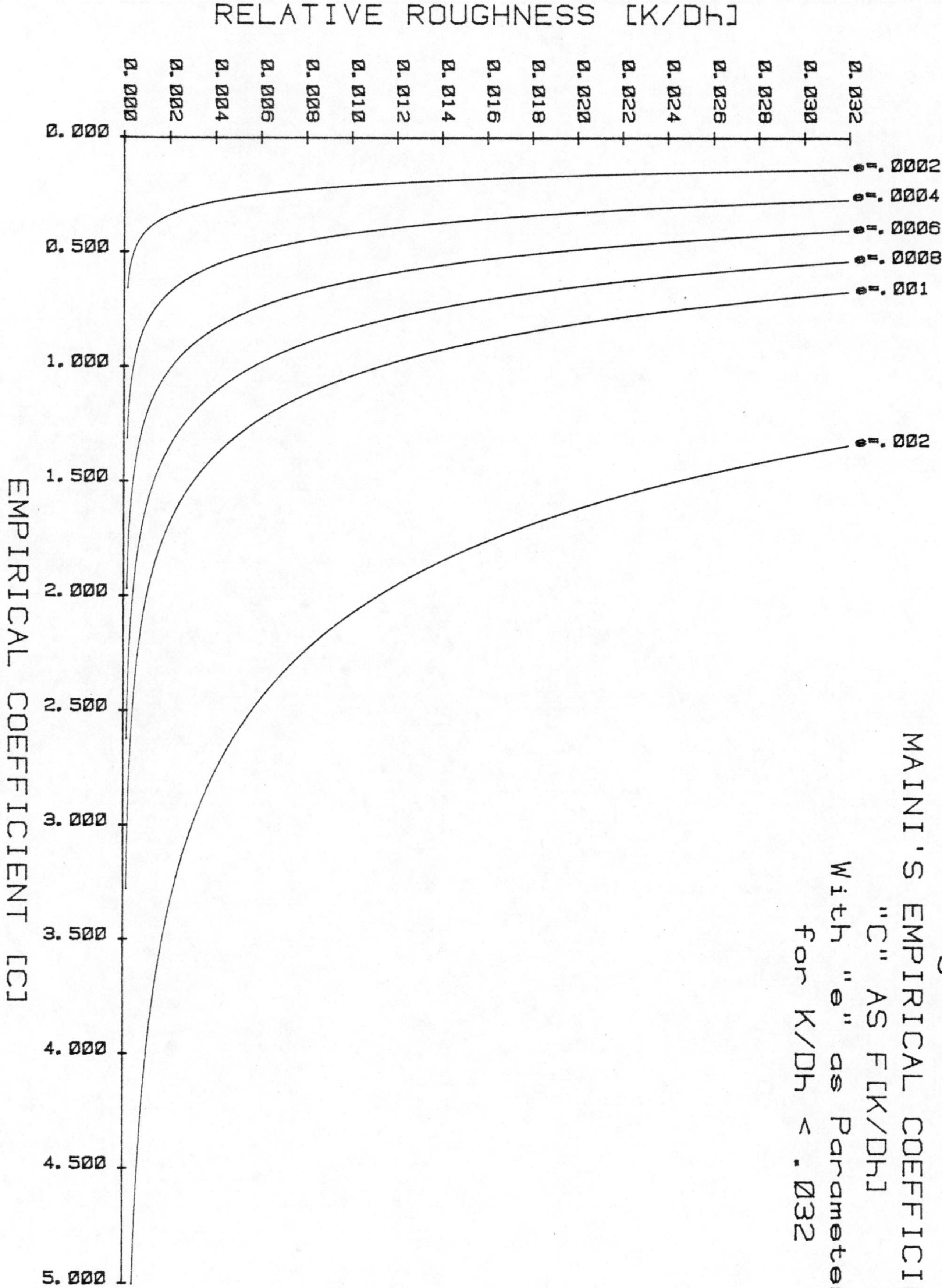
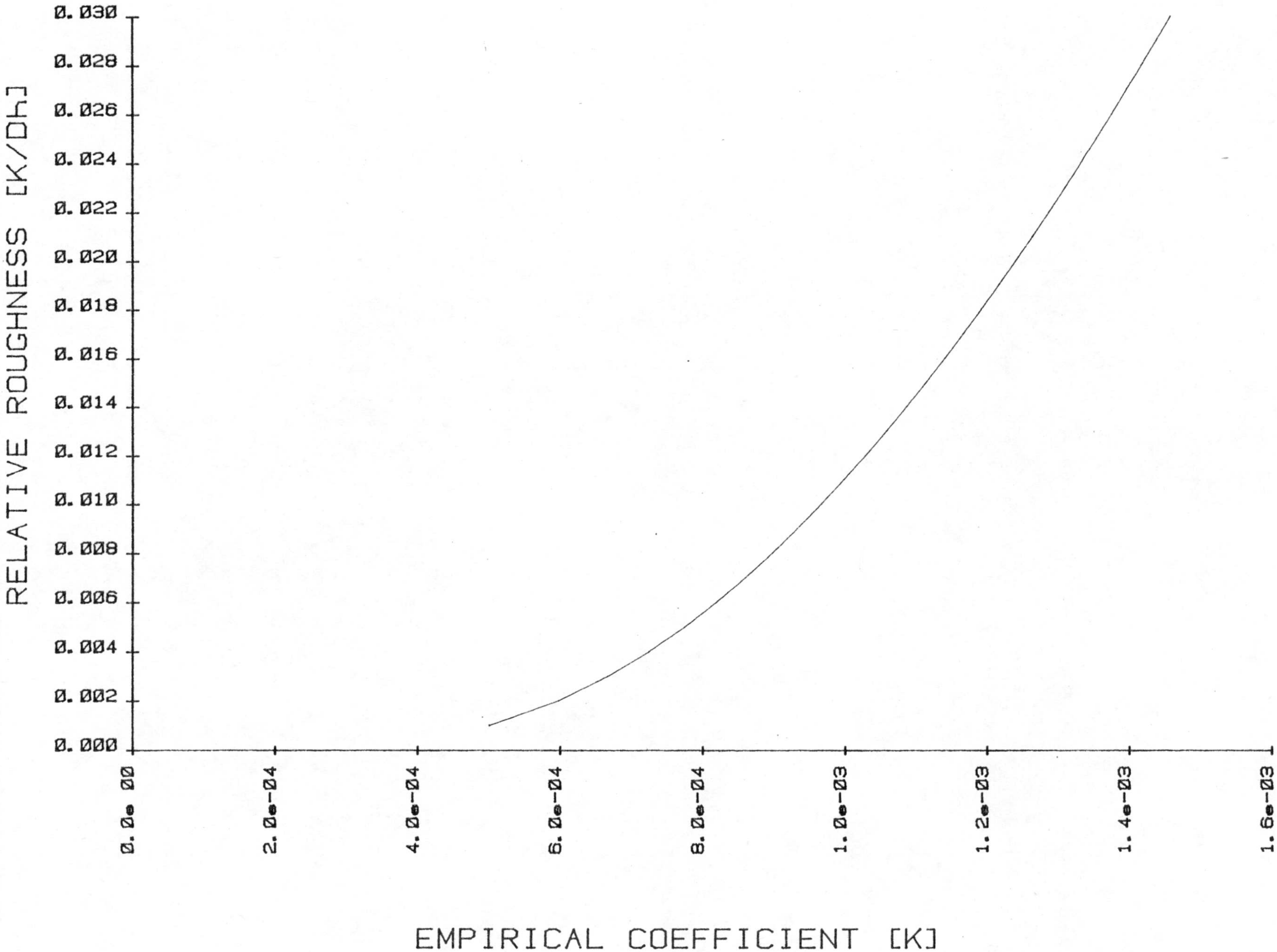


Figure 13
MAINI'S EMPIRICAL COEFFICIENT
"C" AS F[K/Dh]
With "e" as Parameter
for K/Dh < .032

Figure 4
BAKER'S EMPIRICAL COEFFICIENT AS F[K/Dh]
K/Dh < 0.032



flow for various roughness conditions. This is shown graphically on figure 15 as a function of flow rate, with relative roughness as a parameter. It is of interest to note the very sharp break in these curves at a flowrate of 10^{-2} m^3/sec. Up to this flowrate the extent of the turbulent zone never exceeds seven meters for any roughness, and for $k/D_h < 0.032$ does not exceed two meters. Hence if one wishes to ignore the kinetic energy term, then observations should be made at large values of R. However, if this is done, then quite probably any influence due to turbulent flow will not be observed. For flowrates that might be expected in normal rock engineering (5×10^{-3} m^3/sec or less), and especially for a single fracture, the effects of turbulence would most probably only be observed in measurements from the injection hole proper since observation wells would be at least 5 meters distant. In such conditions, if nonlinear phenomenon were observed in an observation well, then kinetic energy effects should be checked prior to attributing the effect to turbulence.

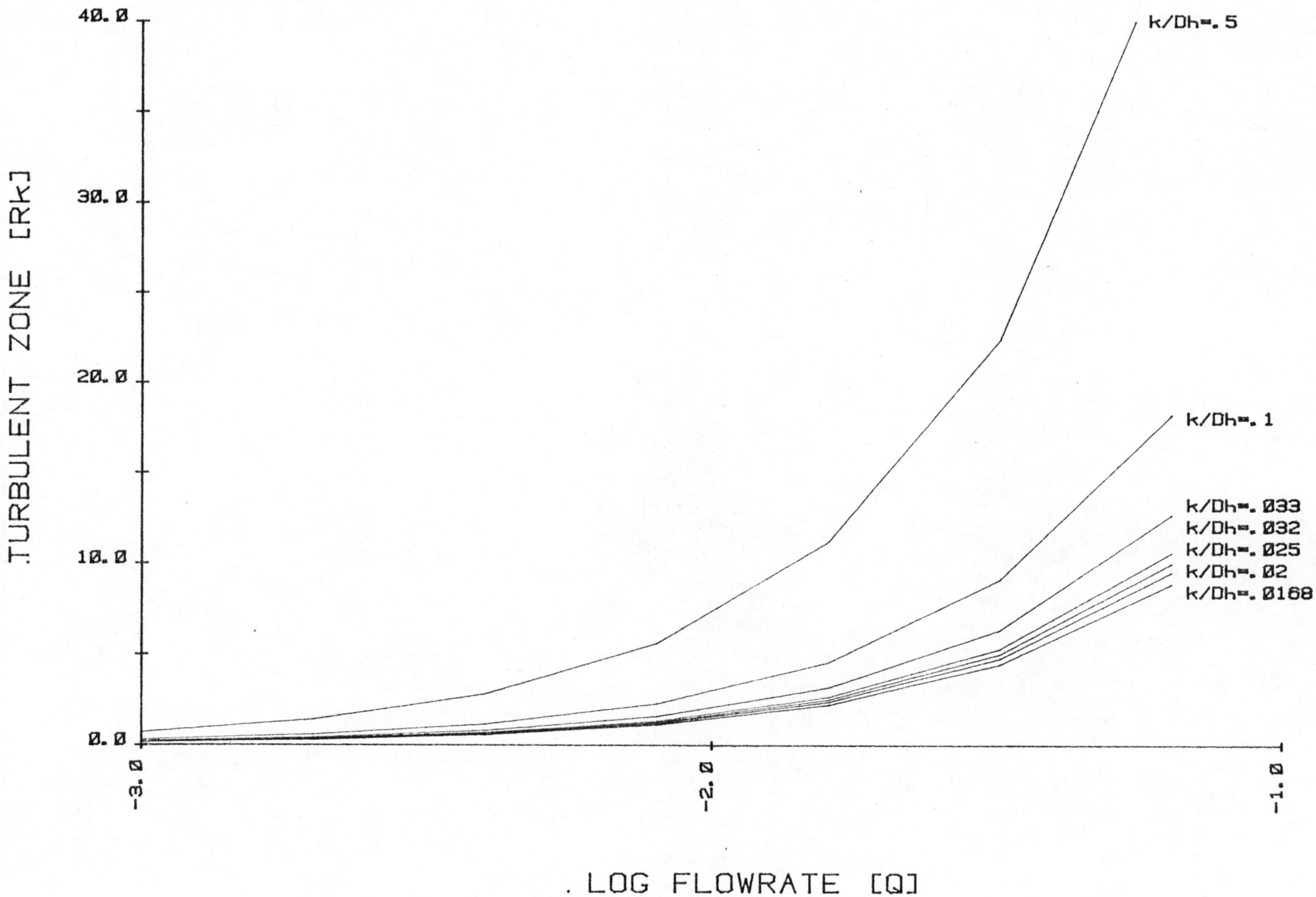
If we revaluate Baker's empirical coefficient k including the kinetic energy loss, then the full right-hand side of equation (3-16) ean be equated to (3-23), or

$$k = \frac{e}{2\,g}\left\{\frac{1}{8\,(\log\frac{1.9}{k/D_h})^2} - (\frac{1}{R_0{}^2} - \frac{1}{R_1{}^2})(\frac{R_0 - R_1}{R_0 \cdot R_1})\right\} \qquad (3\text{-}28)$$

It is obvious from this equation that if

$$R_1 >> R_0$$

Figure 15
EXTENT OF TURBULENT ZONE VERSUS FLOWRATE
with k/Dh as a parameter



then

$$\left(\frac{1}{R_0^2} - \frac{1}{R_1^2}\right)\left(\frac{R_0 - R_1}{R_0 \cdot R_1}\right) \to 0$$

and equation (3-28) becomes equal to equation (3-24).

Now, if R_1 is only slightly larger than R_0, then the right-hand side of the bracketed term in equation (3-28) will be a very small positive number,

i.e. if $R_1 = 2R_0$

then the above term $= + \frac{3}{8R_0^3}$

Hence the effect of the kinetic energy term on the empirical coefficients C and k can be practically neglected.

Returning to figure 14 one may conclude that in normal field packer tests it is very unlikely that turbulent effects will be picked up in observation wells. If such effects are to be monitored then very sensitive downhole pressure transducers must be employed to avoid errors involved in line losses, etc. This concept is fundamental to Rissler's thesis and may be extremely important to the proper interpretation of field permeability tests as discussed below.

The previous evaluation of the empirical constants C and k was based on the fully turbulent case, and the case of full turbulence and very rough walls for Rissler. Analogous calculations could be done for the other fully turbulent cases presented by Rissler (i.e. $k/D_h < 0.032$ and $k/D_h = 0$). Although the absolute values of C and k would change somewhat, the trends shown in figures 10 through 13 would not change. Rissler also presents the case with a transitional change from streamline to fully turbulent, however, this cannot be directly correlated to Baker and Maini since the value to assign n in Miessbach's law is unknown.

3.3 Evaluation of Hydraulic Fracture Parameters from Pump Tests

In his thesis, Rissler (1978) stated that standard field packer tests can be used to determine both fracture aperture e and relative roughness k/D_h, the two most important hydraulic fracture properties necessary to determine fracture conductivity. The basis of Rissler's theory rests on being able to determine the fracture aperture from the "critical energy head", that is, from the energy head at which laminar flow in the fissure changes to turbulent flow directly at the borehole wall.

In Chapter II of this report, Rissler's derivation for the aperture calculation were presented, concluding with equation (2-101) given again below.

$$e = \sqrt[3]{B \cdot C}$$

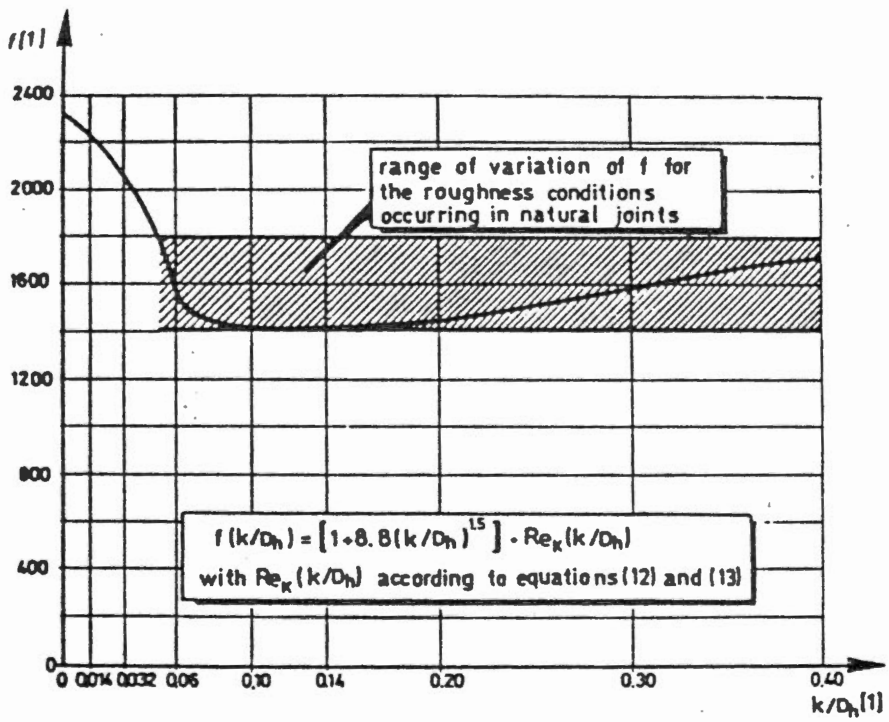
where $$B = \frac{6}{g \cdot H_{oK}} \cdot f(k/D_h)$$

$$C = \nu^2 \cdot r_o \cdot \ln (R/r_o)$$

Rissler evaluates the function $f(k/D_h)$ in B (shown in figure 16) and demonstrates that for the following range of roughness

$$0.06 < k/D_h < 0.4$$

the aperture determined - without consideration of k/D_h - introduces only an



$f(k/D_h)$ plotted versus k/D_h for $0 \le k/D_h \le 0.4$

(After Rissler 1978)

Figure 16

8 percent error in the most unfavourable case.

In order to ease the use of equations (2-99) through (2-101), Rissler evaluates these and presents the results graphically (figure 17). He assumed a value R = 100 m for these charts. Hense varying kinematic viscosities and borehole radii can be taken into account to determine the coefficient C. B is then determined, with a certain inaccuracy due to the influence of k/D_h, from the initial energy head H_{oK}. Having B and C the aperture can then be determined from the lower plot in figure 17.

The most important aspect for the applicability of the method is the determination of the critical energy head, H_{oK}, from the values H_o and Q measured in the field test. It is evident that this can only be achieved when the function $H_o = f(Q)$ is determined with sufficient accuracy. Rissler recommends that for evaluation, the test results be represented either as

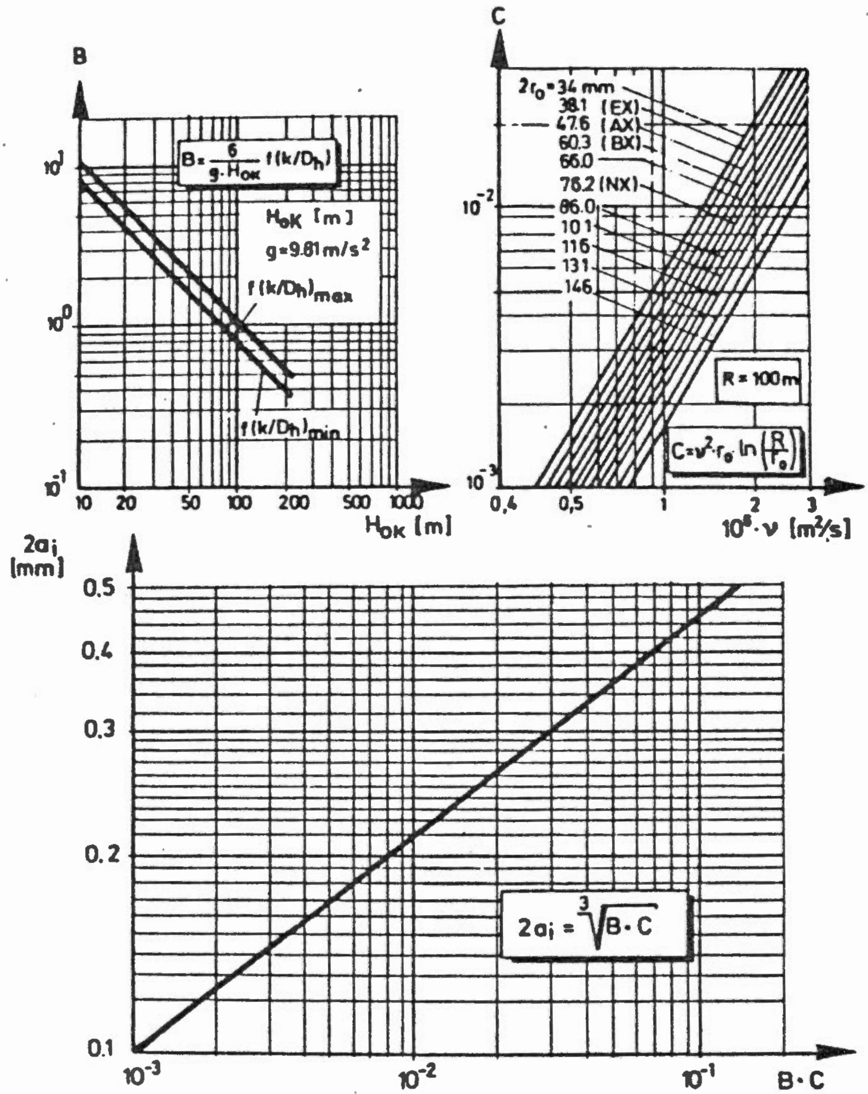
$$\frac{H_o}{Q} = f(H_o)$$

or

$$\frac{Q}{H_o} = f(H_o) \tag{3-29}$$

In both of these cases a horizontal line for the laminar range and a clean break on reaching the critical energy head occurs as shown in figure 18.

Now, with the aperture assumed known, Rissler next determines the actual relative roughness. This can either be done physically on core samples, or by substituting any value pair of H_o and Q measured in a test into equations



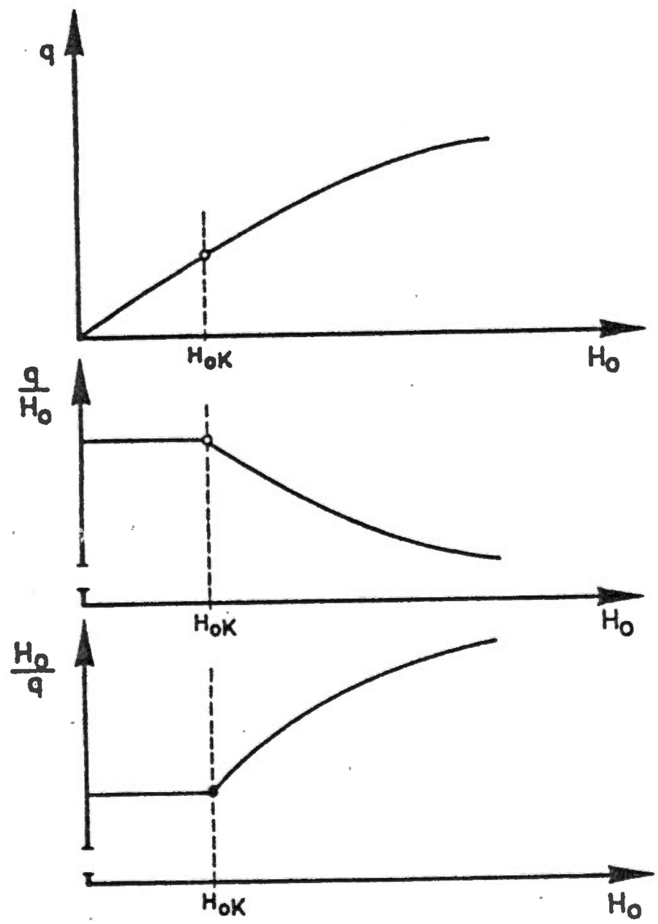
Determination of the aperture $2a_i$ from the critical energy head H_{oK}

(After Rissler 1978)

Figure 17

(2-87) and (2-94) through (2-96), solving for (k/D_h). A graphical solution is presented in figures 19, 20 and 21.

Hence using Rissler's assumptions and ensuring that very careful field measurements are taken, one can determine the two initial hydraulic parameters, e and k/D_h, from a field pump test. These parameters can then be used to determine the hydraulic conductivity of the fracture which can be employed in numerical models for simulation of full-scale engineering projects.



Representation of the test results for determination of the critical energy head H_{oK}

(After Rissler 1978)

Figure 18

Figure 19

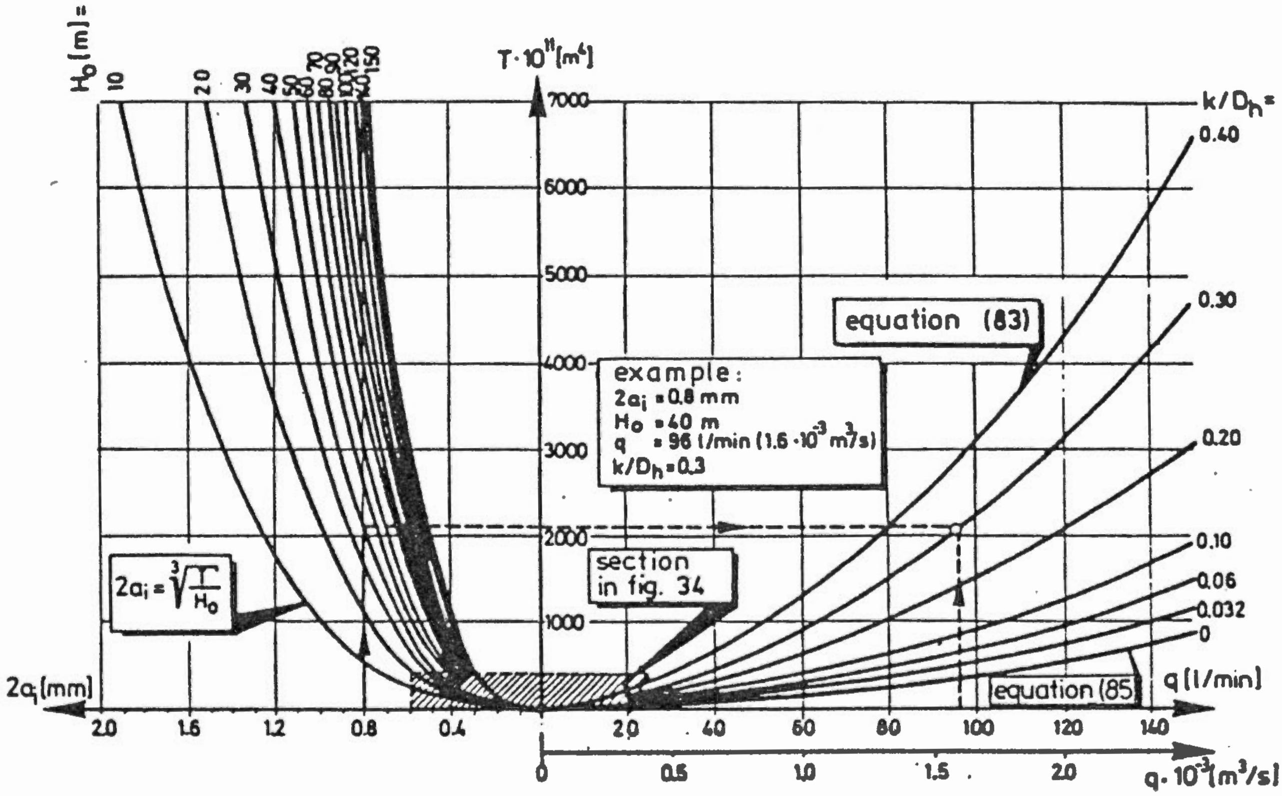


Diagram for determining the relative roughness k/D_h for large apertures $2a_i$

(After Rissler 1978)

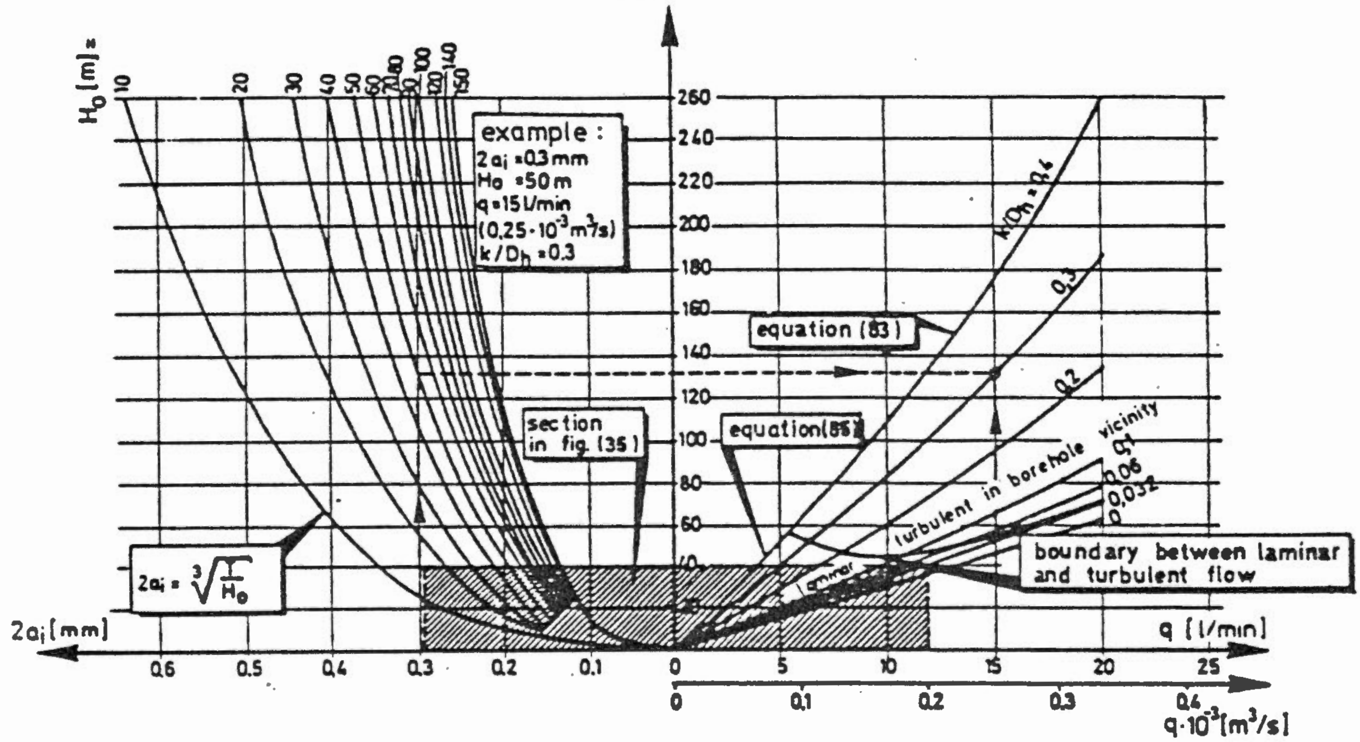


Figure 20 Diagram for determining the relative roughness k/D_h for medium apertures $2a_i$

(After Rissler 1978)

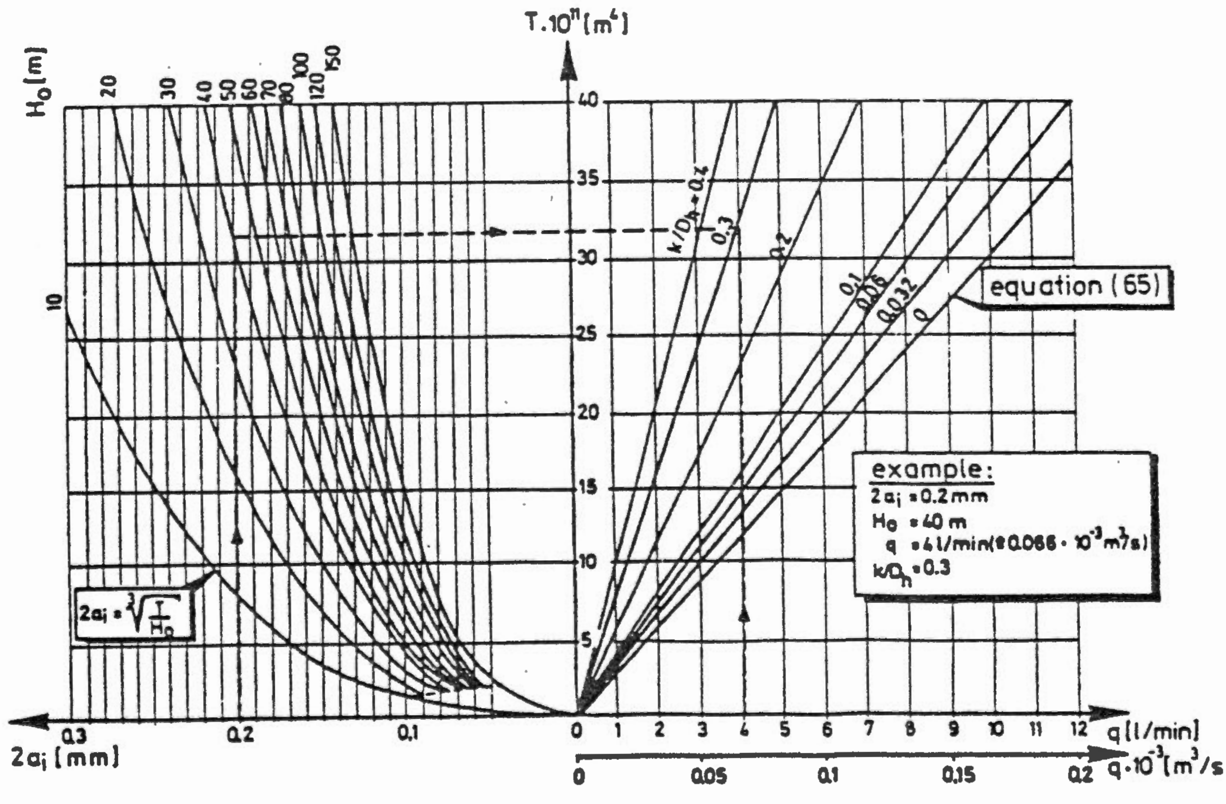


Figure 21 Diagram for determining the relative roughness k/D_h for small apertures $2a_i$ (After Rissler, 197

CHAPTER IV

DISCUSSION AND CONCLUSIONS

a) Streamline (laminar) radially divergent flow can be analysed using the following relation:

$$Q = \frac{\pi \cdot g \cdot e^3}{6\,\nu\,[1 + 8.8\,(k/D_h)^{1.5}]}\;\frac{H_o - H}{\ell n\, r/r_o} \qquad (4\text{-}1)$$

For the case of perfectly smooth parallel plates the previous equation reduces to

$$Q = \frac{\pi \cdot g \cdot e^3}{6\,\nu}\;\frac{H_o - H}{\ell n\, r/r_o}$$

which can be derived from either the Navier-Stokes' equation or the general law of friction losses in pipes of any cross-sectional area.

b) In radial flow both kinetic energy and inertial effects may, under certain conditions, influence the results. For most practical cases these effects can be ignored.

Iwai (1976) claims, however, that results will depart from the Darcy approximation when the ratio of inertia to head gradient exceeds 0.5. This ratio is given as

$$\eta_i = \frac{\rho\, Q\, e}{20\,\mu\,\pi\, r_i^{\,2}} \qquad (4\text{-}2)$$

It is of interest to note that Baker and Maini base their developments on the Navier-Stokes' equation for one-dimensional flow. Then substituting ($- \frac{dH}{dr}$) for the gradient and using the continuity equation for flow from a borehole into a fracture they transpose to radial flow and calculate the average velocity in the fracture. Substituting this into Bernoulli's equation, they derive the kinetic energy head loss.

Iwai, however, begins by writing the Navier-Stokes' equation in cylindrical coordinates, which include a truly inertial term $V_r(\delta V_r/\delta r)$, and develops his model from there. In order to solve this, however, he first ignores the inertia term, then having calculated V_r he resubstitutes to determine the inertia term. It is not certain what error may be involved in this process but he is the only author to deal with this.

Rissler develops all of his theory from the basic law for friction losses in pipes of any cross-section. This formula inherently includes the kinetic energy term of Bernoulli but does not include any inertia effects. It further assumes that the basic formula for pipes remains applicable for essentially an infinite fracture.

As noted in (a), all of the authors derive the same relation for streamline radial flow and hence one may assume that kinetic energy and inertia effects are of secondary importance. Since both of these factors are proportional to $\left(\frac{1}{R}\right)^2$, then this would seem true as long as measurements are taken at some distance from the borehole.

c) Turbulent flow is dismissed by most authors as being insignificant for practical rock engineering. Although this may be true in the analysis of most large-scale rock engineering projects where the flow can generally be modelled as one-dimensional, in packer tests where radial flow occurs, turbulence near the borehole may have considerable significance.

In analysing radial turbulent flow both Baker (1955) and Maini (1971) base their theory on the Miessbach law and derive equations of the form

$$P = A\,Q + B\,Q^2 \qquad (4\text{-}3)$$

which satisfy the parabolic form of the H-Q diagram. However, the coefficients A and B are empirical and do not supply conclusions as to the underlying causes of these relations (i.e. aperture, spacing, etc.). Hence such formulations are of limited use.

Rissler (1978), based on work by Wittke and Louis showing the velocity profile for one-dimensional and radial flow conditions do not differ significantly, derives turbulent radial flow laws using the one-dimensional flow laws of Louis et al. Rissler derives, using this method, the same law for streamline radial flow as previous authors, but for turbulent flow he derives laws that are not empirically based. Thus Rissler's laws may be used for numerical parametric studies to determine the results of various parameters on packer test results, thus aiding in developing a sound understanding of phenomenon observed in the field.

d) Comparison of empirical turbulent laws with those of Rissler, (for the case of fully turbulent flow), show that the empirical constants from Baker and Maini are functions of e and k/D_h, the two most fundamental hydraulic fracture parameters. It is then obvious that these values will be strictly test dependent.

e) Rissler's thesis is critical in that it allows determination of both e and k/D_h from standard field packer tests. Accurate determination of these key in-situ hydraulic parameters has previously been one of the major stumbling blocks to the advancement of fracture flow analysis.

One criticism of Rissler's work is that he does not quantify the effect of possible fracture deformation on his results. Roegiers et al. (1979) demonstrate that for one-dimensional flow in fine smooth fractures (k/D_h < 0.033) very high gradients are required to develop turbulence in one-dimensional flow. If similar conditions exist in radial turbulent flow then fracture deformation may be important since the aperture calculated using Rissler's technique could be largely a function of the test rather than the truly in-situ case. It is the author's belief that this phenomenon deserves further experimental study.

f) Rissler demonstrates, using a numerical study, that energy losses in the immediate vicinity of the borehole may be quite considerable. He shows, for example, that at 25 cm from the borehole wall for a fracture with k/D_h = 0.4 the energy head is only 20 percent of the head H_o at the borehole wall. For strictly laminar flow the energy head would be 66 percent of H_o. These figures are in basic agreement with similar results presented in the literature.

The importance of these results is to point out that packer tests are indicative of conditions in the immediate vicinity of the hole and must not be considered as a large-scale test. Hence a large number of field tests will be required in order to evaluate the statistical property bounds for each fracture set of interest.

g) Finally, there appears to be some discrepancy in the literature concerning the critical Reynolds' number for the initiation of turbulence.

Baker (1955) claims that for his experiments this value ranges from 4,000 to 8,000. Iwai (1976) claimed, however, that his results varied from theory if Re > 100.

The value quoted in the literature as critical for one-dimensional flow is Re = 2300 (Louis 1969). Hence both values quoted for radial flow are vastly different than for one-dimensional flow. For the two authors in question Baker used an artificial fracture and apertures from 0.127 cm to 1.8 cm. Iwai, however, used tension fractures created in rock samples with apertures ranging from 0 to o.025 cm. Thus it is difficult to discern whether the results from these two authors are comparable.

The Reynolds' number for fracture flow is defined as

$$Re = \frac{D_h \bar{v}}{\nu} \qquad (4\text{-}4)$$

where D_h = hydraulic diameter = 2e

$\bar{v}$ = average velocity

ν = kinematic viscosity

Many authors have commented, however, on the difficulty of defining a true Reynolds' number for fracture flow since the cross-sectional flow area may vary so radically from one location to another. Considering the importance of this factor in radial flow, however, especially if Rissler's theorems are to be used, the author believes that further research should be devoted to this topic.

CHAPTER V

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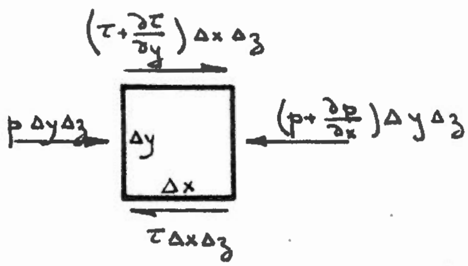
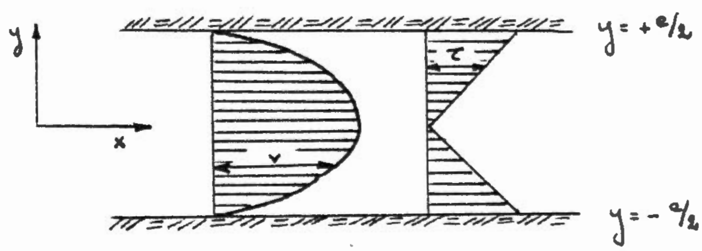
APPENDIX A

EFFECT OF INERTIA ON LINEAR FLOW SYSTEM

(after Maini and Baker)

Consider a free body of length Δx, width Δz, and thickness Δy as shown below.





Equilibrium considerations give

$$P\Delta y\Delta z - \left(P + \frac{dP}{dx}\Delta x\right)\Delta y\Delta z - \tau\Delta x\Delta z$$

$$+ \left(\tau + \frac{d\tau}{dy}\Delta y\right)\Delta x\Delta z = 0 \qquad \text{A-1}$$

which reduces to

$$\frac{dP}{dx}\Delta x\Delta y\Delta z = \frac{d\tau}{y}\Delta x\Delta y\Delta z \qquad \text{A-2}$$

or,

$$\frac{dP}{dx} = \frac{d\tau}{dy} \qquad \text{A-3}$$

Equation A-3 shows that, in the absence of inertia forces, the pressure variation in the direction of flow between parallel plates is equal to the variation in shear in a direction perpendicular to the flow.

Now, integrating equation A-3 with respect to y gives

$$\tau = \frac{dP}{dx}(y+c) \qquad \text{A-4}$$

Now solve for integration constant 'c' using the following boundary condition, $\tau = 0$ @ $y = 0$

$\therefore \quad c = 0$

$$\tau = \frac{dP}{dx}\, y \qquad \text{A-5}$$

Now, using Newton's Shear Law $\tau = \mu \frac{dv}{dy}$

$$\therefore \quad \frac{dv}{dy} = \frac{1}{\mu}\frac{dP}{dx}\, y \qquad \text{A-6}$$

$$v = \frac{1}{\mu}\frac{dP}{dx}\left(\frac{y^2}{2} + c\right) \qquad \text{A-7}$$

Solving for 'c', knowing the following boundary condition

$v = o$ @ $y = \pm\, e/2$

$\therefore \quad c = -\, e^2/8$

or

$$v = \frac{1}{\mu}\frac{dP}{dx}\left(\frac{y^2}{2} - \frac{e^2}{8}\right) \qquad \text{A-8}$$

Now, to obtain the mean square of velocities between fissure faces (i.e. $y = e/2$, $y = -e/2$)

$$(v_m^2) = \frac{1}{e}\int_{-e/2}^{+e/2} \underline{(y^2/2 - e^2/8)}^2 \,.\, dy \,.\, \frac{1}{\mu^2} \,.\, \left(\frac{dP}{dx}\right)^2$$

$$(v_m^2) = \frac{1}{\mu^2}\left(\frac{dP}{dx}\right)^2 \cdot \frac{1}{e}\int_{-e/2}^{e/2}\left(y^4/4 - \frac{y^2e^2}{8} + \frac{e^4}{64}\right)dy$$

$$= \frac{1}{\mu^2}\left(\frac{dP}{dx}\right)^2 \cdot \frac{1}{e}\left|\frac{y^5}{20} - \frac{e^2}{8}\frac{y^3}{3} + \frac{e^4}{64}y\right|_{-e/2}^{e/2}$$

$$= \frac{1}{\mu^2}\left(\frac{dP}{dx}\right)^2\left(\frac{6e^4}{320} - \frac{e^4}{96}\right) = \frac{1}{\mu^2}\left(\frac{dP}{dx}\right)^2\left(\frac{256e^4}{30720}\right)$$

$$\boxed{(v_m^2) = \frac{1}{\mu^2}\left(\frac{dP}{dx}\right)^2\frac{e^4}{120}} \qquad \text{A-9}$$

Equation A-9 matches Baker's equation for the mean velocity.

APPENDIX B

RADIAL FLOW FORMULATION

(after Baker, 1955)

B.1 GENERALITIES

Assumptions

- fissure of uniform size
- horizontal plane
- small aperture compared to width

Baker uses the following equation for steady state flow in self consistent units;

$$P_f = \frac{L \rho V_m{}^2}{T} \phi \left(\frac{\eta}{V_m T \rho}\right) \qquad \text{B-1}$$

where

P_f = pressure in psi = $h \gamma$

ρ = density in pcf = γ

$\eta = \mu = \nu \rho_m$

$\gamma = g \cdot \rho_m$

ρ_m = mass density

T = aperture

- For laminar flow

$$\phi \left(\frac{\eta}{V_m T \rho}\right) = k \left(\frac{\eta}{V_m T \rho}\right) \quad \text{where } k = 12$$

$$\therefore \quad P_{fs} = \frac{L V_m}{T} \cdot 12 \left(\frac{\eta}{T}\right)$$

$$P_{fs} = \frac{12 L \eta V_m}{T^2} \qquad \text{B-2}$$

$$\therefore \quad h \cdot g = \frac{12\, L\, \nu\, V_m}{T^2}$$

$$\frac{h_{fs}}{L} = \frac{12\, \nu}{g\, T^2} V_m \qquad \text{B-3}$$

<u>or</u>

$$P_{fs} = \frac{12\, \mu\, L\, Q}{W\, T^3} \qquad \text{B-4}$$

where $\begin{cases} Q = \text{volume flow rate} \\ W = \text{fracture width} \end{cases}$

- For Turbulent Flow Baker assumes

$$P_{ft} \propto V_m^n$$

such that $\phi\left(\frac{\eta}{V_m T \rho}\right) = k\left(\frac{\eta}{V_m T \rho}\right)^{2-n}$

where k and n are experimental constants.

$$\therefore \quad P_{ft} = \frac{k\, L\, \rho\, V_m^2}{T}\left(\frac{\eta}{V_m T \rho}\right)^{2-n} \qquad \text{B-5}$$

From analogy with pipeflow, n = 2 for full turbulence and equation B-5 becomes

$$\boxed{P_{ft} = \frac{k\, L\, \rho\, V_m^2}{T}} \qquad \text{B-6}$$

or $$\frac{h_{ft}}{L} = \frac{k\ V_m^2}{T}$$

$$\boxed{V_m^2 = \frac{T}{k}\ \frac{h_{ft}}{L}} \qquad \text{B-7}$$

Miessback's Law for turbulence is

$$V_m^n = C\ \frac{\partial P}{\partial r}$$

B.2 CONVERGENT RADIAL FLOW

Baker gives the following initial formula.

$$dp_{ft} = \frac{12\ \rho\ Q\ dr}{2\pi r\ T^3} \qquad \text{B-8}$$

But, after Maini (using Polar Coordinates), Darcy's Law applied to parallel plates is given by

$$V = -K_j\ \frac{\partial P}{\partial r} \qquad \text{B-9}$$

Flow from the cavity into the fissure is

$$Q = 2\pi re\ \overline{V} \qquad \text{B-10}$$

Substituting into equation B-9 leads to:

$$\frac{Q}{2\pi re} = -K_j \frac{\partial P}{\partial r} \text{ or } dP = -\frac{12\ \nu\ Q}{2\pi r\ ge^3}\ dr \qquad \text{B-11}$$

$$\int_{r_o}^{r_1} \frac{dr}{r} = -\int_{P_o}^{P_1} \frac{2\pi e}{Q} \cdot K_j \cdot dP$$

$$\text{or} \quad \ell n\ r_1/r_o = \frac{2\pi e}{Q}\ K_j\ (P_o - P_1)$$

$$= \frac{2\pi e}{Q}\frac{ge^2}{12\nu}(P_o - P_1) = \frac{\pi\ g\ e^3}{6\ \nu\ Q}(P_o - P_1) \qquad \text{B-12}$$

In Maini's case P = h

Knowing that $g = \gamma/\rho_m$ and $\mu = \nu\rho_m$, equation B-11 can be written as:

$$dP = \frac{12\ \nu\ Q}{2\pi r\ \gamma/\rho_m\ e^3}\ dr = \frac{12\ \mu\ Q}{2\pi r\ \gamma e^3}\ dr \qquad \text{B-13}$$

An equation which can be identified to B-8.

Substituting $P_{ft} = h_{ft}\gamma$ and η for ρ in B-8 gives

$$dh_{ft}\gamma = \frac{12\ \eta\ Q}{2\pi r\ \ T^3}\ dr = \frac{12\ \nu\ \rho_m\ Q}{2\pi r\ \ T^3}\ dr \quad \therefore\ dh_{ft} = \frac{12\ \nu\ \ Q}{2\pi r\ g\ T^3}\ dr \qquad \text{B-14}$$

Equations (B-14) and (B-11) are identical, except that P = h.

Therefore, equation (B-8) should be written

$$dP_{ft} = \frac{12\ \eta\ Q}{2\pi r\ T^3}\ dr$$

Integrating this last equation between the outer radius and the inner radius leads to:

$$P_{fs} = \frac{6\ \eta\ Q}{\pi\ T^3}\ \ell n\ R_1/R_2 \qquad \text{B.15}$$

or

$$\boxed{h_{fs} = \frac{6\ \nu\ Q}{\pi\ g\ T^3}\ \ell n\ R_1/R_2} \qquad \text{B-16}$$

Hence equations B-16 and B-12 are identical and for Maini, P is the fluid head not pressure.

For turbulent flow (i.e. assuming n = 2), one obtains.

$$dP_{ft} = \frac{k\ \rho\ Q^2}{4\pi^2\ r^2\ T^3}\ dr \qquad \text{B-17}$$

Integrating

$$P_{ft} = \frac{k\ \rho\ Q^2}{4\pi^2\ T^3}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \qquad \text{B-18}$$

or

$$\boxed{h_{ft} = \frac{k\ Q^2}{4\pi^2\ T^3}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)} \qquad \text{B-19}$$

It should be pointed out that equations B-16 and B-17 consider friction losses only. For radial flow into a well kinetic losses also important due to acceleration at the wellbore wall. Calculation of kinetic energy losses requires the knowledge of the velocity distribution in the fissure It can be computed for streamline but not for turbulent conditions.

Baker assumes this distribution curve to be flat; i.e. at any given radius the velocity is constant at all points between fissure faces and equal to the mean flow velocity.

$\therefore$ <u>For streamline flow</u>: velocity v at distance t from the axial plane of the fissure is given by

$$v = \frac{P_{fs}}{\eta\, L} \left(\frac{T^2}{8} - \frac{t^2}{2}\right) \qquad \text{B-20}$$

or $$v = \frac{h_{fs}\, g}{\nu\, L} \left(\frac{T^2}{8} - \frac{t^2}{2}\right) \qquad \text{B-21}$$

The mean square of velocities between fissure faces, is given by:

$$(v^2)_m = \frac{P_{fs}^2\ T^4}{120\ \eta^2\ L^2} \qquad \text{B-22}$$

or $$= \frac{h_{fs}^2\ g^2\ T^4}{240\ \nu^2\ L^2} \qquad \text{B-23}$$

Head drop due to kinetic energy is, (after Baker)

$$P_{ks} = \frac{1}{2}\,\rho\,(v^2)\,m = \frac{\rho\ P_{fs}^2\ T^4}{240\ \eta^2\ L^2} \qquad \text{B-24}$$

However, from Bernouilli's equation,

$$\frac{v_1^2}{2g} + \frac{P_1}{\gamma} + z_1 = \frac{v_2^2}{2g} + \frac{P_2}{\gamma} + z_2 = \text{constant} \qquad \text{B-25}$$

or $$\frac{v_1^2}{2g} + h_1 = \frac{v_2^2}{2g} + h_2 = \text{constant}$$

for gas one can neglect the elevation head z

$$\therefore \quad \frac{1}{2}\rho_1 v_1^2 = \frac{1}{2}\rho_2 v_2^2$$

Hence Baker used the above equation which is only valid for gas flow.

For fluid flow

$$P_{ks} = \frac{v^2}{2g} = \frac{1}{2g}\,\frac{h_{fs}^2\, g\, T^4}{120\, \nu^2\, L^2}$$

$$P_{ks} = \frac{h_{fs}^2\, g\, T^4}{240\, \nu^2\, L^2} \qquad \text{B-26}$$

Now substituting for h_{fs} from equation B-3

$$\boxed{P_{ks} = \frac{3\, v_m^2}{5g}} \qquad \text{B-27}$$

For radial flow conditions, using equation B-10

$$v_m^2 = \frac{Q^2}{4\pi^2 r^2 T^2} \qquad \text{B-28}$$

Substituting into equation B-27

$$\boxed{P_{ks} = \frac{3\,Q^2}{20\,g\,\pi^2\,T^2}\left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right)} \qquad \text{B-29}$$

note P_{ks} is in L^n units.

<u>For Turbulent Flow</u>, assuming $(v^2)_m = v_m^2$ we may write

$$v_m = \frac{Q}{2\pi r T} \qquad \text{B-30}$$

$$\boxed{P_{kT} = \frac{v^2}{2g} = \frac{Q^2}{8\,g\,\pi^2\,T^2}\left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right)}$$

Hence the <u>total head drop for streamline radial flow</u> is given by summing the expressions given by equations B-16 and B-29.

$$\boxed{h_s = \frac{6\,\nu\,Q}{\pi\,g\,T^3}\,\ell n\,R_1/R_2 + \frac{3Q^2}{20\,g\,\pi^2\,T^2}\left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right)} \qquad \text{B-31}$$

Also, <u>the total head drop for radial turbulent flow</u> is given by summing the expressions given by equations B-19 and B-32

$$h_T = \frac{k\,Q^2}{4\,\pi^2\,T^3}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + \frac{Q^2}{8\,g\,\pi^2\,T^2}\left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right) \qquad \text{B-32}$$

At higher velocities flow changes from streamline to turbulent at some radius 'R' intermediate between R_1 and R_2 and under these conditions head drop 'P' will be given by

$$P = \frac{6\,\nu\,Q}{\pi\,g\,T^3}\,\ell n\,R_1/R + \frac{3\,Q^2}{20\,g\,\pi^2\,T^2}\left(\frac{1}{R^2} - \frac{1}{R_1^2}\right)$$

$$+ \frac{k\,Q^2}{4\,\pi^2\,T^3}\left(\frac{1}{R_2} - \frac{1}{R}\right) + \frac{Q^2}{8\,g\,\pi^2\,T^2}\left(\frac{1}{R_2^2} - \frac{1}{R^2}\right) \qquad \text{B-33}$$

APPENDIX C

RADIAL FLOW FORMULATION

(after Maini, 1971)

C.1 DERIVATION

Starting with equation A-9 (see Appendix A)

$$v_m^2 = \frac{1}{\mu^2} \left(\frac{dP}{dx}\right)^2 \frac{e^4}{120}$$

where $\begin{cases} P = \text{pressure} = h \cdot \gamma \\ \mu = \text{absolute viscosity} = \nu \cdot \rho_m \end{cases}$

$$\therefore \quad v_m^2 = \frac{g^2 \, e^4}{120 \, \nu^2} \left(\frac{dh}{dx}\right)^2 \qquad \text{C-1}$$

Maini gives the following

$$v = \frac{(P_1 - P_2) \, g}{\mu \, L} \left\{ \frac{e^2}{8} - \frac{E^2}{2} \right\} \qquad \text{C-2}$$

where μ = kinematic viscosity = ν

P_1, P_2 = head *

The mean square of velocities is then obtained by:

$$(v_m)^2 = \int_{-e/2}^{e/2} \frac{(e^2/8 - E^2/2)^2}{e^2} \cdot dE \cdot (h_1 - h_2)^2 \cdot g^2 \cdot \frac{1}{\nu^2 L^2}$$

---

* It should be noted that Maini made a mistake by stating that P_1 and P_2 were pressures.

$$= \frac{g^2}{\nu^2 L^2} (h_1 - h_2)^2 \left| \frac{e^3}{64} E - \frac{e}{8} \frac{E^3}{3} + \frac{E^5}{20.e} \right|_{-e/2}^{e/2}$$

$$= \frac{(h_1 - h_2)^2 \ g^2 \ e^4}{120 \ \nu^2 \ L^2} \qquad \text{C-3}$$

Now, from Bernoulli's equation, the pressure drop due to the change in kinetic energy is given by:

$$P_k = \frac{(v^2)_m}{2g} \qquad \text{C-4}$$

or

$$\boxed{h_k = \frac{(h_1 - h_2)^2 \ g \ e^4}{240 \ \nu^2 \ L^2}} \qquad \text{C-5}$$

From Poiseuille's law for streamline flow,

$$v_m = \frac{e^2 \ g}{12 \ \nu} \left(\frac{\Delta h}{\Delta L}\right)$$

or

$$\left(\frac{\Delta h}{\Delta L}\right)^2 = \frac{144 \ \nu^2 \ v_m^2}{e^4 \ g^2} \qquad \text{C-6}$$

Substituting into equation C-5 gives:

$$\boxed{h_k = \frac{3 \ v_m^2}{5 \ g}} \qquad \text{C-7}$$

In the case of radial flow, we can write that

$$v = K_J \frac{\delta p}{\delta r} \qquad \text{C-8}$$

For flow into a fissure

$$Q = 2 \pi r \cdot e \cdot \bar{v}$$

or

$$\bar{v}^2 = \frac{Q^2}{4 \pi^2 r^2 e^2} \qquad \text{C-9}$$

Substituting this last equation into C-7 gives:

$$h_K = \frac{3 Q^2}{20 \cdot g \pi^2 e^2} \left[\frac{1}{r_1^2} - \frac{1}{r_0^2} \right] \qquad \text{C-10}$$

where r_0 = radius of hole

r_1 = radius where head has dropped to a minimum

This equation gives the energy loss due to the sudden acceleration at the joint entrance.

Generalizing the case to 'n' fractures,

$$h_{Kn} = \frac{3 Q^2}{20 g \pi^2 e^2 n^2} \left[\frac{1}{r_1^2} - \frac{1}{r_0^2} \right] \qquad \text{C-11}$$

which may also be expressed as

$$h_{Kn} = \frac{C Q^2}{e^2 n^2} \qquad \text{C-12}$$

where c = constant

Then for a given discharge Q, as the fracture aperture decreases, the energy loss increases drastically.

The final relation between flow and energy loss is then

$$h = \frac{Q \ \ln (r_1/r_0)}{2\pi \ K_J \ ne} + \frac{3 \ Q^2}{20 \ g \ \pi^2 \ e^2 \ n^2} \left(\frac{1}{r_1^2} - \frac{1}{r_0^2}\right) \qquad \text{C-13}$$

or

$$h = \frac{6 \ \nu \ Q}{\pi \ g \ e^3 n} \ln (r_1/r_0) + \frac{3 \ Q^2}{20 \ g \ \pi^2 \ e^2 \ n^2} \cdot \left(\frac{1}{r_1^2} - \frac{1}{r_0^2}\right) \qquad \text{C-14}$$

or

$$h = A \ Q + B \ Q^2 \qquad \text{C-15}$$

For the turbulent flow case Maini bases his development on Miessbach's law

$$(V_m)^n = C \frac{\delta p}{\delta r} \qquad \text{C-16}$$

where
- n = degree of non-linearity, $1 < n < 2$, $n = 2$ - assumes fully turbulent
- c = constant - depends on ν, medium, determined experimentally in field

Now assume V_m = mean-velocity

For flow from cavity into single fissure

$$Q = 2\pi re\ V_m \qquad \text{C-17}$$

where $\begin{cases} r = \text{radius at any point} \\ e = \text{effective aperture} \end{cases}$

Substitute C-17 for C-16

$$\left[\frac{Q}{2\pi re}\right]^n = C\,\frac{\delta P}{\delta r} = C\,\frac{\delta h}{\delta r}$$

$$\therefore \int_{r_0}^{r_1} \frac{dr}{r} = \frac{C(2\pi e)^n}{Q^n} \int_{h_0}^{h_1} dh$$

$$\left(\frac{1}{-n+1}\right)\frac{1}{r^{n-1}}\Big|_{r_0}^{r_1} = \frac{C(2\pi e)^n}{Q^n}\,(h_1 - h_0)$$

$$-\quad \frac{1}{r_0^{\,n-1}} - \frac{1}{r_1^{\,n-1}} = \frac{C(2\pi e)^n}{Q^n}\,(h_1 - h_0)(1-n) \qquad \text{C-18}$$

$$-\quad \frac{1}{r_0^{\,n-1}} - \frac{1}{r_1^{\,n-1}} = -\,\frac{C(2\pi e)^n}{Q^n}\,(h_0 - h_1)(1-n) \qquad \text{C-19}$$

Maini writes as C-18. Therefore one must assume that he takes the negative sign on the R.H.S. into the constant C.

$$\boxed{\therefore\ Q^n = C(2\pi e)^n \left[\frac{r_1^{\,n-1} \cdot r_0^{\,n-1}}{r_1^{\,n-1} - r_0^{\,n-1}}\right] (h_0 - h_1)(1-n)} \qquad \text{C-20}$$

We may rewrite C-20 as

$$E \cdot Q^n = (h_0 - h_1) \qquad \text{C-21}$$

and

$$n \log Q + \log E = \log\,(h_0 - h_1) \qquad \text{C-22}$$

For non-linear flow if one plots $\log(h_0 - h_1)$ vs $\log Q$, a straight line results

slope of line = n

Intercept on P axis = C_e "non-linear permeability function"

APPENDIX D

RADIAL FLOW FORMULATION

(after Iwai)

D.1 BASIC EQUATIONS

For viscous flow of incompressible fluid (i.e. Navier-Stokes' equation)

$$\frac{D\bar{v}}{Dt} = \bar{F} - \frac{1}{\rho} \nabla P + \nu(\nabla^2 \bar{v}) \qquad \text{D-1}$$

Assumptions

a) Flow governed only by mechanical and thermal energy within the system

b) Flow is isothermal

c) Flow is Newtonian and homogeneous

d) Stokes' equation is valid.

Equation of continuity can be written as:

$$\nabla \cdot \bar{v} = 0 \qquad \text{D-2}$$

The above equations include 4 unknowns: v_x, v_y, v_z and P. Therefore, it should theoretically be possible to solve for appropriate initial boundary conditions. Experimental results generally show good agreement with the proposed equation (D-1).

The empirical equation for viscous flow in porous media is given by

$$Q = K A \frac{\Delta h}{L} \qquad \text{D-3}$$

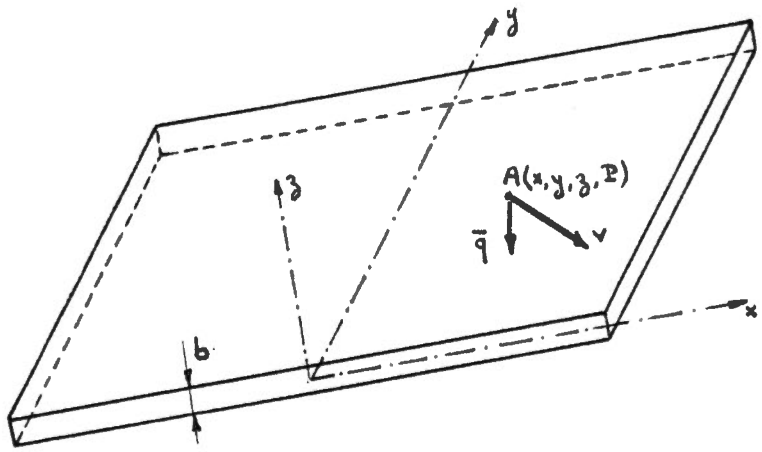
Darcy's Law can be derived from Navier-Stokes' equation by taking an average flow velocity rather than velocities for each fluid particle, provided inertia forces are negligible and steady state conditions prevail. However, one must note that the assumptions in Navier-Stokes' derivation are also inherent in Darcy's approach. Therefore, fracture flow can be investigated using Darcy's law as long as basic assumptions of Navier-Stokes' are met.

Terminology

Fracture permeability - K_f

System permeability - K_s

D.2 APPLICATION OF BASIC FRACTURE EQUATIONS TO VISCOUS FLOW



Consider a fracture with variable opening b and walls which may be assumed smooth and parallel in a global sense. The fracture opening has arbitrary axes 0_x and 0_y in the fracture plane, and 0_z is taken perpendicular to this plane.

If gravity is introduced as a body force,

$$\bar{F} = \bar{g} = -g\nabla z \qquad \text{D-4}$$

where ∇z represents the unit vector in the z- direction, then, the equation for flow @ point 'A' can be written as

$$\frac{D\bar{v}}{Dt} = -g\,\nabla\left(z + \frac{P}{\gamma}\right) + \nu\nabla^2\bar{v} \qquad \text{D-5}$$

where $(z + \frac{P}{\gamma})$ is called potential energy, piezometric head or velocity potential and is denoted by 'h'

$-\nabla h \equiv$ hydraulic gradient

Equation (D-5) can also be rewritten as:

$$\frac{D\bar{v}}{Dt} = g(-\nabla h) + \nu\nabla^2\bar{v} \qquad \text{D-6}$$

Now assuming the following

1) steady state flow conditions
2) variation in 'b' so small that $v_z = 0$
3) change in velocities in x- and y- directions are negligibly small compared to the velocity change in the z- direction.

Equation (D-6) then becomes

$$\begin{cases} 0 = -g\dfrac{dh}{dx} + \nu\dfrac{d^2v_x}{dz^2} \\ 0 = -g\dfrac{dh}{dy} + \nu\dfrac{d^2v_y}{dz^2} \\ 0 = -g\dfrac{dh}{dz} \end{cases} \qquad \text{D-7}$$

Hence h is independent of z.

Integrating, v_x becomes:

$$v_x = \frac{g}{\nu}\frac{dh}{dx}\frac{1}{2}z^2 + c_1 z + c_2$$

where c_1 and c_2 are integration constants.

Boundary conditions

$$v_x = 0 \quad @ \; z = \pm\, b/2 \quad \text{gives}$$

$$v_x = \frac{1}{2}\,\frac{g}{\nu}\,\frac{dh}{dx}\left(z^2 - \frac{b^2}{4}\right) \qquad \text{D-8}$$

the maximum velocity @ $z = 0$ is thus

$$v_{max} = -\,\frac{\gamma b^2}{8\mu}\,\frac{dh}{dx}$$

and the average velocity over the fracture aperture is given by:

$$v_x = \frac{1}{b}\int_{-b/2}^{+b/2} v_x \, dz = -\,\frac{\gamma b^2}{12\mu}\,\frac{dh}{dx} \qquad \text{D-9}$$

Similarly $\quad v_y = -\,\frac{\gamma b^2}{12\mu}\,\frac{dh}{dy}$ D-10

Therefore, the average velocity vector is given by:

$$\bar{v} = \frac{\gamma b^2}{12\mu}\,(-\nabla h) \qquad \text{D-11}$$

and the flow rate per unit width in direction $\bar{v}$ for a single fracture is:

$$q = \frac{\gamma b^3}{12\mu}\,(-\nabla h) \qquad \text{D-12}$$

By analogy with Darcy's approach let's define fracture permeability

$$K_f = \frac{\gamma b^2}{12\mu} \qquad \text{D-13}$$

If the fracture spacing is s, then the system permeability is

$$K_s = \frac{b}{s} K_f = \frac{\gamma b^3}{12\mu} \frac{1}{s} \qquad \text{D-14}$$

This last equation implicitly assumes that the flow of the system is equal to the sum of flows within each individual fracture.

D.3 FRACTURES WITH SMOOTH WALLS - AXISYMMETRIC COORDINATE SYSTEM

Navier-Stokes' equation for incompressible fluid in a cylindrical coordinate system is:

$$\left\{\begin{array}{l} \left(\frac{Dv_r}{Dt} - \frac{v_\phi^2}{r}\right) = F_r - \frac{1}{\rho}\frac{dP}{dr} + \nu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{dv_\theta}{d\theta}\right) \\ \left(\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r}\right) = F_\theta - \frac{1}{\rho}\frac{1}{r}\frac{dP}{d\theta} + \nu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{dv_r}{d\theta} - \frac{v_\theta}{r^2}\right) \\ \frac{Dv_z}{Dt} = F_z - \frac{1}{\rho}\frac{dP}{dz} + \nu\nabla^2 v_z \end{array}\right. \qquad \text{D-15}$$

where

$$\left\{\begin{array}{l} \frac{D}{Dt} = \frac{d}{dt} + v_r\frac{d}{dr} + \frac{v_\theta}{r}\frac{d}{d\theta} + v_z\frac{d}{dz} \\ \nabla^2 = \frac{d}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d^2}{d\theta^2} + \frac{d^2}{dz^2} \end{array}\right.$$

The equation of continuity for an incompressible fluid in this system is given by

$$\frac{1}{r}\frac{d}{dr}(rv_r) + \frac{1}{r}\frac{dv_\theta}{d\theta} + \frac{dv_z}{dz} = 0 \qquad \text{D-16}$$

Assuming that axisymmetric steady state conditions prevail the previous equations reduce to:

$$\left\{\begin{aligned} v_r \frac{dv_r}{dr} &= -g\frac{dh}{dr} + \nu\left(\frac{d^2v_r}{dr^2} + \frac{1}{r}\frac{dv_r}{dr} + \frac{d^2v_r}{dz^2} - \frac{v_r}{r^2}\right) && \text{D-17} \\ 0 &= -g\frac{dh}{dz} \\ \frac{d}{dr}(r\,v_r) &= 0 && \text{D-18} \end{aligned}\right.$$

It should be noted that equation (D-17) contains a nonlinear term, $v_r \frac{dv_r}{dr}$ which is dependent of the variation of v_r in r-direction -- one of the important characteristics of radial flow.

The previous set of equations is difficult to solve analytically.

Assuming that the inertial term in equation (D-17) may be ignored and recognizing that:

$$v_r = \frac{F(z)}{r} \qquad \text{D-19}$$

equation (D-17) becomes

$$0 = -g \frac{dh}{dr} + \nu \frac{1}{r} \frac{d^2F(z)}{dz^2} \qquad \text{D-20}$$

It can easily be seen from equation (D-17) that h is independent of z, and therefore,

$$\frac{d^2F(z)}{dz^2} = \text{constant}$$

or,

$$F(z) = \frac{1}{2} c_3 z^2 + c_4 z + c_5$$

Substituting for the boundary conditions, i.e. $v_r = 0$ @ $z = \pm b/2$, one obtains

$$\begin{cases} 2\,c_5 = c_3 (b/2)^2 \\ c_4 = 0 \end{cases}$$

and, therefore,

$$F(z) = \frac{1}{2} c_3 \left[z^2 - (b/2)^2 \right] \qquad \text{D-21}$$

Now substituting into equation (D-20) leads to

$$\frac{dh}{dr} = \frac{\mu}{\gamma} \cdot \frac{c_3}{r}$$

or,

$$h = \frac{\mu}{\gamma} c_3 \ln r + c_6$$

Boundary conditions:

$$\begin{cases} h = h_i & @ \; r_i \\ h = h_o & @ \; r_o \end{cases}$$

thus,

$$\begin{cases} h_i = \frac{\mu}{\gamma} \; c_3 \; \ln r_i + c_6 \\ h_o = \frac{\mu}{\gamma} \; c_3 \; \ln r_o + c_6 \end{cases}$$

or

$$h_i + h_o = \frac{\mu}{\gamma} \; c_3 \left(\ln r_i + \ln r_o \right) + 2c_6$$

$$c_6 = \frac{1}{2} \left[h_i + h_o - \mu/\gamma \; c_3 \left(\ln r_i + \ln r_o \right) \right]$$

Similarly,

$$h_i - h_o = \mu/\gamma \; c_3 \left(\ln \; r_i/r_o \right)$$

$$c_3 = - \frac{\gamma}{\mu} \frac{(h_i - h_o)}{\ln(r_o/r_i)}$$

which leads to:

$$c_6 = \frac{h_i \; \ln r_o - h_o \; \ln r_i}{\ln \; r_o/r_i}$$

Also, from equation (D-21)

$$F(z) = -\frac{\gamma}{2\mu} \; \frac{(h_i - h_o)}{\ln \; r_o/r_i} \; \left[z^2 - (b/2)^2 \right] \qquad \text{D-22}$$

and finally, equation (D-19) becomes

$$v_r = -\frac{\gamma}{2\mu r}\frac{(h_i - h_o)}{\ln r_o/r_i}\left[z^2 - (b/2)^2\right] \qquad \text{D-23}$$

$$v_{r,max} = \frac{\gamma}{8\mu r}\frac{(h_i - h_o)}{\ln r_o/r_i} b^2$$

$$\frac{dh}{dr} = \frac{\mu \cdot c_3}{r \cdot \gamma} = -\frac{1}{r}\frac{(h_i - h_o)}{\ln r_o/r_i} \qquad \text{D-24}$$

$$h = \frac{\mu}{\gamma} \cdot c_3 \ln r + c_6$$

$$= \frac{1}{\ln r_o/r_i}\left(h_i \ln \frac{r_o}{r} + h_o \ln \frac{r}{r_i}\right) \qquad \text{D-25}$$

The average velocity over the fracture aperture is

$$\bar{v}_r = \frac{1}{b}\int_{-b/2}^{+b/2} v_r \, dz$$

$$= \frac{\gamma b^2}{12\mu r}\frac{(h_i - h_o)}{\ln r_o/r_i} \qquad \text{D-26}$$

and the flow rate is given by:

$$Q = 2\pi rb\, v_r = \frac{\pi\gamma b^3}{6\mu}\, \frac{(h_i - h_o)}{\ell n\, r_o/r_i} \qquad \text{D-27}$$

This last equation may be rewritten in terms of $(\frac{\partial h}{\partial r})$ and $(2\pi rb)$ as

$$Q = \frac{\gamma b^2}{12\mu}\,(2\pi rb)\,(-\frac{\partial h}{\partial r}) \qquad \text{D-28}$$

therefore, when the inertial force is negligible, the analogy with Darcy's approach gives the fracture permeability as

$$K_f = \frac{\gamma b^2}{12\mu} \qquad \text{D-29}$$

Important Remark

In cases where both velocity and (or), its gradient are significant the effect of the inertia term must be considered. This effect may be inferred by writing equation D-17 as:

$$A = B + C$$

and therefore $v_r \left(\partial v_r/\partial r\right)$ is always negative, tending to zero for the limit of Darcy's case.

If one assumes that B and C are always opposite in sign, then:

For Divergent Flow B is positive and $\frac{\partial h}{\partial r} < 0$

Consequently, C must be negative and $|C| > |B|$

For Convergent Flow B is negative and $\frac{\partial h}{\partial r} > 0$

Consequently, C must be positive and $|C| < |B|$

Therefore, for divergent flow conditions ⟶ apparent larger K

for convergent flow conditions ⟶ apparent smaller K

It should be pointed out that, since variations in velocity and its gradients are greatest at the inner boundary one can avoid these erroneous values of K due to inertia effects by measuring $\partial h/\partial r$ at an appropriate distance from the inner boundary.

Iwai in his work determined the upper limit of applicability of Darcy's Law such that inertial effects could be neglected. This was accomplished by assuming that, as a first approximation, v_r and $\partial v_r/\partial r$ can be determined from Darcy's law using equation D-23.

$$\frac{\partial v_r}{\partial r} = \frac{\partial}{\partial r}\left\{\frac{\gamma}{2\mu}\,\frac{(h_i-h_o)}{\ell n\; r_o/r_i}\,\frac{1}{r}\,\left[\left(\frac{b}{2}\right)^2 - z^2\right]\right\}$$

$$= \frac{\gamma}{2\mu}\,\frac{\Delta h}{\ell n\; r_o/r_i}\,\left(-\frac{1}{r^3}\right)\left[\left(\frac{b}{2}\right)^2 - z^2\right]^2$$

and, therefore,

$$v_r\,\frac{\partial v_r}{\partial r}\bigg|_a = \left(\frac{\gamma}{2\mu}\,\frac{\Delta L}{\ell n\; r_o/r_i}\right)^2\left(-\frac{1}{r^3}\right)\left[(b/2)^2 - z^2\right]^2$$

Now taking the average over the fracture aperture

$$v_r\,\frac{\partial v_r}{\partial r}\bigg|_a = \frac{1}{b}\int_{-b/2}^{b/2} v_r\,\frac{\partial v_r}{\partial r}\,dz$$

Letting

$$\left(\frac{\gamma}{\partial\mu}\,\frac{\Delta h}{\ell n\; r_o/r_i}\right)^2 = A \text{ and } \left(-\frac{1}{r^3}\right) = B$$

$$v_r \frac{\partial v_r}{\partial r}\bigg|_a = \frac{1}{b}\cdot A\cdot B \int_{-b/2}^{b/2} \left[(b/2)^4 - 2\,(b/2)^2\, z^2 + z^4\right] dz$$

$$= A\cdot B\left(\frac{6b^4}{80} - \frac{b^4}{24}\right) = \left(\frac{\gamma}{2\mu}\,\frac{\Delta L}{\ell n\; r_o/r_i}\right)^2 \left(-\frac{1}{r^3}\right)\left(\frac{b^4}{30}\right) \qquad \text{D-30}$$

Now from equation D-24 we obtain the head gradient

$$-g\,\frac{\partial h}{\partial r} = \frac{g}{r}\,\frac{\Delta h}{\ell n\; r_o/r_i}$$

and the ratio between the inertia term and the head gradient term is given by:

$$N_i = \frac{v_r \frac{\partial v_r}{\partial r}\Big|_a}{-g\,\frac{\partial h}{\partial r}} = \frac{1}{g}\left(\frac{\gamma}{2\mu}\right)^2 \left(-\frac{1}{r^2}\right)\left(\frac{\Delta h}{\ell n\; r_o/r_i}\right)\left(\frac{b^4}{30}\right)$$

But (Eq. D-27)

$$Q = \frac{\pi\gamma}{6\mu}\, b^3 \left(\frac{\Delta h}{\ell n\; r_o/r_i}\right)$$

and, consequently,

$$\therefore\; N_i = \frac{\rho\, Q\, b}{20\,\mu\,\pi\, r_i^2} \qquad \text{D-31}$$

This inertia factor may be useful in determining effects of inertia on radial flow.

D.4 FURTHER CONSIDERATIONS ON FRACTURE FLOW

Two dimensionless parameters

$$\begin{cases} Re & - \text{ Reynolds Number} \\ \psi & - \text{ Friction factor} \end{cases}$$

may be used to generalize the flow law.

Define

$$Re = \frac{VD}{\nu} \qquad \text{D-32}$$

$$\psi = D \frac{(-\nabla h)}{v^2/2g} \qquad \text{D-33}$$

where

$$\begin{cases} D = 4\ R_h = 4\ \frac{A}{G} \ ^{*} \\ A = \text{cross-sectional area} \\ G = \text{wetted perimeter} \end{cases}$$

The physical meaning of the Reynold's number is that it represents the ratio of inertia forces to viscous forces present in the flow system. As the Reynold's number increases, the inertia becomes more significant to the point where turbulence may occur. However, turbulence is also promoted by eddy currents due to surface roughness. Therefore, the

---

* In the particular case of a fracture of aperture 'b' and width 'w',

$$D = \frac{4}{2} \frac{(b \cdot w)}{(b+w)} \simeq 2b$$

$$R_h = b/2$$

governing flow laws must be established for different values of Re and the relative roughness.

For example, the analytical solution of equation (D-12) can be written as:

$$\Psi = \frac{D\,(-\nabla H)}{v^2/2g} = 2b \frac{(-\nabla h)}{\frac{\gamma b^2}{12}\;(-\nabla h)\;\frac{V}{2g}} = \frac{48}{\frac{\rho b}{\mu}\cdot V} = \frac{96}{Re} \qquad \text{(D-35)}$$

It should be pointed out that this dimensionless expression represents the flow law only for cases where the Reynold's number and the relative roughness are negligible. When these conditions are not met, the flow law can only be established experimentally (see Louis, 1969).

In the case of radial flow, an additional problem occurs due to the fact that the Reynold's number cannot be uniquely defined for the complete flow region due to continuous changes in the cross-sectional area. Baker (1955) used the Reynolds number computed at a certain radius within which turbulent flow conditions were assumed to take place to describe the overall flow character, i.e.

$$Re = \frac{V(2b)}{\nu} = \frac{Q\cdot 2b}{\nu\cdot 2\pi R b} = \frac{\rho Q}{\pi\mu R} \qquad \text{(D-36)}$$

Following Baker's idea, the Reynolds' number may be computed at the inner boundary if viscous forces predominate the inertial terms. Similarly,

$$\Psi = D \frac{(-\nabla h)}{V_{r,i}^2 / 2g} \tag{D-37}$$

Baker found that the critical Reynolds' number under radial flow conditions, varied from 4,000 to 8,000, values which are very high when compared to data generated in the case of linear flow.

Few workers have suggested approaches for radial, turbulent flow conditions.

Maini (1971) started from Missbach's law for flow between parallel plates.

$$(V)^n = c \frac{\partial P}{\partial r} \tag{D-38}$$

where n is the degree of nonlinearity which varies from 1 to 2 (n = 2 corresponding to fully turbulent conditions), and c is a constant depending both on the medium and the viscosity of the fluid.

Maini derived the following expression:

$$Q^n = c(2\pi b)^n \left(\frac{r_i^{n-1} \; r_o^{n-1}}{r_o^{n-1} - r_i^{n-1}} \right) (P_i - P_o) \; (1-n) \tag{D-39}$$

Effect of Kinetic Energy on Flow

Nonlinear flow rate characteristics have been noted prior to onset of turbulence. Maini suggested that the main cause was the presence of "dead spaces". He also presented corrections for both linear and radial flow conditions.

Linear Flow:

$$h_v = \frac{3}{5g} v^2 = \frac{3}{5b^2 g} q^2 \tag{D-40}$$

$$\Delta h = \frac{12\mu L}{\gamma b^3} q + \frac{3}{5b^2 g} q^2 \tag{D-41}$$

Radial Flow:

$$h_{v,i} = \frac{3Q^2}{20\pi^2 b^2 g} \frac{1}{r_i^2} \tag{D-42}$$

$$\Delta h = \frac{6\mu}{\pi\gamma b^3} \ell n \left(\frac{r_o}{r_i}\right) Q + \frac{3Q^2}{20\pi^2 b^2 g} \frac{1}{r_i^2} \tag{D-43}$$

Baker (1955) incorporated equation (D-43) in a semi-empirical flow rate equation.

$$P = \frac{6\mu Q}{\pi b^3} \ell n\left(\frac{r_o}{R}\right) + \frac{3\rho Q^2}{20\pi^2 b^2} \left(\frac{1}{R^2} - \frac{1}{r_o^2}\right) +$$

$$\frac{\rho Q a}{4\pi^2 b^3} \left(\frac{1}{r_i} - \frac{1}{R}\right) + \frac{\rho Q^2}{8\pi^2 b^2} \left(\frac{1}{r_i^2} - \frac{1}{R^2}\right) \tag{D-44}$$

where a is an experimental constant, and R is radius to turbulent-laminar transition.

APPENDIX E

RADIAL FLOW FORMULATION

(after Rissler, 1978)

APPENDIX E

Table of Contents

E.1 ONE-DIMENSIONAL LAMINAR FLOW IN A FISSURE WITH SMOOTH WALLS

Under those conditions, the Navier-Stokes equation can be written as:

$$\frac{dv}{dt} = P - \frac{1}{\rho} \cdot \text{grad } p + \nu (\Delta v) + \frac{\nu}{3} \cdot \left\{ \text{grad div } (v) \right\} \qquad \text{E-1}$$

where
- v = flow velocity
- P = mass force
- ρ = density
- p = pressure
- ν = kinematic viscosity, dependent on water temperature.

Integrating this expression twice shows that the mean flow velcity $\bar{v}$ is proportional to the gradient I, in vector form:

$$\left\{\bar{v}\right\} = \frac{g(2a_i{}^2)}{12\ \nu} \cdot \left\{I\right\} = \bar{k} \cdot \left\{I\right\} \qquad \text{E-2}$$

where
- g = acceleration of gravity
- $\bar{k}$ = permeability coefficient of single fracture

E.2 FLOW LAWS FOR ONE-DIMENSIONAL FLOW IN A FISSURE

Energy losses in pipes of any cross-sectional area are given by

$$I = \lambda \cdot \frac{1}{D_h} \cdot \frac{v^2}{2g} \qquad \text{E-3}$$

where
λ = friction coefficient
D_h = hydraulic diameter of pipe
$\frac{v^2}{2g}$ = kinetic energy relative to the unit weight

For a fissure aperture = $2a_i$

$$D_h = 2 \cdot (2a_i) = 4a_i, \qquad \text{E-4}$$

Reynold's number can be written as $Re = \frac{D_h \cdot \bar{v}}{\nu}$ E-5

or $\lambda = \frac{96}{Re}$ E-6

This last equation is valid for roughnesses $0 < k/D_h \leqslant 0.032$.

In the case of one-dimensional flow between non-parallel walls (i.e. $k/D_h > 0.032$), this equation is slightly transformed and becomes:

$$\lambda = \frac{96}{Re}\left[1 + 8.8\,(k/D_h)^{1.5} \right] \qquad \text{E-7}$$

In the turbulent range

a) hydraulically smooth ($k/D_h = 0$)

$$\lambda = 0{,}316 \cdot Re^{-\frac{1}{4}} \qquad \text{E-8}$$

b) completely rough

i) $k/D_h \leqslant 0{,}032$ $\qquad \frac{1}{\sqrt{\lambda}} = -2 \log \frac{k/D_h}{3.7}$ E-9

ii) $k/D_h > 0.032$ $$\frac{1}{\sqrt{\lambda}} = -2 \log \frac{k/D_h}{1.9}$$ E-10

In the laminar-turbulent transition for $k/D_h < 0.032$

- use flow law from Colebrook-White

$$\lambda = 0.316\ R^{-0.25}$$

λ can also be approximated using (E-9) and (E-10).

Louis, 1969, suggested the existence of a critical Reynold's number for parallel flow and $k/D_h < 0.0168$

$$Re_k = 2300 \qquad \text{E-11}$$

as characterizing the laminar to turbulent transition. For $k/D_h > 0.0168$, $Re_k = f(k/D_h)$. On a λ versus Re diagram, this can be approximated as a straight line (ref. Rissler, Figure 4). Eliminating λ between the equation of this line and (E-9) and (E-10) leads to

1) For $0.0168 \leqslant k/D_h \leqslant 0.032$

$$\log Re_k = \frac{1}{1.76}\left[\log\ 142000\ (\log \frac{3.7}{k/D_h})^2\right] \qquad \text{E-12}$$

2) For $k/D_h > 0.032$

$$\log Re_k = \frac{1}{1.76}\left[\log\ 142000\ (\log \frac{1.9}{k/D_h})^2\right] \qquad \text{E-13}$$

Transition from hydraulically smooth (Blasius (E-8)) to completely rough (Nikuradse (E-9)) depends on Re and K/D_h. If designate Re at transition as Re_{k1}, eliminating λ from (E-8) and (E-9) gives:

$$Re_{k1} = 2.552 \left(\log \frac{3.7}{k/D_h}\right)^8 \qquad \text{E-14}$$

It should be pointed out that all of the previous flow laws have been determined experimentally on artificial joints.

Wittke and Louis showed that the velocity profile in the case of laminar, divergent radial flow varies only slightly from that corresponding to one-dimensional flow conditions. They concluded that flow laws valid for this last case can also be used as a good approximation for divergent radial flow and can be handled by potential theory.

Water pressure tests, using both linear and non-linear relationships, have resulted in the observation of over-proportional and under-proportional relationships (over-proportional corresponding to an increase in flow with pressure due to either expansion or cracking of joint; under-proportional corresponding to turbulent conditions). In the literature, attempts have been made to describe curves by parabolic formulae such as:

$$p = A \cdot Q + B \cdot Q^2 \qquad \text{E-15}$$

where
- p = pressure in B.H. at point of entrance
- Q = flow rate
- A,B = coefficients (with differing dimensions)

A and B - empirically determined from tests therefore usable conclusions about causes of these relations (i.e. aperture, spacing, etc.) not made.

E.3 ENERGY LOSSES DURING WATER PRESSURE TESTS

The knowledge of the water head at joint entrance is required for reliable evaluation of water pressure tests. If the pressure is measured at the bottom of the hole, energy losses existing between the gauge and the entrance to the joint must be calculated.

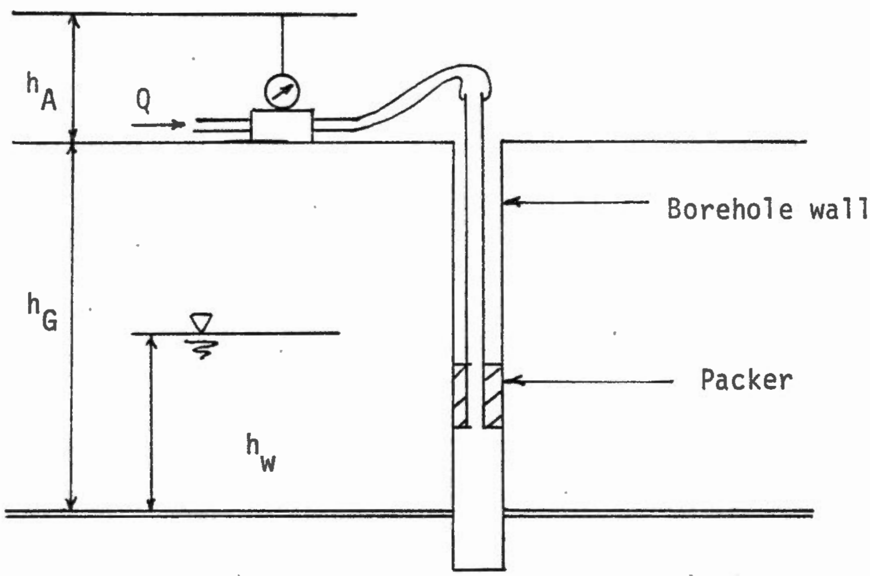


Fig. E-1 Pump Test Setup - Typical (after Rissler 1978).

If gravity is the only acting mass force, then the total energy head of a fluid element is given by Bernoulli's equation:

$$H = z + \frac{p}{\gamma} + \frac{\bar{v}^2}{2g} \qquad \text{E-16}$$

where
- z - geometric head
- p - static pressure
- γ - density of the fluid (γ_w for H_2O)

$\bar{v}$ - mean flow velocity

g - acceleration of gravity

In practical rock engineering, the kinematic term is often very small; hence, the velocity head may be neglected.

The energy losses will be illustrated using Figure 1. Selecting the mid-plane of the fracture as a reference, z = 0. The energy head effective in the test section, neglecting energy losses, is given by:

$$H = h_A + h_G - h_W \qquad \text{E-17}$$

If the water table is situated below the joint, $h_W = 0$ and the total loss of energy between the gauge and the joint consists of:

h_1: losses due to bends in tube

h_2: friction losses in pressure tube (h_2^{1}) and packer rod (h_2^{2})

h_3: losses due to enlargement in cross-sectional area below packer

h_4: losses due to bending and contracting at entrance

Then, <u>the energy head acting at the joint is</u>

$$\boxed{H_0 = h_A + h_G - h_W - h_1 - h_2 - h_3 - h_4} \qquad \text{E-18}$$

$$h_1 = S_K \cdot \frac{\bar{v}^2}{2g} = S_K \cdot \frac{1}{2g} \left(\frac{4 \cdot Q}{D^2 \pi}\right)^2 \qquad \text{E-19}$$

For bends at 90°, Truckenbrodt gives values for loss coefficient (S_K) with respect to bend radius and pipe diameter (r_K/D). Bend losses (h_1) can then be calculated.

Friction losses (h_2) are important at high flow rates, and can be calculated using equation (E-3). They are also given in nomograph form on Fig. 14, Page 44 of Rissler.

Losses due to cross-sectional changes below the packer are given by

$$h_3 = \left(1 - \frac{D_1^2}{D_2^2}\right)^2 \cdot \frac{1}{2g} \left(\frac{4\,Q}{D_1^2 \cdot \pi}\right)^2 \qquad \text{E-20}$$

where D_1 = packer rod diameter
D_2 = borehole diameter

This is given in nomograph form on Fig. 15, Page 46 of Rissler.

E.3.1 Energy Losses due to Bending and Contraction at the Joint Entrance: h_4

These losses must be calculated for all pressure tests:

Assuming (i) a vertical borehole, (ii) a horizontal joint and (iii) a sharp-edged entrance (i.e. no special investigation required for bending and contraction at entrance. Since the hole radius $r_a \gg 2a_i$ the problem is reduced to a one-dimensional case:)

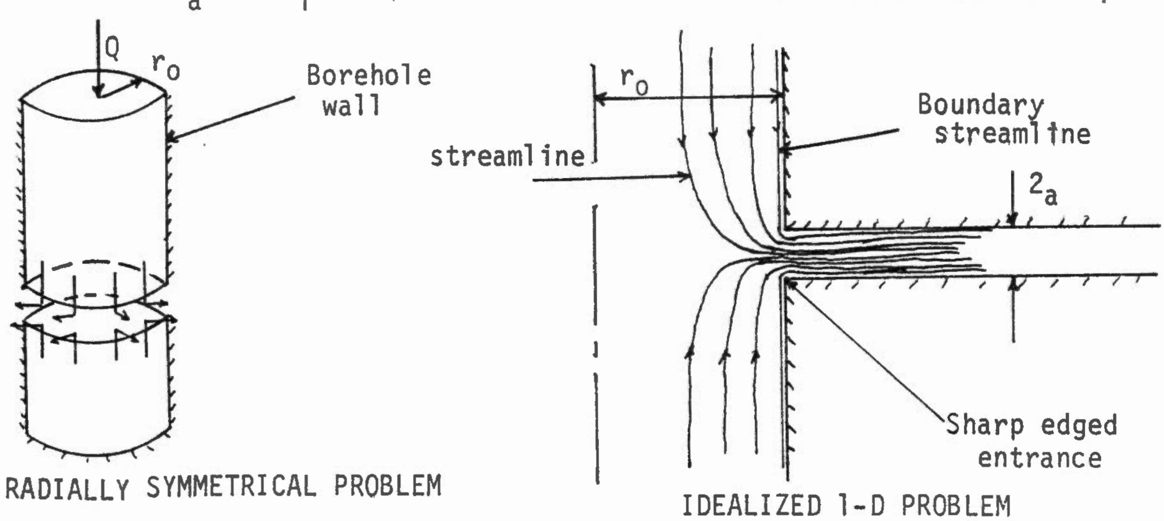


Figure E-2 Statement of Problem for Determining Energy Losses (h_4) (after Rissler 1978)

The one-dimensional problem was studied by Hahremann and Ehret for steady-state conditions, constant temperature and smooth walls (i.e. $k/D_h = 0$). This same approach will be used here for rough joints and non-parallel walls.

At the joint entrance, the water is deflected 90° which leads first to a contraction. The fracture walls will be approached after a finite length, a function of the energy losses. Further losses are also incurred in forming the velocity profile corresponding to the Reynolds number.

E.3.2 Energy Losses for Laminar Flow

The length x_a of disturbed flow at the joint entrance is given by

$$x_a = 0.00923 \cdot D_h \cdot Re \qquad \text{E-21}$$

Along this length, energy reduction is constant due to:

(i) losses at the entrance

(ii) normal laminar radial flow friction losses.

The energy loss at any cross-section within the distrubed zone is Δh from which a mean friction coefficient $\lambda_{x,1}$ results

$$\frac{\Delta h}{x} = \lambda_{x,1} \cdot \frac{1}{D_h} \cdot \frac{\bar{v}^2}{2g} \qquad \text{E-22}$$

$$\lambda_{x,1} = \frac{\Delta h}{x} \cdot D_h \cdot \frac{2g}{\bar{v}^2} \qquad \text{E-23}$$

but the unknowns, $\lambda_{x,1}$ and Δh, are functions of x.

The relationship $\lambda_{x,1} = F(x)$ is determined experimentally and can be written in a dimensionless form as:

$$Re \cdot \lambda_{x,1} = F\left(\frac{x}{D_h \cdot Re}\right) \qquad \text{E-24}$$

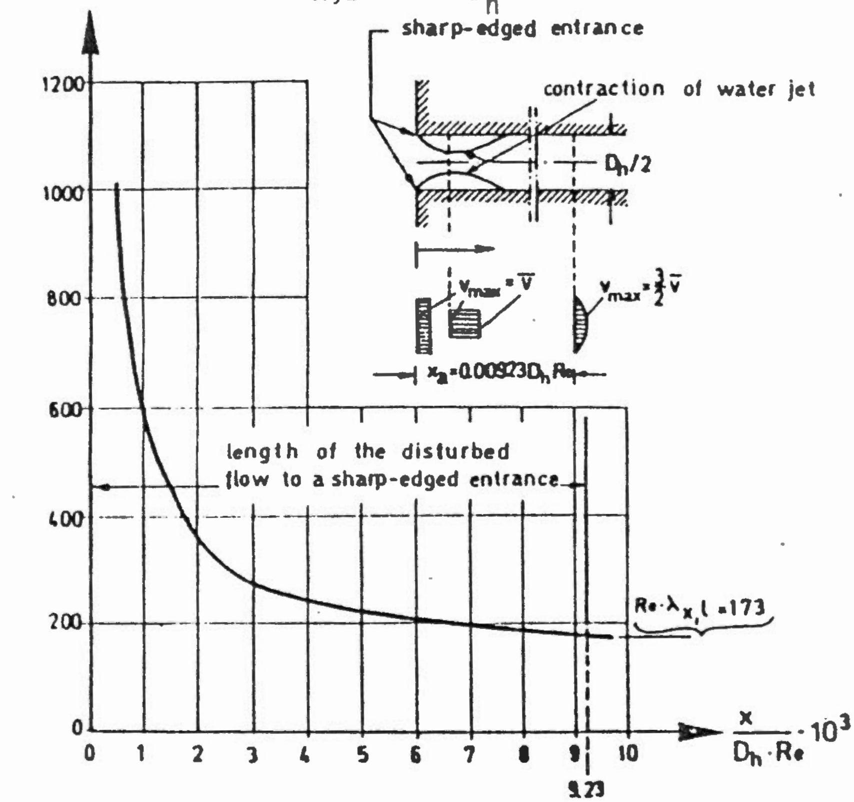


Fig. E-3 Mean friction coefficient $\lambda_{x,1} = f(x)$ of the laminar, parallel, hydraulically smooth flow in a fissure with a sharp-edged entrance (After Rissler, 1978)

The mean friction coefficient between the entrance and the end of the disturbed flow area is given by:

$$\lambda_{xa,1} = \frac{173}{Re} \qquad \text{E-25}$$

Reducing equation (E-25) by λ for friction losses in hydraulically smooth channels ($\lambda = \frac{96}{Re}$),

$$\lambda_E = \lambda_{xa,1} - \lambda = \frac{173}{Re} - \frac{96}{Re} = \frac{77}{Re} \qquad \text{E-26}$$

where λ_E is the friction coefficient decisive for bending and contraction at the entrance

$$\therefore \quad h_4 = \lambda_E \cdot \frac{x_a}{D_h} \cdot \frac{\bar{v}^2}{2g} \qquad \text{E-27}$$

using the value for x_a given from (E-21), this last equation reduces to:

$$h_4 = 0.711 \cdot \frac{\bar{v}^2}{2g} \qquad \text{E-28}$$

or introducing the loss coefficient

$$S_1 = 0.711$$

$$\text{then } h_4 = S_1 \cdot \frac{\bar{v}^2}{2g} \qquad \text{E-29}$$

E.3.3 Energy Losses for Turbulent Flow

Losses for entrance into a joint with turbulent, hydraulically smooth conditions and sharp edges are determined in a similar manner.

Length of disturbed flow:

$\begin{cases} x_v & \text{- laminar foresection} \\ x_a & \text{- following section necessary to develop turbulence} \end{cases}$

$$\frac{x_v}{D_h} = \frac{1250}{(Re - 3000)}{}^{2/3} \qquad \text{E-30}$$

for $4000 < Re < 18{,}000$, but

$$x_a = 0.33 \cdot Re^{\frac{1}{4}} \cdot D_h \qquad \text{E-31}$$

and therefore, the length of the disturbed flow at the entrance is given by

$$\frac{x_v + x_a}{D_h} = \frac{1250}{(Re - 3000)^{2/3}} + 0.33 \cdot Re^{\frac{1}{4}} \qquad \text{E-32}$$

The mean friction coefficient between the entrance and a random cross-section is:

$$\lambda_{x,t} = \frac{D_h}{x} (B_1 \cdot C_1 + A_1 + \frac{\Delta P}{q_{xv}}) = F(Re) \qquad \text{E-33}$$

where $\begin{cases} B_1 \ \& \ C_1 = F(Re, x) \\ A_1 \ \& \ \frac{\Delta P}{q_{xv}} = F(Re) \end{cases}$

All four parameters were determined experimentally. Now substituting the length of the disturbed flow for $(x_v + x_a)$ from equation (E-32), one obtains the mean friction coefficient between the entrance and the end of the length where disturbed flow conditions prevail $(\lambda_{xva,t})$. Multiplying by $(\frac{x_v + x_a}{D_h})$, one finally obtains the loss coefficient (S_G) for total energy loss between entrance and the point where disturbed flow ends:

$$S_G \ (Re) = \lambda_{xva,t} \ (Re) \ [x_v \ (Re) + x_a \ (Re)] \cdot \frac{1}{D_h} = \Delta h \cdot \frac{2g}{v^2} \qquad \text{E-34}$$

This expression is composed of a term related to the energy loss during entrance $[S_t \ (Re)]$ and a term expressing the friction losses along the length of disturbed flow [S_R (Re)]

$$\text{i.e.} \quad S_G\,(Re) = S_t\,(Re) + S_R\,(Re) \qquad \text{E-35}$$

If hydraulically smooth conditions were to be assumed, S_R (Re) could be expressed by Blasius' law

$$S_R\,(Re) = 0.316 \cdot Re^{-\frac{1}{4}} \cdot \frac{x_v + x_a}{D_h} \qquad \text{E-36}$$

Equations (E-35) and (E-36) combined with Figure 20 of Rissler can be used to determine S_t (Re); and consequently, S_t is found to be almost constant, i.e.

$$\boxed{S_t = 0.415} \qquad \text{E-37}$$

E.3.4 Summary

Using two loss coefficients S_1 and S_t for laminar and turbulent regimes, the corresponding head losses can be determined from the following:

$$h_4 = S_1 \cdot \frac{\bar{v}}{2g} = 0.711 \cdot \frac{1}{8 \cdot g \cdot \pi^2} \cdot \frac{1}{r_0^2} \cdot \left(\frac{q}{2a_i}\right)^2 \quad \left(\begin{matrix}\text{laminar}\\ \text{flow}\end{matrix}\right) \qquad \text{E-38}$$

$$h_4 = S_t \cdot \frac{\bar{v}^2}{2g} = 0.415 \cdot \frac{1}{8 \cdot g \cdot \pi^2} \cdot \frac{1}{r_0^2} \cdot \left(\frac{q}{2a_i}\right)^2 \quad \left(\begin{matrix}\text{turbulent}\\ \text{flow}\end{matrix}\right) \qquad \text{E-39}$$

For further details concerning the friction loss calculations refer to Rissler (1978).

E.4 STEADY RADIAL SYMMETRICAL FLOW IN HORIZONTAL JOINTS

When performing water pressure tests, the following parameters are usually determined or known:

- Q - flow rate
- H_o - energy head at the joint entrance
- t - water temperature
- ν - fluid viscosity
- r_o - borehole radius

The purpose of such tests is to evaluate fracture aperture ($2a_i$) and relative roughness (k/D_h).

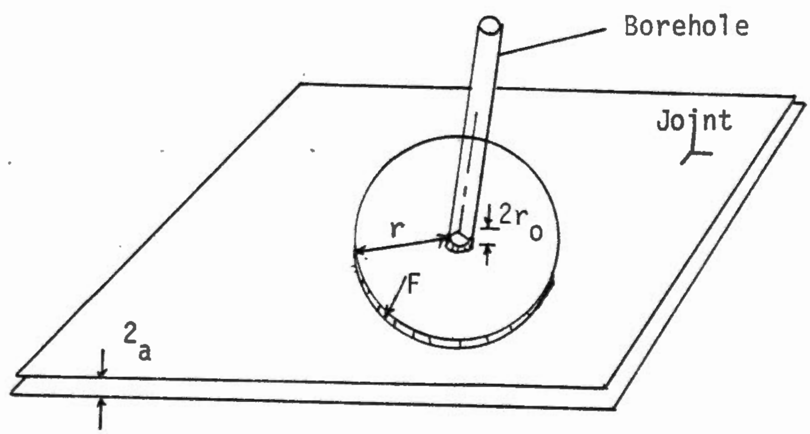


Figure #-4 Radial Flow in a Horizontal Joint (After Rissler 1978)

Considering a circular cut of radius r and laminar flow conditions, the continuity equation can be written as:

$$q = \bar{v} \cdot F = \bar{v} \cdot 2\pi r \cdot 2a_i \qquad \text{E-40}$$

or

$$\bar{v} = \frac{q}{2\pi r \cdot 2a_i} \qquad \text{E-41}$$

Introducing Louis' flow law

$$\lambda = \frac{96}{Re}\left[1 + 8.8\ (k/D_h)^{1.5}\right] \qquad \text{E-42}$$

into equation (E-3) and replacing I by ($-\frac{dH}{dr}$), Rissler states that the following relation can be derived:

$$\bar{v} = -\frac{g\ (2a_i{}^2)}{12 \cdot \nu \cdot [1 + 8.8\ (k/D_h)^{1.5}]} \cdot \frac{dH}{dr} \qquad \text{E-43}$$

This derivation is checked below.*

By equating equation (E-41) and (E-43), the following expression

---

* Substitute (E-42) into (E-3)

$$I = \frac{96}{Re}[1 + 8.8\ (k/D_h)^{1.5}] \cdot \frac{1}{D_h} \cdot \frac{\bar{v}^2}{2g}$$

or

$$-\frac{dH}{dr} = \frac{96}{Re}[1 + 8.8\ (k/D_h)^{1.5}] \cdot \frac{1}{D_h} \cdot \frac{\bar{v}^2}{2g} \qquad \text{E-44}$$

but,

$$Re = \frac{D_h\ \bar{v}}{\nu} \quad \text{and} \quad D_h = 4a_i$$

Therefore,

$$\bar{v} = -\frac{(2a_i)^2 \cdot g}{12\ \nu\ [1 + 8.8\ (k/D_h)^{1.5}]} \cdot \frac{dH}{dr} \qquad \text{E-45}$$

results

$$\frac{dH}{dr} = - \frac{12\ \nu\ [1 + 8.8\ (k/D_h)^{1.5}]}{g \cdot (2a_i)^2} \cdot \frac{q}{2\pi r (2a_i)} \qquad \text{E-46}$$

Integrating

$$H(r) = - \frac{6\ \nu\ [1 + 8.8\ (k/D_h)^{1.5}]}{g \cdot (2a_i)^3} \cdot \frac{q}{\pi} \cdot \ln r + C \qquad \text{E-47}$$

Using the following boundary conditions:

$$H(r_o) = H_o \qquad \text{E-48}$$

$$C = H_o + \frac{6\ \nu\ [1 + 8.8\ (k/D_h)^{1.5}]}{g \cdot (2a_i)^3} \cdot \frac{q}{\pi} \cdot \ln r_o \qquad \text{E-49}$$

and,

$$\boxed{H = H_o - \frac{6\ \nu\ [1 + 8.8\ (k/D_h)^{1.5}]}{g \cdot (2a_i)^3} \cdot \frac{q}{\pi} \cdot \ln \frac{r}{r_o}} \qquad \text{E-50}$$

This last equation describes the energy head H in the joint as $f(H_o, r_o, q, \nu, 2a_i, k/D_h)$ which are constants for each test.

As can easily be seen, H decreases with r, tending to zero as r tends to infinity. Generally it is sufficient to introduce $H = 0$ at $r = R$ as a boundary condition if R is very large. Using this, a linear relation between the energy head at the joint entrance (H_o) and flowrate (q) can be obtained.

$$H_o = q \cdot \frac{6\,\nu}{g \cdot \pi \cdot (2a_i)^3}\,[1 + 8.8\,(k/D_h)^{1.5}]\,\ell n\,\frac{R}{r_o} \qquad \text{E-51}$$

This last equation contains measurable values ν and r_o as well as those parameters sought, $2a_i$ and (k/D_h) which are decisive to the fracture permeability. A similar function has been derived by Baker and Wittke and Louis.

E.5 TURBULENT FLOW NEAR THE BOREHOLE - SMOOTH AND COMPLETELY ROUGH AND NONPARALLEL WALLS

The following is valid for

$$k/D_h = 0 \quad \text{and} \quad k/D_h > 0.0168.$$

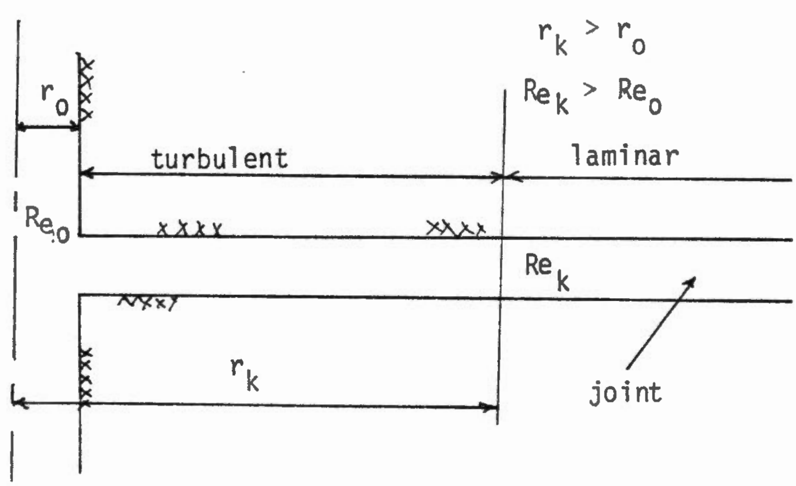
For $0 < k/D_h < 0.0168$, further considerations are necessary as discussed later.

a) Extent of turbulent zone

From the continuity equation (E-41) it can easily be seen that the mean flow velocity decreases as r increases. Therefore, the Reynolds number also decreases with r. If flow adjacent to the borehole wall is turbulent, Re decreases with r until it reaches a critical value r_K when a change from turbulent to laminar flow conditions occurs.

Figure E-5

Definition for r_k



To compute r_K, the following substitutions are required

$$\left\{\begin{array}{ccc} Re_K & \rightarrow & Re \\ \frac{q}{2\pi r \cdot 2a_i} & \rightarrow & \bar{v} \\ r_K & \rightarrow & r \end{array}\right.$$

to finally obtain

$$Re_K = \frac{q}{\pi \cdot r_K \cdot \nu}$$

or

$$\boxed{r_K = \frac{q}{\pi \cdot \nu \cdot Re_K}} \qquad \text{E-52}$$

Substituting into equations (E-11) through (E-18) leads to:

- for $k/D_h < 0{,}0168$

$$r_K = \frac{q}{\pi \cdot \nu \cdot 2300} \qquad \text{E-53}$$

- for $0.0168 \leq k/D_h \leq 0.032$

$$r_K = \frac{q}{\nu \cdot \pi} [142{,}000 \ (\log \frac{3.7}{k/D_h})^2]^{-\frac{1}{1.76}} \qquad \text{E-54}$$

- for $k/D_h > 0.032$

$$r_K = \frac{q}{\nu \cdot \pi} [142{,}000 \ (\log \frac{1.9}{k/D_h})^2]^{-\frac{1}{1.76}} \qquad \text{E-55}$$

These three equations show that the extent of the turbulent zone r_K for given parameters q and ν is not a constant, but is dependent on the relative roughness (k/D_h).

E.6 DISTRIBUTION OF ENERGY HEAD IN SECTION OF TURBULENT FLOW AND TRANSITION CONDITIONS

For case $k/D_h > 0{,}032$

The general relation for energy losses in radially symmetrical flow can be written:

$$-\frac{dH}{dr} = \lambda \cdot \frac{1}{D_h} \cdot \frac{\bar{v}^2}{2g} \qquad \text{E-56}$$

From the continuity equation

$$\bar{v} = \frac{q}{2\pi r \cdot 2a_i} \qquad \text{E-57}$$

and the coefficient λ can be replaced by

$$\frac{1}{\sqrt{\lambda}} = -2 \log \frac{k/D_h}{1.9} \; * \qquad \text{E-58}$$

Therefore, equation (E-56) becomes

$$\frac{dH}{dr} = -\frac{q^2}{64\,(2a_i)^3 \cdot g \cdot \pi^2 \left(\log \frac{1.9}{k/D_h}\right)^2} \cdot \frac{1}{r^2} \qquad \text{E-59}$$

---

* $\frac{1}{\sqrt{\lambda}} = 2 \log \frac{1.9}{k/D_h}, \quad \lambda = \frac{1}{4\left(\log \frac{1.9}{k/D_h}\right)^2}$

Integrating gives

$$H = \frac{q^2}{64\ (2a_i)^3 \cdot g \cdot \pi^2\ (\log \frac{1.9}{k/D_h})^2} \cdot \frac{1}{r} + C \qquad \text{E-60}$$

The constant C is determined from the boundary conditions

$$H(r_o) = H_o$$

or

$$C = H_o - \frac{q^2}{64\ (2a_i)^3 \cdot g \cdot \pi^2\ (\log \frac{1.9}{k/D_h})^2} \cdot \frac{1}{r_o}$$

and, therefore,

$$\boxed{H = H_o - \frac{q^2}{64\ (2a_i)^3 \cdot g \cdot \pi^2\ (\log \frac{1.9}{k/D_h})^2} \left(\frac{1}{r_o} - \frac{1}{r}\right)} \qquad \text{E-61}$$

Hence

$$H = f(r).$$

In order to extend this approach into the laminar zone, it will be necessary to determine first the energy head $H = H_K$ prevailing at the outer boundary of the turbulent zone using equations (E-55) and (E-61). This will then be applied as the inner boundary conditions for the laminar flow region. The energy head at the turbulent boundary may be written as

$$H_K = H_o - \frac{q^2}{64\ (2a_i)^3 \cdot g \cdot \pi^2\ (\log \frac{1.9}{k/D_h})^2} \left\{\frac{1}{r_o}\right.$$

$$- \frac{\nu \cdot \pi}{q} \left[142{,}000 \left(\log \frac{1.9}{k/D_h}\right)^2\right]^{\frac{1}{1.76}} \Bigg\} \qquad \text{E-62}$$

E.7 RELATIONSHIP BETWEEN THE ENERGY HEAD EFFECTIVE AT THE JOINT ENTRANCE AND THE FLOW RATE q

The function $H = f(r)$ for laminar flow was derived previously. It is now necessary to substitute the right-hand side of equation (E-62) into equation (E-50).

$$H = H_o - \frac{6\,\nu\,[1 + 8.8\,(k/D_h)^{1.5}]}{g\,(2a_i)^3} \cdot \frac{q}{\pi} \cdot \ln \frac{r}{r_o}$$

H_o - Original Pressure Head $\qquad$ H_k - Head effective of turbulent laminar boundary

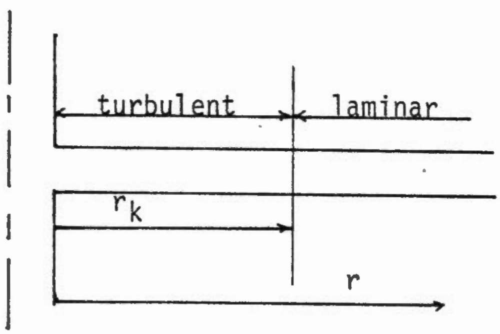


Figure #-6 Energy Head Effective at turbulent-laminar boundary (After Rissler, 1978)

Note: H_k becomes effective H_o for laminar zone and r_k becomes effective r_o

Performing the required substitutions then gives:

$$H = f(r) = H_o - \frac{q^2}{64\,(2a_i)^3 \cdot g \cdot \pi^2 \left(\log \frac{1.9}{k/D_h}\right)^2} \left\{ \frac{1}{r_o} - \right.$$

$$\left. \frac{\nu \cdot \pi}{q} \left[142{,}000 \left(\log \frac{1.9}{k/D_h}\right)^2\right]^{\frac{1}{1.76}} \right\}$$

$$- \frac{6 \cdot \nu \cdot q}{g\ (2a_i)^3} [1 + 8.8\ (k/D_h)^{1.5}]$$

$$\cdot\ \ell n \left\{ \frac{r}{\frac{q}{\nu \cdot \pi} [142{,}000\ (\log \frac{1.9}{k/D_h})^2]} - \frac{1}{1.76} \right\} \qquad \text{E-63}$$

If this equation is considered to be a function $H_o = f(q)$, then constant numerical values must be assigned to H and r.

Substituting the following boundary conditions

$$r = R \quad \text{and} \quad H = 0$$

leads to

$$H_o = \frac{q^2}{64\ (2a_i)^3 \cdot g \cdot \pi^2\ (\log \frac{1.9}{k/D_h})^2} \left\{ \frac{1}{r_o} - \frac{\nu \cdot \pi}{q} [142{,}000\ (\log \frac{1.9}{k/D_h})^2] \frac{1}{1.76} \right\}$$

$$+ \frac{6 \cdot \nu \cdot q}{g\ (2a_i)^3} [1 + 8.8\ (k/D_h)^{1.5}]$$

$$\cdot\ \ell n \left\{ \frac{\nu \cdot \pi \cdot R}{q} [142{,}000\ (\log \frac{1.9}{k/D_h})^2]^{\frac{1}{1.76}} \right\} \qquad \text{E-64}$$

Analogous considerations can be made for completely rough ($0.0168 \leq k/D_h < 0.032$) and smooth ($k/D_h = 0$) conditions. The corresponding flow laws must, of course, be introduced.

- for $0.0168 \leq k/D_h < 0.032$

$$H_o = \frac{q^2}{64\ (2a_i)^3 \cdot g \cdot \pi^2 \left(\log \frac{3.7}{k/D_h}\right)^2} \left\{ \frac{1}{r_o} - \frac{\nu \cdot \pi}{q} \left[142{,}000 \left(\log \frac{3.7}{k/D_h}\right)^2\right]^{\frac{1}{1.76}} \right\} + 6 \frac{\nu \cdot q}{g \cdot \pi \cdot (2a_i)^3} \left[1 + 8.8\ (k/D_h)^{1.5}\right] \cdot \ln \left\{ \frac{R \cdot \nu \cdot \pi}{q} \left[142{,}000 \left(\log \frac{3.7}{k/D_h}\right)^2\right]^{\frac{1}{1.76}} \right\} \qquad \text{E-65}$$

- for $k/D_h = 0$

$$H_o = 0.0263 \sqrt[4]{\frac{\pi \cdot \nu}{q}} \cdot \frac{q^2}{g \cdot \pi^2\ (2a_i)^3} \left[r_o^{-3/4} - \left(\frac{q}{\nu \cdot \pi \cdot 2300}\right)^{-3/4}\right] + \frac{6 \cdot \nu \cdot q}{g \cdot \pi (2a_i)^3} \cdot \ln \left(\frac{2300\ \nu \cdot \pi \cdot R}{q}\right) \qquad \text{E-66}$$

These last three equations represent, for the given roughness ranges, a relationship between the data H_o and q resulting from the test and $2a_i$ and (k/D_h) decisive for the fracture permeability.

E.8 TURBULENT FLOW NEAR THE BOREHOLE - TRANSITION ZONE

Up until now, an abrupt change from hydraulically smooth to rough (turbulent) conditions have been assumed. However, a gradual change may well be the general case, and the following situation prevails:

turbulence (Nikuradse) → transition (Blasius)
→ laminar (Louis, Pouiseuille)

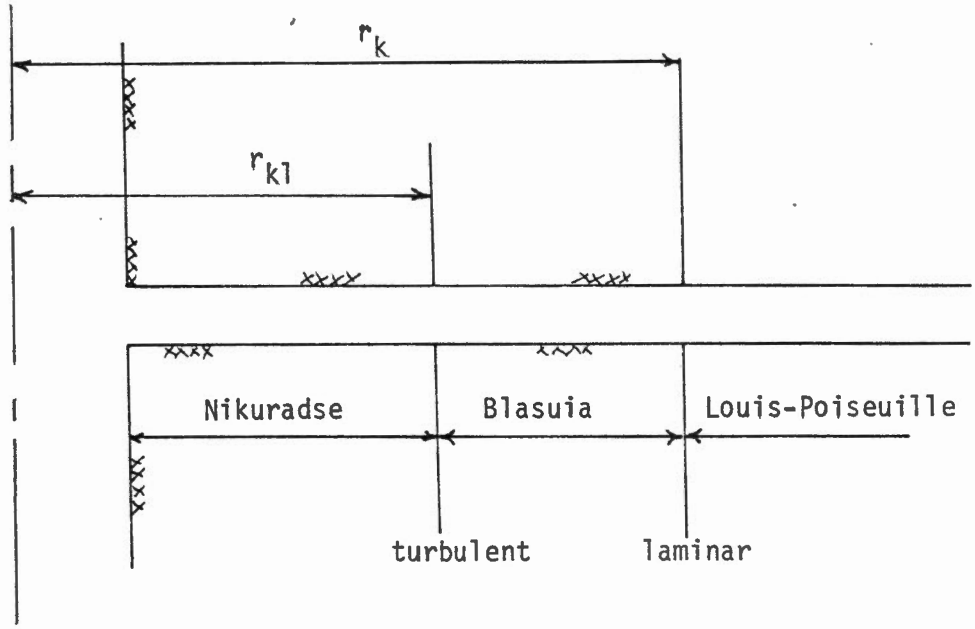


Figure E-7 Transitional Turbulent Zone Near the Borehole (After Rissler 1978)

The radius at which transition from completely rough to hydraulically smooth conditions occur can be determined in a similar manner as $r_K \rightarrow r_{K_1}$ (refer to equation E-14), i.e.

$$r_{K_1} = \frac{q}{\pi \cdot \nu} \cdot \frac{1}{2.552 \quad \log \frac{3.7}{k/D_h} \quad 8} \qquad \text{E-67}$$

Rissler states that for apertures $0.13 > 2a_i >$ 0.40 mm, a numerical evaluation of the theoretical relation revealed that for certain apertures, the laminar turbulent switchover was nearly independent of (k/D_h)(for nearly constant energy head conditions), i.e. see Figure E 8:

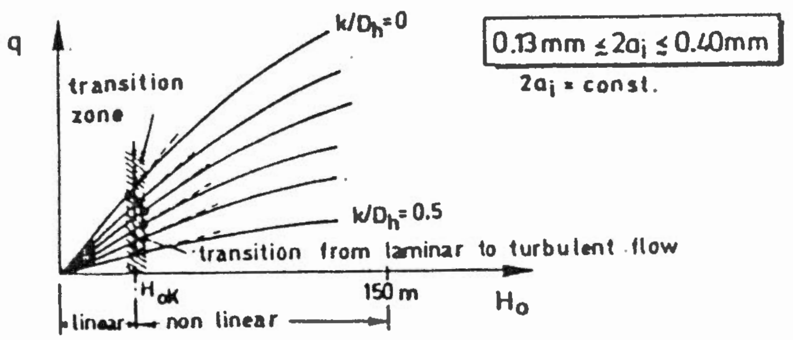


Figure E-8 Characteristic Curve $q = f(H_o)$

This allows the determination of the aperture from critical energy head considerations without knowing the relative roughness. It is necessary, however, to represent the relationship found between H_{oK} and $2a_i$ using equations (E-51) and (E-64) through (E-66).

E.9 DETERMINATION OF THE APERTURE FROM CRITICAL ENERGY HEAD

If H_{oK} represents the critical energy head at which laminar flow in the fissure immediately becomes turbulent at the entrance, this condition can be obtained using equation (E-51) by replacing the flow rate q by the critical Re.

$$Re_K = f(k/D_h)$$

Therefore

$$H_{oK} = \frac{6 \cdot \nu^2 \cdot r_o}{g\ (2a_i)^3} \cdot Re_K\ (k/D_h) \cdot [1 + 8.8\ (k/D_h)^{1.5}] \cdot \ln \frac{R}{r_o} \qquad \text{E-68}$$

It should be noted that this equation has been obtained from a numerical evaluation without any explanation.

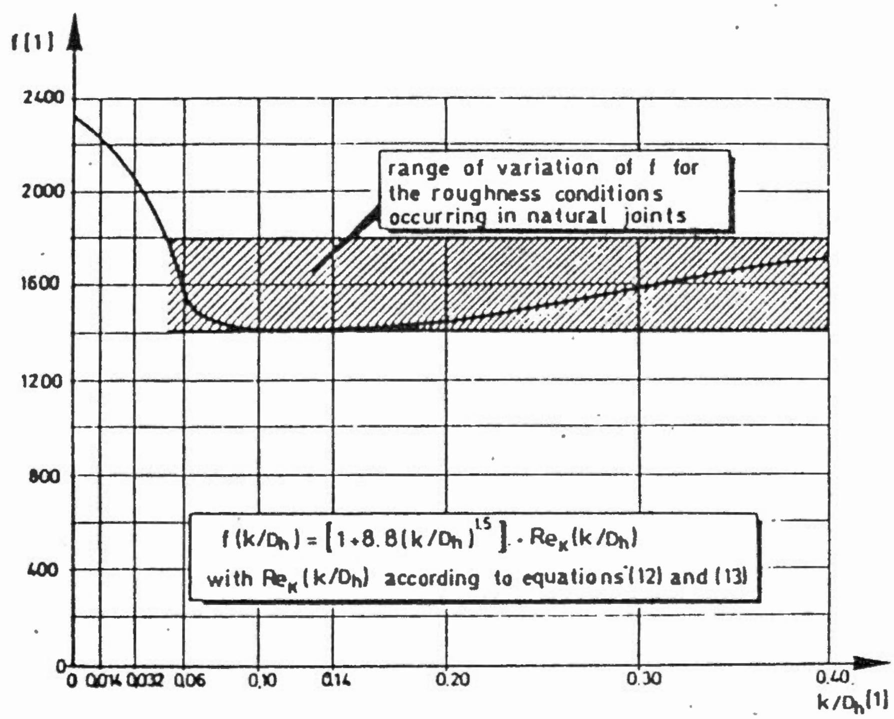


Figure E-9 $f(k/D_h)$ plotted versus k/D_h for $0 \le k/D_h \le 0.4$

Solving (E-68) for $2a_i$, leads to

$$2a_i = \sqrt[3]{\frac{6 \cdot \nu^2 \cdot r_o}{g \cdot H_{oK}} \cdot Re_K\ (k/D_h) \cdot [1 + 8.8\ (k/D_h)^{1.5}] \cdot \ln \frac{R_\circ}{r_o}} \qquad \text{E-69}$$

Introducing the values

$$B = \frac{6}{g \cdot H_{oK}} \cdot Re_K\ (k/D_h) \cdot [1 + 8.8\ (k/D_h)^{1.5}] = \frac{6}{g \cdot H_{oK}} \cdot f(k/D_h)$$

and $C = \nu^2 \cdot r_o \cdot \ln\left(\frac{R}{r_o}\right)$

one obtains

$$\boxed{2a_i = \sqrt[3]{B \cdot C}} \qquad \text{E-70}$$

Rissler has shown that the influence of (k/D_h) on the relation between H_{oK} and $2a_i$ is limited - if one evaluates $f(k/D_h)$ for roughness between .06 and 0.4 it is found to vary from 1400 to 1800. Hence, the error is given by

$$e = 1 - \sqrt[3]{\frac{1400}{1800}} = 8\%$$

Determination of Relative Roughness

Since $2a_i$ is now assumed known, we can calculate k/D_h. Substituting any given H_o, q pair from a test plus $2a_i$ previously calculated, and solve the equations given previously.

Rissler presents diagrams to simplify these calculations. These are shown in figure 17 in the text of this report.