Energy, Mines and Resources Canada
Énergie, Mines et Ressources Canada

Earth Physics Branch
Direction de la physique du globe

1 Observatory Crescent
Ottawa Canada
K1A 0Y3

1 Place de l'Observatoire
Ottawa Canada
K1A 0Y3

Geothermal Service of Canada

Service géothermique du Canada

SOME APPLICATIONS OF THE BOUNDARY INTEGRAL TECHNIQUE IN STRESS ANALYSIS

T.D. Wiles and J.C. Roegiers

Earth Physics Branch Open File No. 81/8

Ottawa, Canada, 1981

73 p.

Price/Prix: $23.50

UNIVERSITY OF TORONTO
Department of Civil Engineering

ISBN 0-7727-7011-5
PUBLICATION 80-07

SOME APPLICATIONS OF THE BOUNDARY INTEGRAL TECHNIQUE IN STRESS ANALYSIS

T.D. WILES
J.-C. ROEGIERS

JULY 1980

ABSTRACT

In this report, the mathematical formulation and some of the advantages and possible uses for the boundary element process have been reviewed. The choice of singularity problem has been addressed and some real advantages of certain singularities over others have been discussed. Constant, linear and quadratic elements have been compared. In addition, the problem areas in analysis and their effect on the resulting answer is presented.

RESUME

Dans ce rapport on fait la révision de la formulation mathématique et de quelques-uns des avantages et des emplois possibles de la méthode des éléments de borne. Le choix du problème de singularité est abordé et les avantages réels de certaines singularités sont discutés. Les éléments constants, linéaires et du second degré sont comparés. En plus les difficultés de l'analyse et leur influence sur le résultat sont présentées.

ABSTRACT

In this report, the mathematical formulation and some of the advantages and possible uses for the boundary element process have been reviewed. The choice of singularity problem has been addressed and some real advantages of certain singularities over others have been discussed. Constant, linear and quadratic elements have been compared. In addition, the problem areas in analysis and their effect on the resulting answer is presented.

RESUME

Dans ce rapport on fait la révision de la formulation mathématique et de quelques-uns des avantages et des emplois possibles de la méthode des éléments de borne. Le choix du problème de singularité est abordé et les avantages réels de certaines singularités sont discutés. Les éléments constants, linéaires et du second degré sont comparés. En plus les difficultés de l'analyse et leur influence sur le résultat sont présentées.

UNIVERSITY OF TORONTO GEOTECHNICAL REPORT

A report prepared for

Dr. A.M. Jessop,
Department of Energy, Mines and Resources,
Division of Seismology & Geothermal Studies,
Earth Physical Branch,
1 Observatory Crescent,
OTTAWA, Ontario. K1A 0Y3.

by

T.D. Wiles and Professor J.-C. Roegiers,
Department of Civil Engineering,
University of Toronto,
Galbraith Building,
35 St. George Street,
TORONTO, Ontario. M5S 1A4.

Contract Number: ISU80-0007 July 1980

FOREWORD

This report summarizes and discusses some of the advantages of the boundary integral method when applied to stress analysis. Various new developments are presented and discussed in detail as far as their advantages and disadvantages are concerned.

This report also contains the listing of the program developed at the University of Toronto and presents the various test cases which were run. It finally discusses the application to modelling of pressurized cracks.

Any opinions expressed in this report are those of the authors and the Earth Physics Branch takes no responsibility neither does it endorse the findings.

Toronto, July 1980.

T.D. Wiles
J.-C. Roegiers

TABLE OF CONTENTS

		Page
	SUMMARY	iv
	NOMENCLATURE	v
I.	INTRODUCTION	1
II.	THE BOUNDARY INTEGRAL METHOD	2
	2.1 Indirect Approach	2
	2.2 Direct Approach	5
III.	NUMERICAL APPROACH	8
IV.	SINGULAR SOLUTION	11
	4.1 Point Load on the Surface of Semi-Infinite Medium	11
	4.2 Point Load in an Infinite Medium	11
	4.3 Displacement Discontinuity in an Infinite Medium	13
	4.4 Discussion	15
V.	INTEGRATION OF SINGULAR SOLUTION	16
	5.1 Generalities	16
	5.2 Additional Considerations	19
VI.	TEST CASES	23
	6.1 Generalities	23
	6.2 Rigid Body Translation	24
	6.3 Pure Shear	26
	6.4 Homogeneous Compression	31
	6.5 Pressurized Cavities	36
	6.6 Beam with End Shear	36
	6.7 Pressurized Crack	41
VII.	A BIE PROGRAM	43
	7.1 Generalities	43
	7.2 Program Documentation	43
	7.3 Program Listing	46
	7.4 Sample Problem: Pressurized Crack	62
VIII.	CONCLUSIONS	65
IX.	BIBLIOGRAPHY	67

APPENDIX I

A	POINT LOAD SINGULARITY	
A.1	Constant Variation	
A.2	Linear Variation	
A.3	Quadratic Variation	

B DISCONTINUTIY IN DISPLACEMENT SINGULARITY

B.1 Constant Variations
B.2 Linear Variation
B.3 Quadratic Variation

SUMMARY

In this report, the mathematical formulation and some of the advantages and possible uses for the boundary element process have been reviewed. The choice of singularity problem has been addressed and some real advantages of certain singularities over others have been discussed. Constant, linear and quadratic elements have been compared. In addition, the problem areas in analysis and their effect on the resulting answer is presented.

NOMENCLATURE

σ_i	stress at point i
u_i	displacement at point i
q_j	force applied at point j
f_i^j	effect of unit point force or discontinuity applied at point j
$f\sigma_i^j$	on stress
fu_i^j	on displacement
b_k	body force applied at point k
ϕ_i	potential at point i
r_{ij}	distance between points i and j
∇^2	Laplace's operator
n	normal to the bounding surface
P	vector of interpolation values for q
$F\sigma_i$	matrix of coefficients expressing stress at point i due to the interpolated representation of q
Fu_i	matrix of coefficients expressing displacement at point i due to the interpolated representation of q
σ	stress vector at boundary interpolation points
u	displacement vector at boundary interpolation points
Q	work done
σ_x, σ_y, τ_{xy}	normal and shear stress components at point (x,y)
u, v	displacement components at point (x,y)
ν	Poisson's ratio
E	Young's modulus
G	shear modulus
X	point force or discontinutiy in x-direction
Y	point force or discontinuity in y-direction

Chapter I

INTRODUCTION

In recent years there has been great interest generated in numerical boundary integration approaches as applied to various engineering problems. The main reason for this interest being that only the boundary location and conditions need to be considered in most cases.

The amount of effort required to prepare the input for large problems is therefore small compared to other techniques such as Finite Elements or Finite Differences, because only the bounding surfaces need to be discretized. In addition, a small number of unknowns may only have to be considered for problems with boundaries extending to infinity.

The boundary integration may be conducted in several ways by making a variety of assumptions and approximations. In this report the point load singularity as well as the singular displacement discontinuity are compared for assumed constant, linear, and quadratic variations of the singularity along the boundary. Both closed form solutions and numerical integration approaches are considered.

CHAPTER II

THE BOUNDARY INTEGRAL METHOD

2.1 Indirect Approach

In order to understand the boundary integral approach, one need only to conceive the principle of superposition. To illustrate this, consider the two-dimensional(*) elasticity problem shown in Figure 2.1.

It is evident that there is no difference between systems (a) and (b), provided that the stress distribution due to the load P is removed by equal and opposite stresses around the apparent boundary. In order to remove these stresses, however, one must apply additional boundary forces, each of which will result in its own stresses at all points around the apparent boundary. Now, if a boundary force distribution

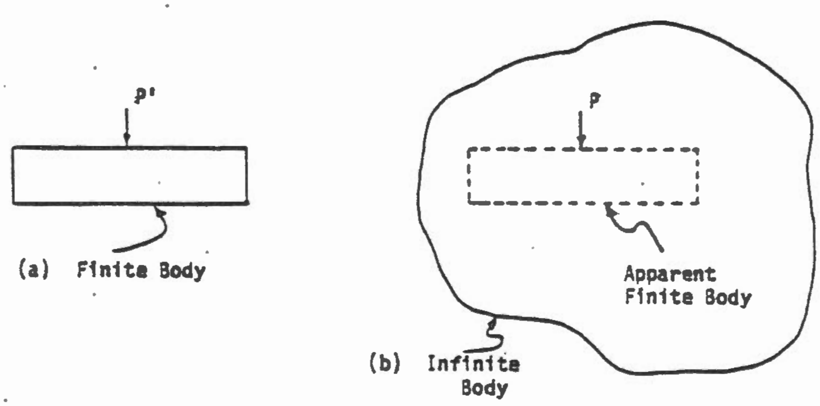


Figure 2.1

(*) Idential arguments can be used for any potential problem in two or three dimensions.

can be found which results in the desired boundary stresses, the overall solution for the stress field within the boundary can be obtained by superimposing(*) the effects of the entire boundary force distribution.

These considerations are equally valid when specifying stresses or displacements along any portion of the apparent boundary or even when considering cavities or inclusions in an infinite medium.

This method has been used to arrive at the closed form solutions for a concentrated force acting on a beam and also for the case of the diametral compression of a circular disk (Timoshenko and Goodier, 1970). Massonnet (1965) also used this approach to determine the force distribution required to obtain a resulting homogeneous compression of any finite body.

Superposition for any applied force distribution results in the following stress and displacement field

$$\sigma_i = \int q_j \, f\sigma_i^j \, dV \quad , \tag{2.1}$$

$$u_i = \int q_j \, fu_i^j \, dV \quad , \tag{2.2}$$

where:

σ_i represents the stress at any point i

u_i represents the displacement at any point i;

q_j represents the applied force at the point j;

(*) Superposition is permissible here as the geometry of the infinite medium does not change with any force application.

f_i^j represents the effect of a unit force applied at point j on the stress (f_σ) or displacement (f_u) component at point i;

dV represents the integral taken over all locations where q acts.

The applied force distribution can be broken down into the forces acting on the apparent boundary (i.e. surface forces), and the forces applied within the body (i.e. body forces) giving

$$\sigma_i = \oint q_j \; f\sigma_i^j \, ds + \oint b_k \; f\sigma_i^k \, dV \quad , \tag{2.3}$$

$$u_i = \oint q_j \; fu_i^j \, ds + \oint b_k \; fu_i^k \, dV \quad . \tag{2.4}$$

where:

$\oint ds$ represents the integral taken over the entire boundary;

q_j represents the force distribution applied at any location j on the apparent boundary;

$\oint dV$ represents the integral taken over the region where b_k acts;

b_k represents the body force distribution applied at any point within the body.

This approach amounts to the application of Green's function theorem. Numerous authors have obtained similar results using various mathematical arguments and have applied it to solve different problems.

Massonnet (1965), and Benjumea and Sikarskie (1972), for example, show that equations (2.1) and (2.2) are Fredholm's equations of the second kind due to the singular nature of the kernel "f". This makes these equations solvable since the effect of the singularity at the point at which it is applied becomes finite and hence can be removed from the integral. Benjumea and Sikarskie (1972) also provided solutions for a point load in a transversely anisotropic (sometimes called orthotropic) medium.

Crouch (1978) obtained the equations (2.1) and (2.2) by application of the Green's function theorem and presented solutions for a displacement discontinuity in both an isotropic and transversely anisotropic media.

2.2 Direct Approach

An alternate approach is available by use of the reciprocal theorem which states that

> "the work done by the forces of the first set acting through the displacements of the second set is equal to the work done by the forces of the second set acting through the displacements of the first set" (Jaeger and Cook, 1969).

If the first set is considered to be a unit point force applied at any point i on the boundary, this theorem can be written from the expression given by Timoshenko and Goodier (1970) as

$$\oint u_j \, f\sigma_j^i \, ds = \oint \sigma_j \, fu_j^i \, ds + \int b_k \, fu_k^i \, dV \quad , \tag{2.5}$$

where:

σ_j represents the stress at any point j on the boundary;

u_j represents the displacement at any point j on the boundary.

This equation is sufficient to solve for the boundary unknowns, but in order to determine stresses and displacements at other points inside the body the Somigliana identity must be used. The required relation can be expressed from the results given by Love (1927) as

$$u_i = -\oint u_j f\sigma_j^i \, ds + \oint \sigma_j \, fu_j^i \, ds + \oint b_k \, f\bar{u}_k^i \, dV \quad . \tag{2.6}$$

Stresses can then be obtained by differentiation and substitution into stress/strain relationships.

Rizzo (1967) and Cruse (1969) also obtained this result and proved that the equations are solvable due to the fact that the integrals are finite at the singularity locations. The integration can be conducted for any chosen singular solution. In this case, however, the remaining integrations must be taken in the sense of Cauchy's Principal Value.

Rizzo and Shippy (1968) extended this solution to include the effects of regular non-homogeneous elastic inclusions by evaluating the integrals around the countour of each inclusion.

Lachat and Watson (1976), applied this result to some three-dimensional problems by evaluating the integrals using numerical processes. Later on, Rizzo (1975) extended this boundary integral process to the solution of Laplace's equation, i.e.

$$\nabla^2\phi_u = 0 \quad , \tag{2.7}$$

where:

ϕ represents the potential at any point;

∇^2 represents Laplace's operator.

By introducing a (1/r) - singularity into Green's theorem the following, equation may be obtained,

$$\phi_i = \frac{1}{4\pi}\oint \left[\frac{1}{r_{ij}}\frac{\partial\phi_j}{\partial n} - \phi_j \frac{\partial(1/r_{ij})}{\partial n}\right] ds \quad , \tag{2.8}$$

where:

r_{ij} represents the distance between points i and j;

ϕ_j represents the potential at any point j on the surface s;

$\int \phi ds$ represents the integration along the entire bounding surface;

$\frac{\partial \phi}{\partial n}$ represents the gradient of ϕ taken in the direction normal to the bounding surface.

This equation can be used in its present form. Rizzo (1975) shows that it is solvable since the integral is finite when evaluated at the singularity location. A similar result was obtained by Brebbia and Dominguez (1977).

Brebbia (1978) applied equations(2.5) to (2.8) to a variety of problems considering both isotropic and orthotropic material properties. In addition he extended equation (2.8) to the more general case of Poisson's equation(*). All of Brebbia's equations are derived using the weighted residual method or the principle of virtual work. Numerical integretation processes are used throughout his work.

(*) $\nabla^2\phi = f(x,y)$

CHAPTER III

NUMERICAL APPROACH

In order to evaluate the system of integral equations (2.3) and (2.4) or (2.5) and (2.6) the boundary can be discretized into several finite line segements (neglecting body forces) giving for equation (2.3)

$$\sigma_i = \sum_{k=1}^{n} \int_k q_j f\sigma_i^j \, ds \quad , \qquad (3.1)$$

where the sum is taken over all boundary elements, and the integrals are evaluated for each segment of the boundary. This equation is exact and numerical approximation is only introduced at this point if the boundary segments do not exactly represent the actual boundary shape.

Integration of the n integrals above is still required. If some distribution for q on each each element is assumed, these can then be evaluated either numerically or following a close form solution.

If some distribution for q on each element is assumed and expressed as a function of the value of q at discrete interpolation points, the result will be of the form

$$\sigma_i = \sum_{k=1}^{n} F\sigma_i \; P \quad . \qquad (3.2)$$

where:

- P represents a vector of interpolation values for q;
- $F\sigma_i$ represents a vector of coefficients expressing the stress at point i due to the interpolation representation of q.

A similar result can be found for the displacements, i.e.

$$u_i = \sum_{k=1}^{n} Fu_i \, P \quad . \tag{3.3}$$

Equations (3.2) and (3.3) can then be evaluated at the location of the apparent boundary, giving

$$\sigma = F_\sigma P \quad , \tag{3.4}$$

$$u = F_u P \quad , \tag{3.5}$$

where:

σ and u represent vectors of values of stress and displacement at the boundary interpolation points;

$F\sigma$ and Fu represent matrices of coefficients expressing the stress and displacement at each boundary interpolation point as a function of the interpolated representation of q.

Considering n boundary points with two degrees of freedom at each point, there are consequently a total of 2n values for each combination of σ, u and P. Since there are 4n equations, therefore 2n unknowns must be supplied to solve the problem.

Equation (2.5) can be set up in the same way by assuming some variations for σ and u on each element, the result will be of the form

$$Fu^T \sigma = F\sigma^T u \quad , \tag{3.6}$$

Where Fu^T and $F\sigma^T$ are the transposes of the same matrices as in equations (3.4) and (3.5).

By substitution of equations (3.4) and (3.5) into the expression (3.6) it can easily be found that

$$Q = Fu^T F\sigma P = F\sigma^T Fu P \quad , \qquad (3.7)$$

Where Q is the work done in the reciprocal theorem for each element.

This relation requires that the product $Fu^T F\sigma$ by symmetric.

The body forces can be included by conducting the required volume integration in equations (2.3) through (2.5), similarly to the approach used in the Finite Element technique.

CHAPTER IV

SINGULAR SOLUTIONS

Although the preceeding discussion has been concerned with the effect of a point load in an infinite medium, the entire discussion is equally valid for many other singular solutions. The body force integrals must use the point load singularity however.

Various authors have used a variety of singularities including a point load on the surface of a semi-infinite medium, point loads in an infinite medium, and discontinuities in displacement in infinite and semi-infinite media. Anisotropic cases have been treated as well.

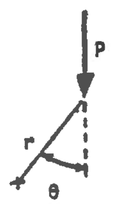
A few of these singularities are discussed below.

4.1 Point Load on the Surface of Semi-Infinite Medium

This singular solution is exceedingly simple for the isotropic case and its closed form solution has been used by Massonnet (1965). It should however be pointed out that since the point load has no effect above the surface of the semi-infinite medium its solution cannot be used in cases where there are reentrant corners or cavities in the geometry. Benjumea and Sikarskie (1972) also mentioned that the use of this singularity is more complex in the anisotropic case than other more useful singularities.

4.2 Point Load in an Infinite Medium

This is by far the most widely used singularity in boundary integral methods. While this singularity is somewhat simpler than a discontinuity in displacement, it has certain drawbacks due to its poor accommodation of rigid body motion and non-zero displacements at infinity. In addition, while well-defined shapes are easily modelled, openings such as mathematically flat cracks cannot be accommodated due to the fact that two point loads acting in opposite direction at the same point simply cancel one another rather than stressing the medium. The result for a point load in an infinite medium is given by Timoshenko and Goodier (1970) for the plane stress situation as



$$\sigma_r = \frac{-(3+\nu)\,P\cos\theta}{4\pi r},$$

$$\sigma_\theta = \frac{(1-\nu)\,P\cos\theta}{4\pi r}, \qquad (4.1)$$

$$\tau_{r\theta} = \frac{(1-\nu)\,P\sin\theta}{4\pi r}.$$

Figure 4.1 Point Load in an Infinite Medium

For the plane strain situation, the complete solution for applied forces in the x and y-directions can be found to be

$$\sigma_x = \frac{X}{4\pi(1-\nu)}\left[\frac{2xy^2}{r^4} - \frac{(3-2\nu)\,x}{r^2}\right] + \frac{Y}{4\pi(1-\nu)}\left[\frac{2y^3}{r^4} - \frac{(1+2\nu)\,y}{r^2}\right],$$

$$\sigma_y = \frac{X}{4\pi(1-\nu)}\left[-\frac{2xy^2}{r^4} + \frac{(1-2\nu)x}{r^2}\right] + \frac{Y}{4\pi(1-\nu)}\left[-\frac{2y^3}{r^4} - \frac{(1-2\nu)y}{r^2}\right],$$

$$\tau_{xy} = \frac{X}{4\pi(1-\nu)}\left[\frac{2y^3}{r^4} - \frac{(3-2\nu)y}{r^2}\right] + \frac{Y}{4\pi(1-\nu)}\left[-\frac{2xy^2}{r^4} - \frac{(1-2\nu)\,x}{r^2}\right],$$

$$u = \frac{X}{4\pi E(1-\nu)}\left[-\frac{(1+\nu)y^2}{r^2} - (3-\nu-4\nu^2)\,\ell n\, r\right] + \frac{Y}{4\pi E}\,\frac{1+\nu}{1-\nu}\,\frac{xy}{r^2}, \qquad (4.2)$$

$$v = \frac{X}{4\pi E}\,\frac{1+\nu}{1-\nu}\,\frac{xy}{r^2} + \frac{Y}{4\pi E(1-\nu)}\left[-\frac{(1+\nu)x^2}{r^2} - (3-\nu-4\nu^2)\,\ell n\, r\right],$$

$$r = (x^2+y^2)^{\frac{1}{2}}$$

where:

> X and Y are point forces applied at the origin;
>
> $\sigma_x, \sigma_y, \tau_{xy}$ are the stress components at the point (x,y);
>
> u and v are respectively the displacement in the x and y-directions at (x,y);
>
> ν is Poisson's ratio;
>
> E is Young's modulus

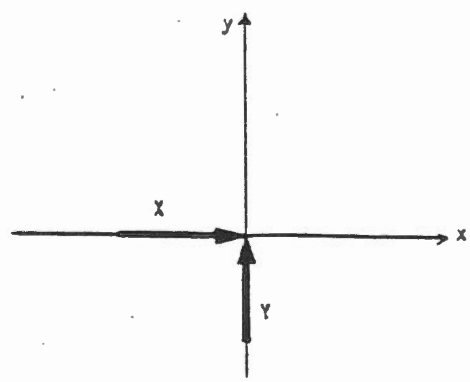


Figure 4.2 Applied Forces in the x and y-directions in an Infinite Medium

It should be noted that although the stresses become zero as r approaches infinity, the displacements do not vanish. Also, the addition of arbitrary rigid body terms will not affect the solution.

4.3 Displacement Disctontinuity in an Infinite Medium

Crouch (1976) suggested using the solution for a constant discontinuity in displacement. The singular solution for this case can be obtained by differentiation of Crouch's constant displacement discontinuity, or can be deduced from the solution for a point load in an infinite medium by setting

$$
\begin{aligned}
u(X) &= -\tau_{xy}(X) \quad , \\
u(Y) &= -\sigma_y(X) \quad , \\
v(X) &= -\tau_{xy}(Y) \quad , \\
v(Y) &= -\sigma_y(Y) \quad ,
\end{aligned}
\tag{4.3}
$$

where:

u and v are the displacements due to point discontinuities at the origin;

τ_{xy} and σ_y are the shear and normal stresses due to point loads at the origin.

The solution can be found to satisfy the equilibrium and compatibility conditions and is given by

$$
\begin{aligned}
\sigma_x &= \frac{XG}{2\pi(1-\nu)} \left(\frac{2y^3x - 6yx^3}{r^6}\right) + \frac{YG}{2\pi(1-\nu)} \left(\frac{x^4 + y^4 - 6x^2y^2}{r^6}\right) \quad , \\
\sigma_y &= \frac{XG}{2\pi(1-\nu)} \left(\frac{2x^3y - 6xy^3}{r^6}\right) + \frac{YG}{2\pi(1-\nu)} \left(\frac{x^4 - 3y^4 + 6x^2y^2}{r^6}\right) \quad , \\
\tau_{xy} &= \frac{XG}{2\pi(1-\nu)} \left(\frac{x^4 + y^4 - 6x^2y^2}{r^6}\right) + \frac{YG}{2\pi(1-\nu)} \left(\frac{2x^3y - 6xy^3}{r^6}\right) \quad , \\
u &= \frac{X}{4\pi(1-\nu)} \left[-\frac{2y^3}{r^4} + \frac{(3-2\nu)\ y}{r^2}\right] + \frac{Y}{4\pi(1-\nu)} \left[\frac{2y^2x}{r^4} - \frac{(1-2\nu)\ x}{r^2}\right] \quad , \\
v &= \frac{X}{4\pi(1-\nu)} \left[\frac{2y^2x}{r^4} + \frac{(1-2\nu)\ x}{r^2}\right] + \frac{Y}{4\pi(1-\nu)} \left[\frac{2y^3}{r^4} + \frac{(1-2\nu)y}{r^2}\right] \quad ,
\end{aligned}
\tag{4.4}
$$

where:

X and Y are points discontinuities in displacement at the origin;

$\sigma_x, \sigma_y, \tau_{xy}, u, v$ are the stresses and displacements at point (x,y);

G is the shear modulus.

It should be noted that the stresses and displacements all tend to be zero as r approaches infinity. Again, the addition of arbitrary rigid body terms will not affect the solution.

Although this solution can be used with either the direct (equations 2.5 and 2.6) or indirect (eouations 2.3 and 2.4) approaches, it will generally be most desirable to use the indirect approach as values of the discontinuity in displacement are required when modelling mathematically flat cracks.

4.4 Discussion

It can be observed in both systems of equations (4.2) and (4.4) that the stresses depend on the elastic constants. This is a result of the fact that the infinite medium in which the singularities are imposed is a multiple-connected body. That is, a section can be cut from the from the application point of the singularity to infinity without dividing the body into two parts. The actual general solution is not single-valued unless the initial stresses due to any dislocation along such a cut is specified. The solutions used here have assumed that all such dislocations are non-existent.

CHAPTER V

INTEGRATION OF SINGULAR SOLUTIONS

5.1 Generalities

Numerical integration of the singular solutions is straightforward at any location (x,y) except at the point (0,0). Special forms of numerical integration formulae which can accommodate certain types of singularities could be used, although for straight elements the integrals are easily evaluated in closed form. Numerical integration is however mandatory if one is to consider curved elements.

Polynomial interpolation of the q distribution in equations (2.3) and (2.4) requires tha q be represented by discrete values at points along the integration interval. It can easily be found that for a linear variation of q the result is given by

$$q(x') = \frac{P_2}{2b}\left[(b+x) + (x'-x)\right] + \frac{P_1}{2b}\left[(b-x) - (x'-x)\right] , \qquad (5.1)$$

where:

x' is the location of q in the (x,y) space;

$\pm b$ are the locations of the interpolation points.

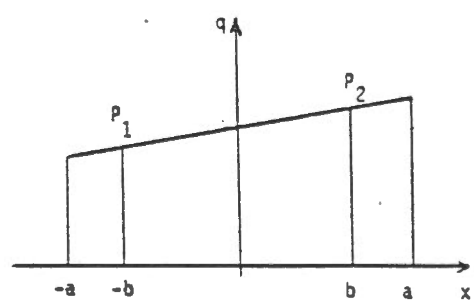


Figure 5.1 Linear variation of q

If a quadratic variation of q is assumed the result is slightly more complex

$$
\begin{aligned}
q(x') = \frac{P_1}{2b^2} & \left[(x'-x)^2 + (2x-b)(x'-x) + x(x-b)\right] \\
& + \frac{P_2}{b^2}\left[-(x'-x)^2 - 2x(x'-x) + (b+x)(b-x)\right] \\
& + \frac{P_3}{2b^2}\left[(x'-x)^2 + (2x+b)(x'-x) + x(x+b)\right] .
\end{aligned}
\tag{5.2}
$$

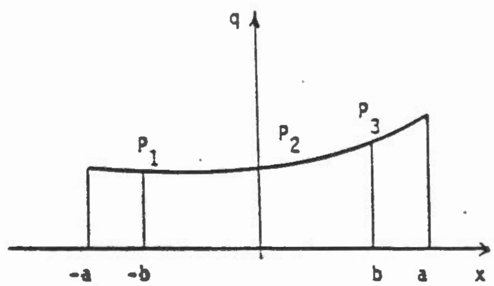


Figure 5.2 Quadratic variation of q

The required integrals in equations (2.3) and (2.4) are

$$
\begin{aligned}
& \int_{-a}^{a} f \, dx' , \\
& \int_{-a}^{a} (x'-x) f \, dx' , \\
& \int_{-a}^{a} (x'-x)^2 f \, dx' ,
\end{aligned}
\tag{5.3}
$$

where f is given by either of the singular solutions (equations 4.2 or 4.4) with (x-x') substituted for x. The results of these integrations are given in Appendix I.

These integrals can then be substituted into the interpolation relationships giving the required expressions for the stresses and displacements.

The choice of the location of the interpolation points must be made such that the error due to polynominal interpolation is minimized. This can be ensured by choosing the zeros of Chebyshev's polynominals (Hornbeck, 1975)

$$x = -a \cos \left[\left(\frac{2n+1}{2n+2} \right) \pi \right] \quad , \qquad (5.4)$$

where:

(n + 1) is the number of interpolation points;

m = 0, 1, 2,···, n is the point number.

The same result can be found for equations (2.5) and (2.6) by interpolating σ and u as polynomials. In this case the distribution of σ and u are well-defined as n^{th} order polynominals and consistent values for distributed loadings are easily arrived at. Unfortunately, for the first case (equations 2.3 and 2.4) the surface stress and displacement variations are not defined at all. For both the point load and the displacement discontinuity singularities the stresses become infinite at the ends of the elements if q is not continuous. This may result in difficulties when applying surface loads which are actually equivalent to the desired loads.

5.2 Additional Considerations

(i) Symmetry

When dealing with symmetrical problems stresses and displacements need only be solved over a portion of the total body. In order for symmetry to exist it is also necessary that portions of q (see equations (2.3) through (2.5)) have the same value. Thus in equations (3.4) and (3.5) the number of unknowns can be restricted to those in one of the symmetric portions.

Considering for example the following problem with one line of symmetry.

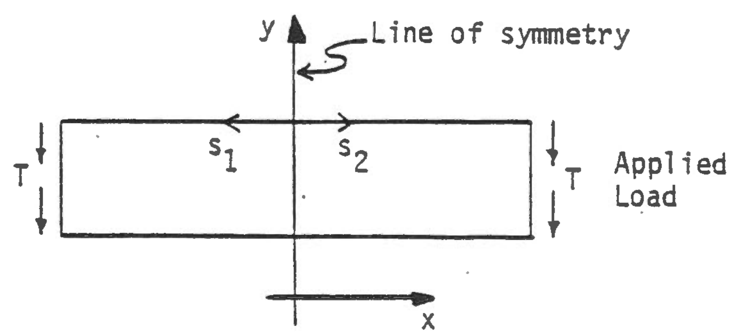


Figure 5.3 Symmetric Problem

The number of boundary conditions can be reduced since

$$\begin{aligned} q_y(s_2) &= q_y(s_1) \quad , \\ q_x(s_2) &= -q_x(s_1) \quad . \end{aligned} \tag{5.5}$$

It should be noted that integration along the surface proceeds in opposite directions. This will affect the results of the integrations in equation (3.1) which must then be evaluated for the q distribution around the entire boundary. However, from the form of equations (5.5) it can be seen that the coefficients evaluated along the symmetric portions of s_1 and s_2 can be added together. For one line of symmetry this results in solving only half of the original equations.

Although F in equations (3.4) and (3.5) is one quarter its original size, one-half as many coefficients need to be evaluated since each coefficient in F represents the sum of two soefficients, one for s_1 and one for s_2.

If two lines of symmetry were to be present the problem would even be further simplified. Even though F would then be one-sixteenth its original size, only one-quarter as many coefficients would need to be evaluated since each coefficient in F represents the sum of four coefficients.

(ii) Initial Stress, Initial Strains and Body Forces

Any field stresses which do not vary with position can be treated as initial stresses by superimposing the solution for an equal and opposite pressure distribution on the boundary. This amounts to subtracting the field components from the stresses everywhere to obtain an equivalent problem with zero field stresses and some modified boundary stresses. After solving this problem the field stresses need to be added back to obtain the solution of the original problem.

Body forces can be directly incorporated as indicated in equations (2.3) to (2.5) by conducting the volume integrations. This will in general require full discrtization of the loaded region.

However, the exact solution of some problems may require integrations extending to infinity in certain directions. This situation can be simplified by considering the problem of body forces applied everywhere (within the bounding surface too), then removing these forces in the bounded area and adding on the effect of any local variations.

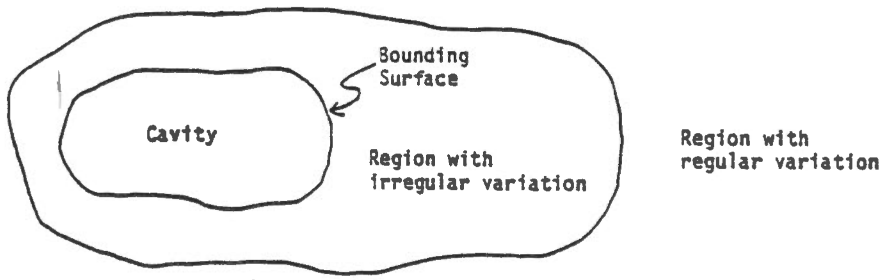


Figure 5.4 Body Forces

Initial stresses and initial strains which vary with position can be incorporated as body force contributions by the method of strain suppression (Timeshenko and Goodier, 1970).

As in the case of body forces, the initial stresses and initial strains will be regular everywhere except in the immediate region of the bounding surface. Therefore, the affects of these local perturbations can be included by considering the effect of initial homogeneous values applied everywhere, then removing the effect within the bounded area and adding any local perturbations.

(iii) Multiple Surfaces

Multiple-connected bodies or multiple openings in an infinite medium can be accommodated by simply considering several separate surfaces simultaneously.

(iv) Multiple Materials

Rizzo and Shippy (1968) also consider non-homogeneous inclusions in an infinite medium. They show that an inclusion of different material properties can be included by solving two problems at the same time. The stresses at the boundary of the inclusion must be in equilibrium with the stresses on the surface of the bounding medium and compatibility or known discontinuities in displacement must be required at the interface surface. A layered medium can be modelled by carrying the interface surface far away from the region of interest.

CHAPTER VI

TEST CASES

6.1 Generalities

Several test cases were analyzed in order to investigate the performance of each singularity. Some singularities were observed to perform better than the others depending on the case considered. The influence of the order of the polynominal interpolation was also compared.

All generated data was evaluated based on the total number of unknowns used to analyze the problem. From now on, the abbreviation PL will be used to represent the point load singularity while DD will represent the displacement discontinuity. With the exception of the beam example (6.6), Young's modulus was set at 10^6 while Poisson's ratio was kept at 1/4. Plane strain conditions were also assumed. All runs were conducted on the University of Toronto IBM 370 computer.

For each test case, a few runs were duplicated using both single (7 significant digits) and double (15 significant digits) precision to try and detect any numerical error. Single precision was found to give accurate results up to about 25 unknowns. In most examples, shapes with straight sides were used so that no error due to the approximation of the boundary shape was introduced. In all cases the closed form integrated solutions were used for the analyses and the problems were solved using Gaussian elimination.

The percentage error in displacements and stresses show identical trends and have almost the same magnitude at all points. The maximum errors occur nearest any non-straight points on the boundary. However, the large errors associated with these corner areas is confined to the immediate vicinity of the kink in the boundary. It should be pointed out, however, that a large error was noted in the tangential boundary stresses for constant elements. This error was reduced by more than an order of magnitude by using higher order elements. It remained, however, considerably higher than for the interior stress values. It should be noted that the surface displacement values show approximately the same error as the interior displacements and stresses.

6.2 Rigid Body Translation

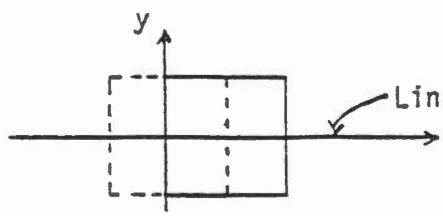


Figure 6.1 Rigid Body Translation

Consider a 2x2 square prism translated by one unit in the positive x-direction. The analysis using the DD singularity resulted in uniform translation of the body with no error in the displacement distribution. In addition, the related stresses (in the range of $uxE/10^5$ for single precision and $uxE/10^{14}$ for double precision) were within the limits of the numerical accuracy of the computer.

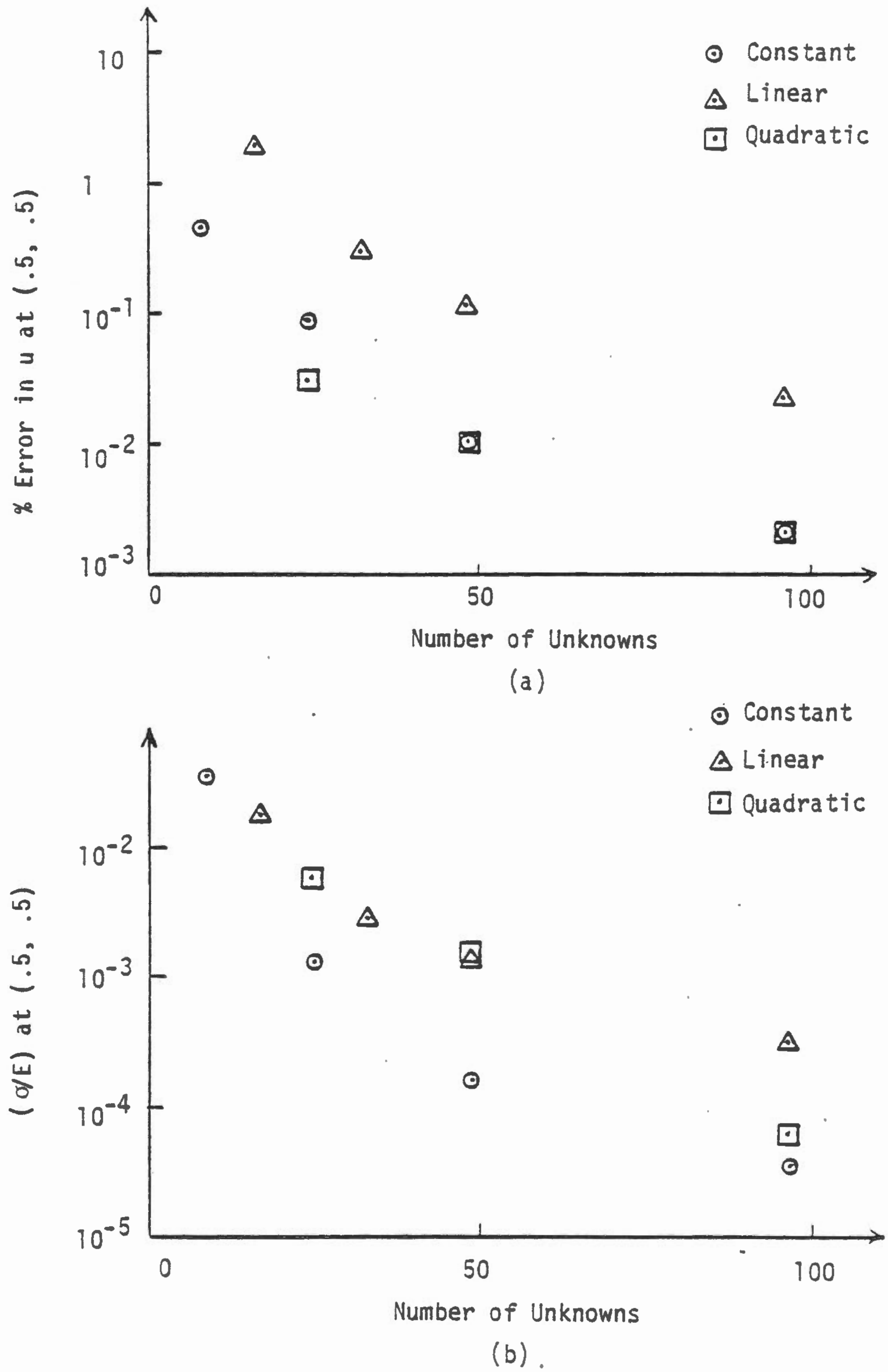


FIGURE 6.2 RIGID BODY TRANSLATION (PL)

Some results for the PL singularity are presented in Figures 6.2. It can be observed that all stresses and displacements, except in the corner areas, are approaching the correct values with an increase in the considered number of unknowns. The stresses in the corner area are approximately equal to u x E.

Love (1927) presented a technique for separating the rigid body component which would alleviate this problem. His approach was not introduced in the program at this stage since rigid body motions would be of second order importance in the scope of this research.

6.3 Pure Shear

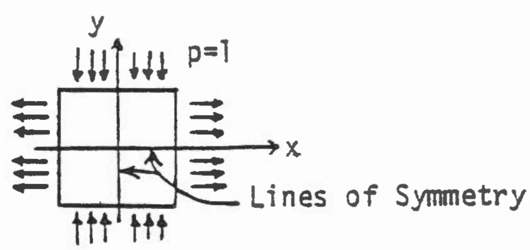
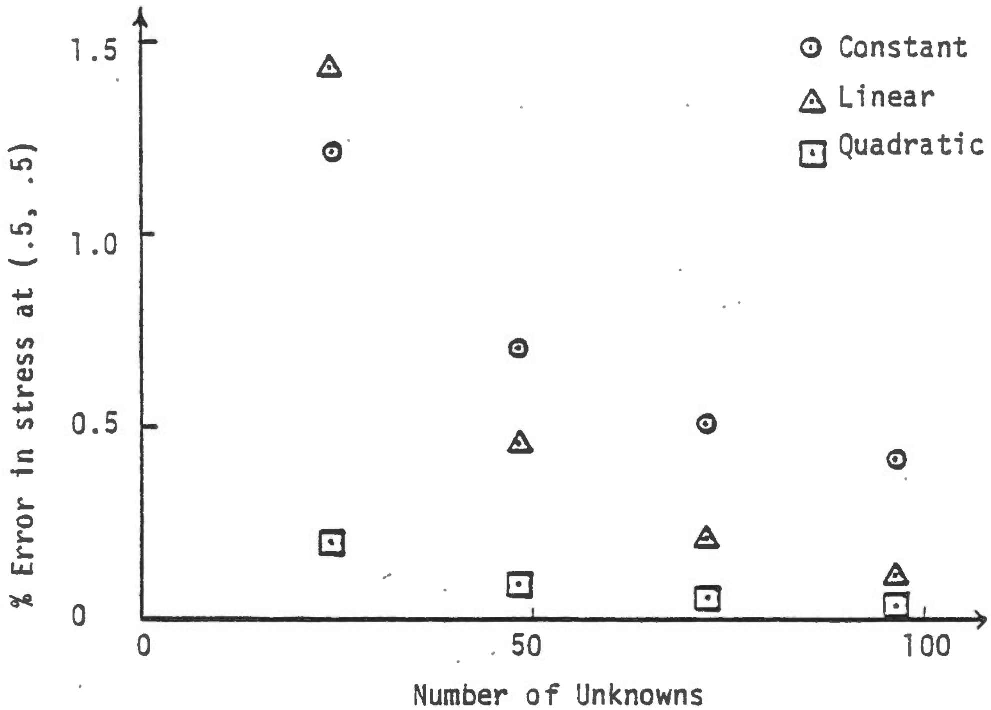


Figure 6.3 Pure Shear Loading

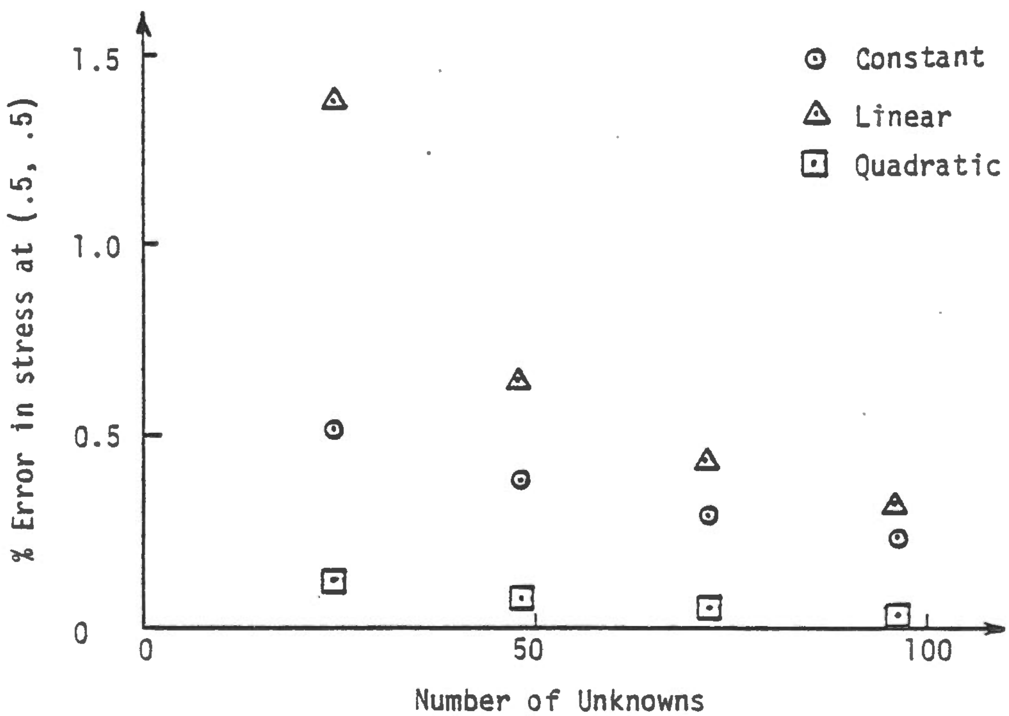
A 2x2 square prism was loaded in pure shear by a unit stress as shown in Figure 6.3.

Figures 6.4 show results comparing the element order for both singularity types. Neither singularity offers any significant improvement in accuracy. The quadratic elements give consistently better accuracy.

The stress and displacement distributions along the x-axis are shown in Figure 6.5. The error at mid-side (1,0) for the case of the tangential stress is shown in Figure 6.6. As mentioned previously, the use of quadratic elements gives a much better correlation. Figure 6.7 represents

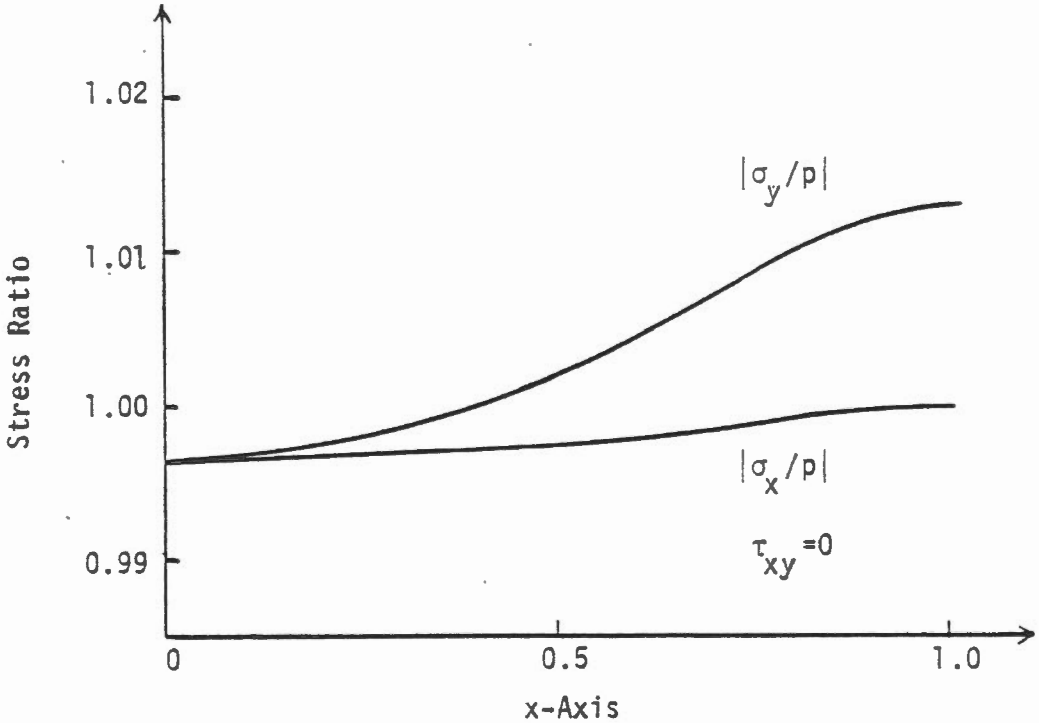


(a) PL

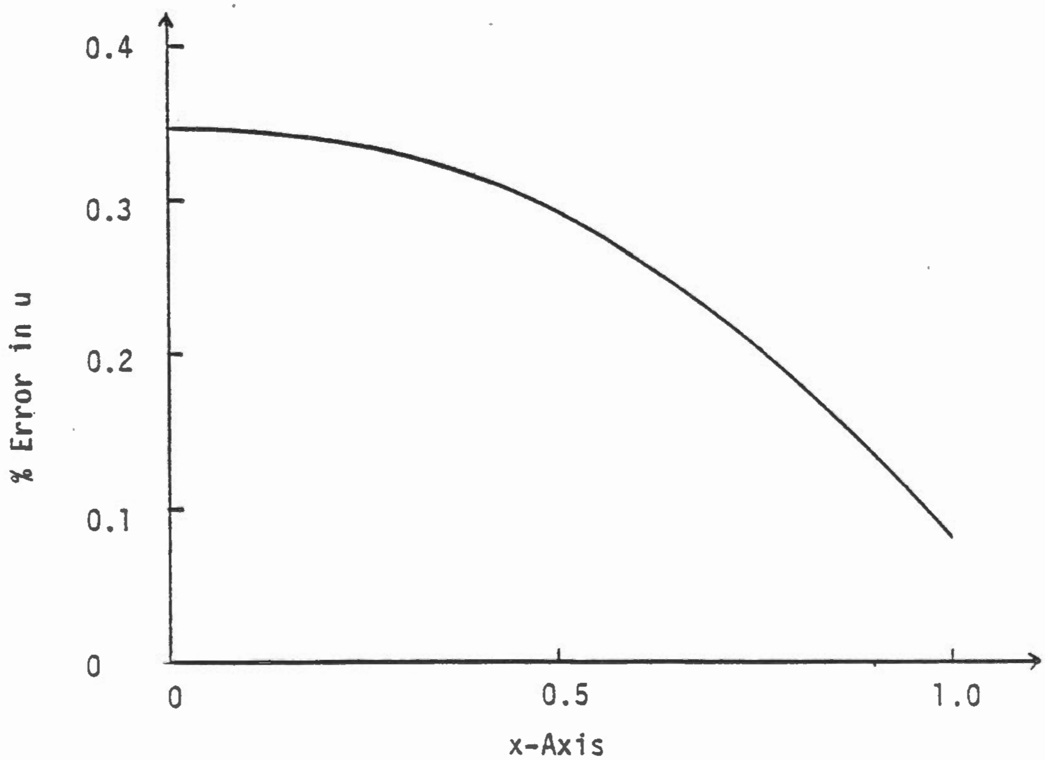


(b) DD

FIGURE 6.4 PURE SHEAR



(a) Stresses on x-Axis



(b) Displacement on x-Axis

FIGURE 6.5 PURE SHEAR (DD), 8 QUADRATIC ELEMENTS, 2 LINES OF SYMMETRY

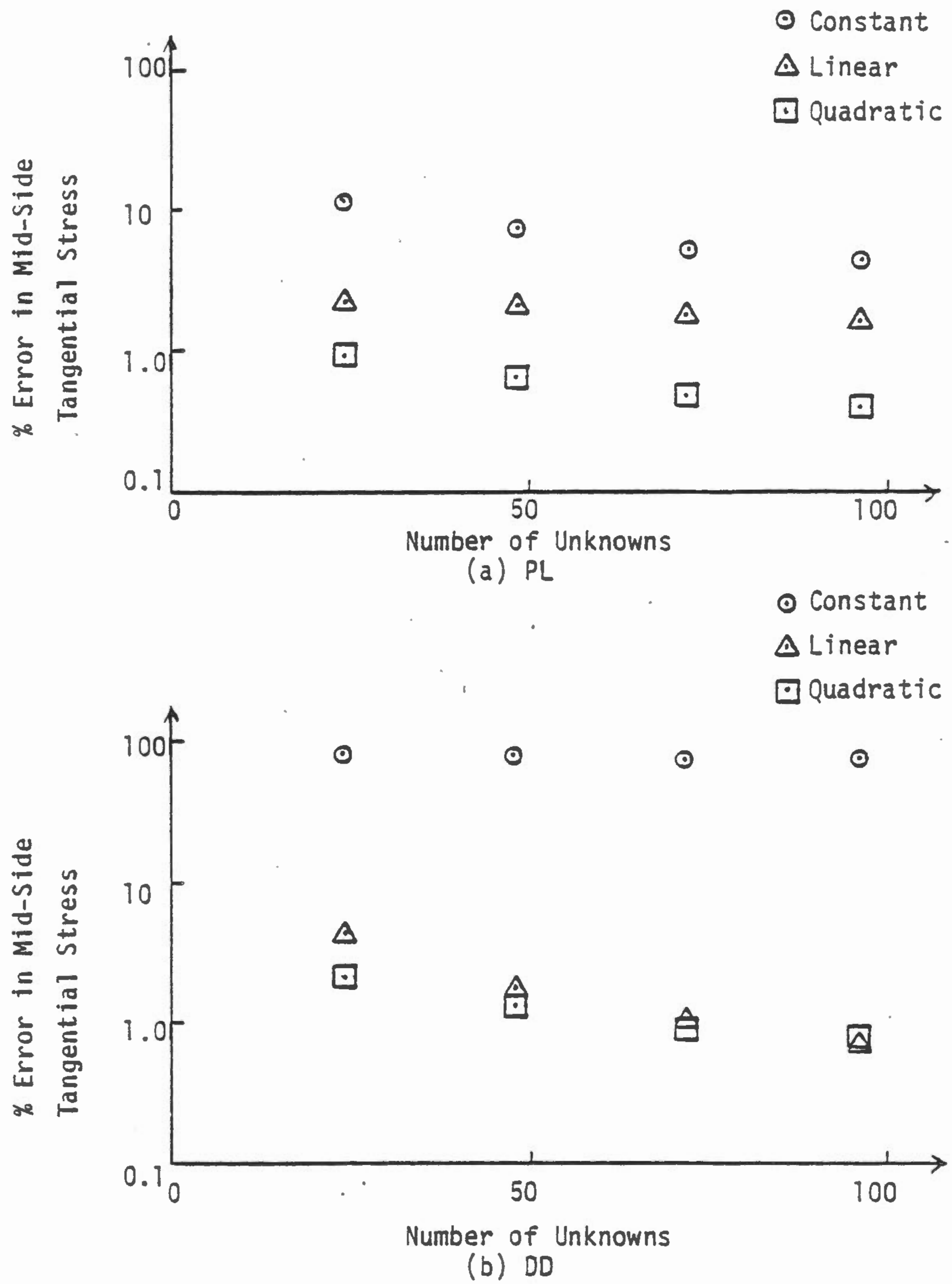
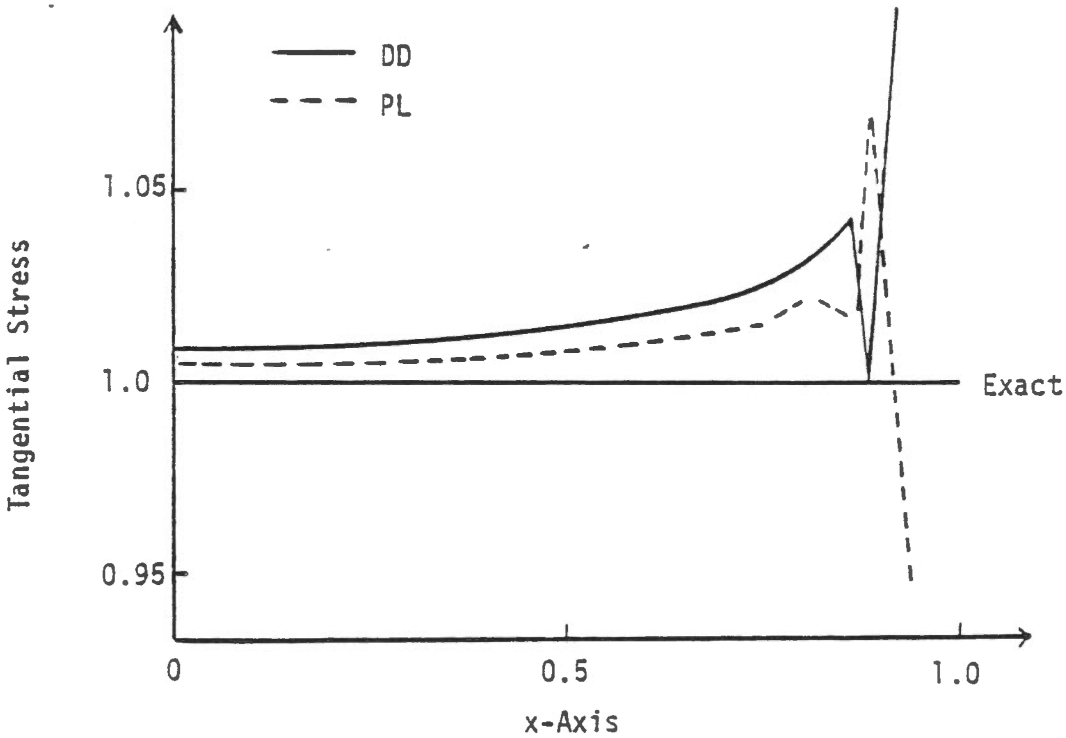
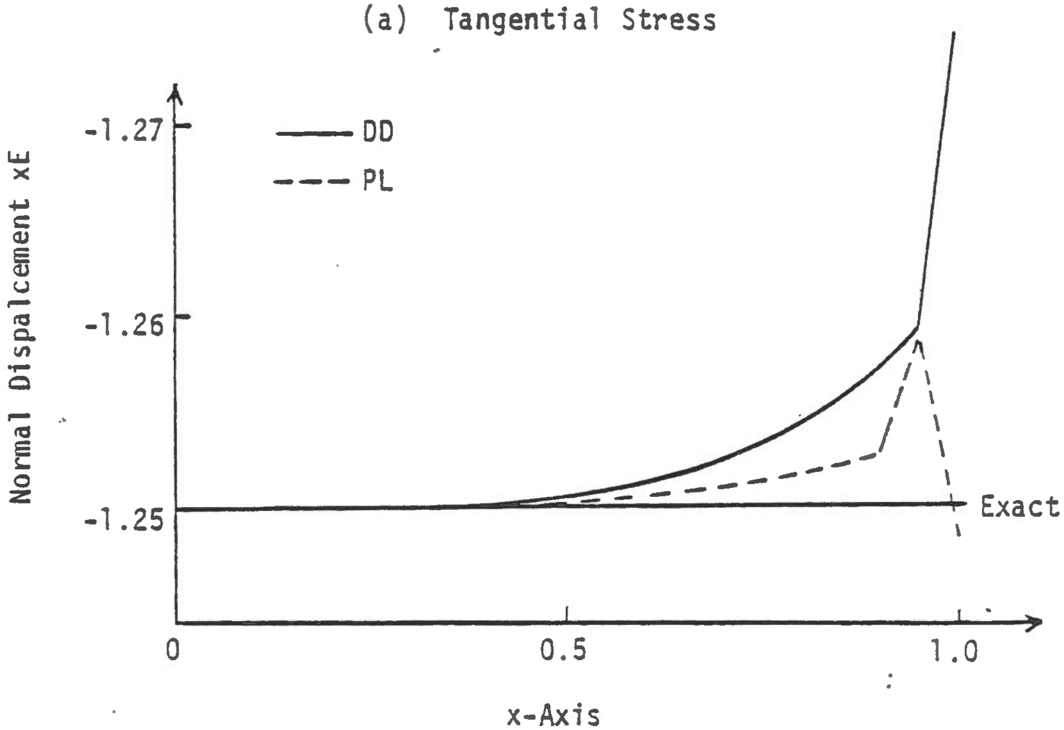


FIGURE 6.6 PURE SHEAR, MID-SIDE TANGENTIAL STRESS



(a) Tangential Stress



(b) Normal Displacement

FIGURE 6.7 PURE SHEAR, 16 QUADRATIC ELEMENTS the surface stresses and displacement distributions along y=1. It can be observed that the poor results are confined to the first few corner nodes. The PL singularity exhibits the best behaviour.

6.4 Homogeneous Compression

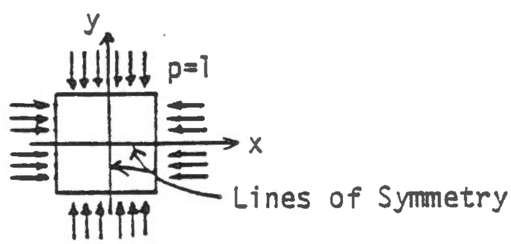


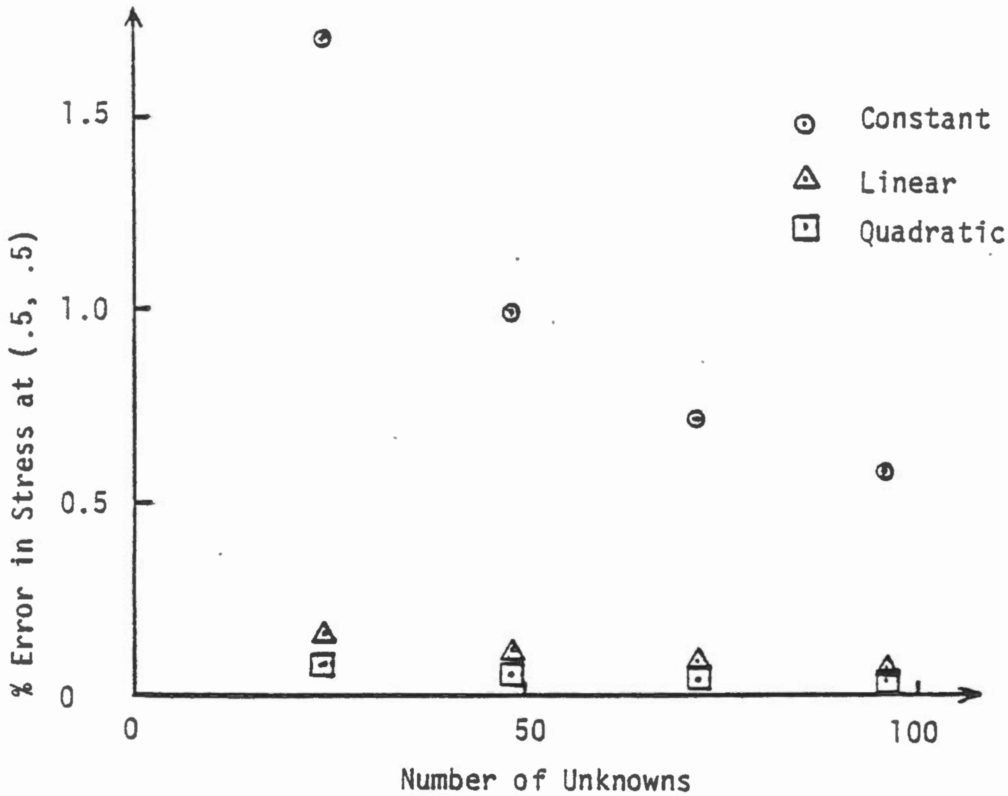
Figure 6.8 Case of Homogeneous Compression

Consider the same square prism submitted to a unit hydrostatic compression (Figure 6.8).

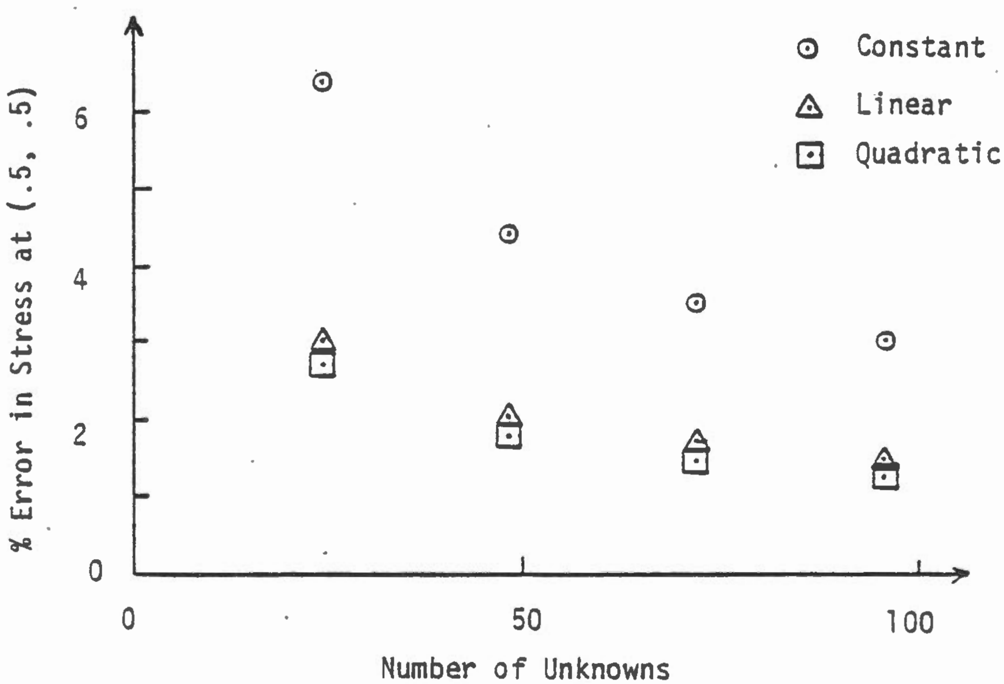
The Results obtained for this case are almost identical to the previous example of pure shear.

As can be seen from the following figures the PL-singularity gave results with significantly smaller errors than the DD-singularity. Also, quadratic and linear elements gave better results compared to constant elements.

In Figures 6.11 the mid-side (1,0) tangential stress is compared for various element orders and show that for both singularities a large improvement in accuracy results by use of higher order elements. The surface displacement and stresses along y=1 are plotted in figures 5.12 and show that the large errors again are confined to the corner areas.

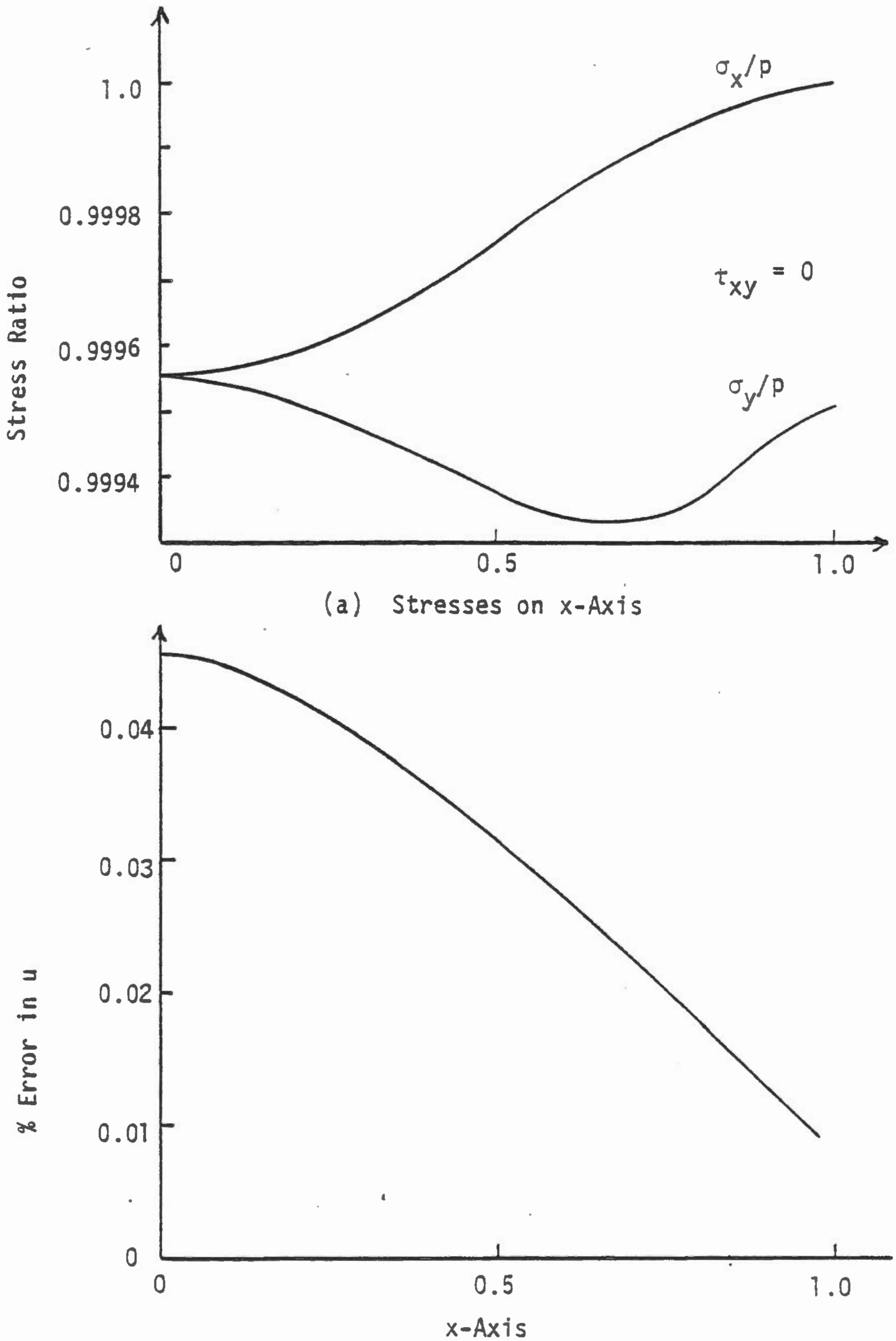


(a) PL



(b) DD

FIGURE 6.9 HOMOGENEOUS COMPRESSION



(b) Displacement on x-Axis

FIGURE 6.10 HOMOGENEOUS COMPRESSION (PL),
8 QUADRATIC ELEMENTS

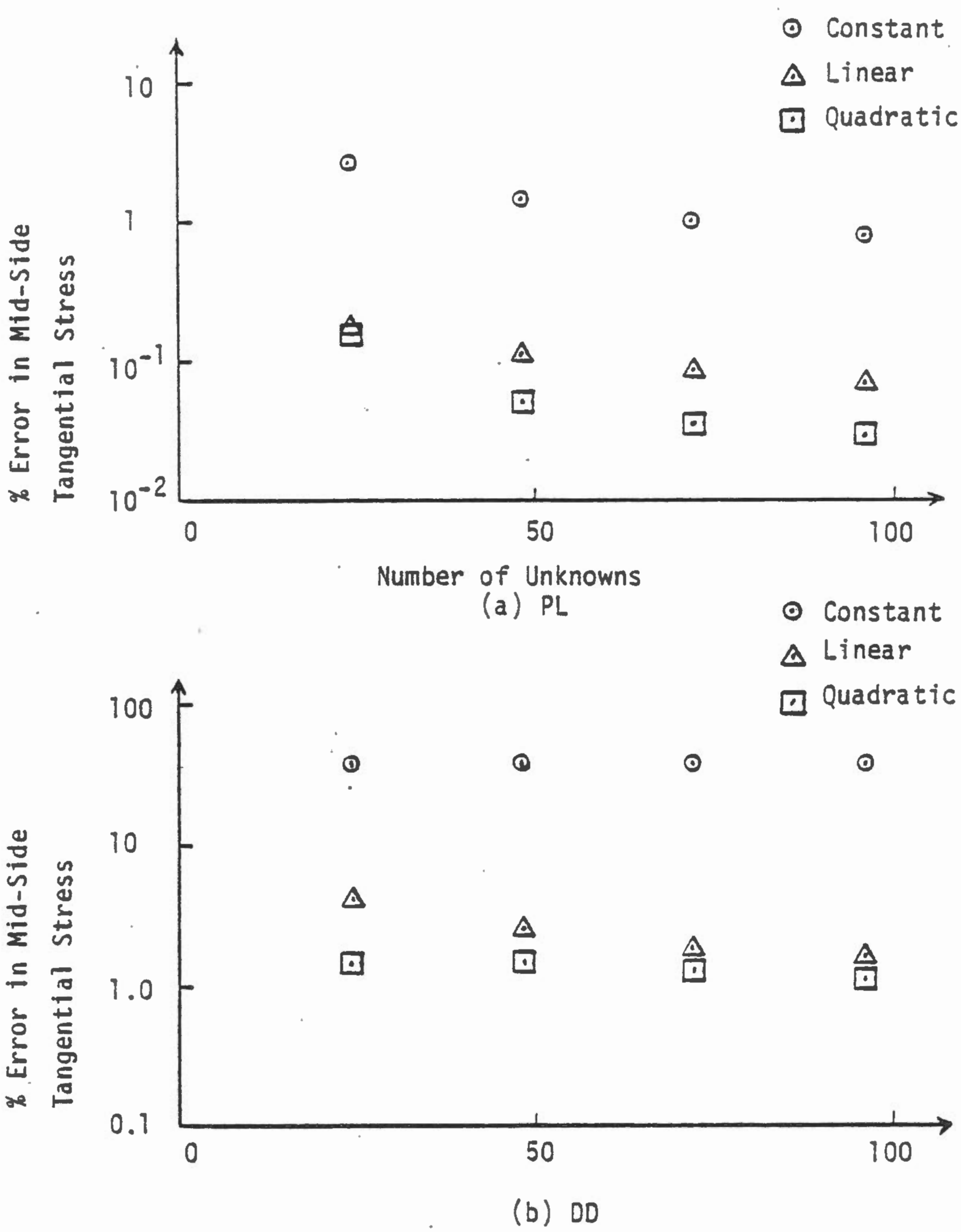
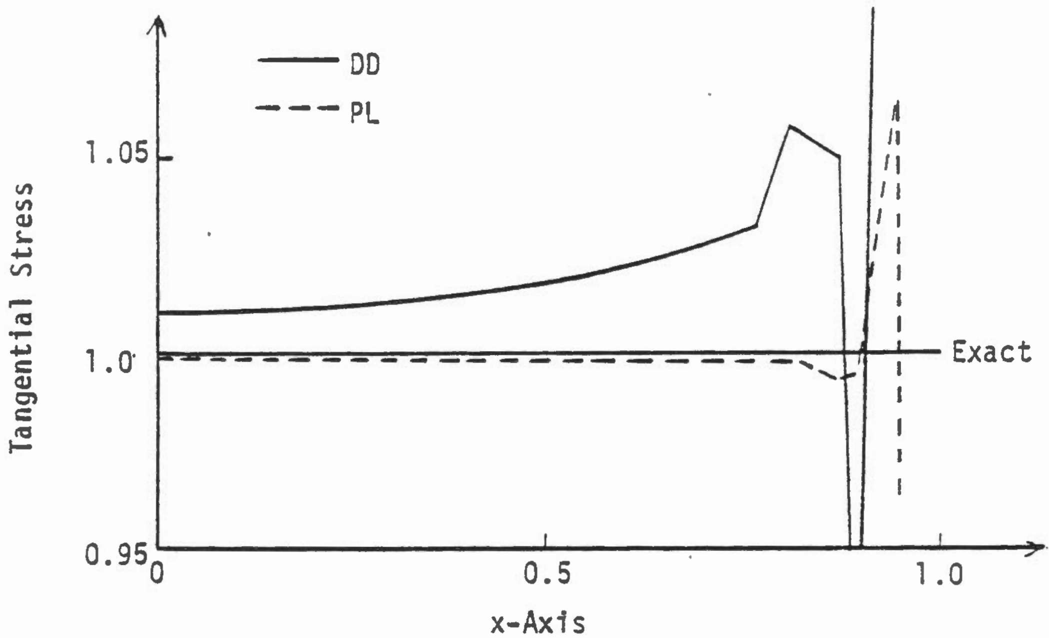
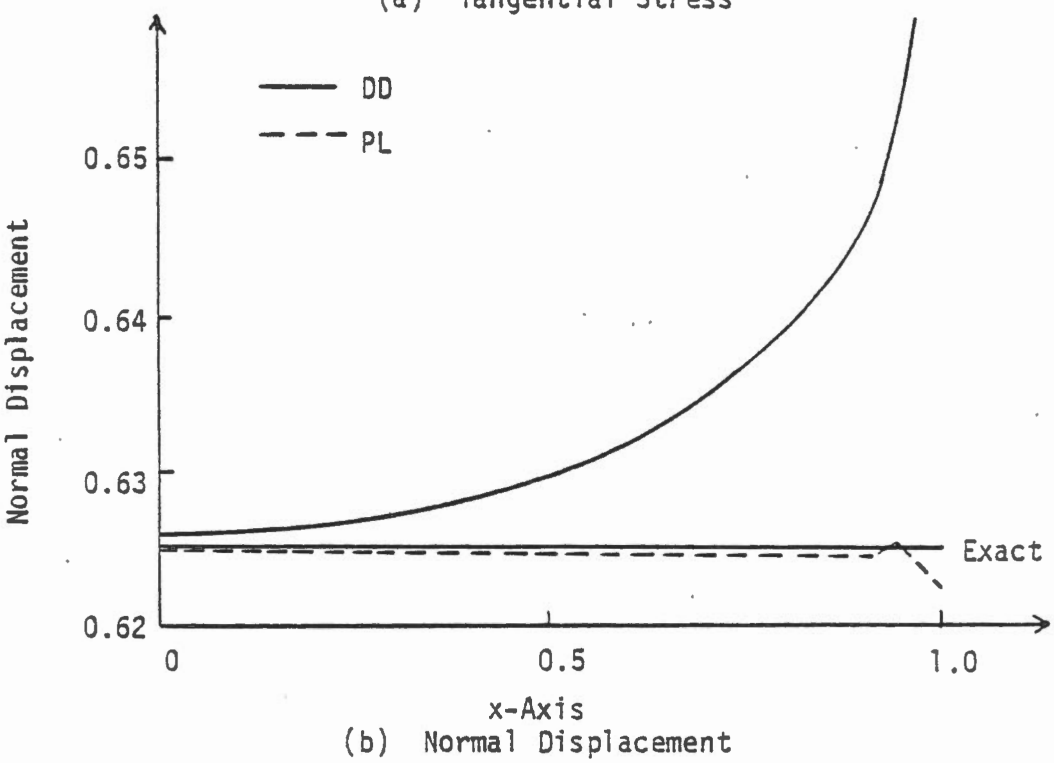


FIGURE 6.11 HOMOGENEOUS COMPRESSION, MID-SIDE TANGENTIAL STRESS



(a) Tangential Stress



(b) Normal Displacement

FIGURE 6.12 HOMOGENEOUS COMPRESSION,
16 QUADRATIC ELEMENTS

6.5 Pressurized Cavity

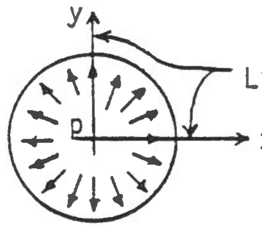


Figure 6.13 Pressurized Cavity

Consider a circular cavity with a unit radius and pressurized as shown in Figure 6.16.

In this example the higher order elements for the DD-singularity gave consistantly better results.

The results for both singularities are presented in Figures 6.14.

The noticeably poor behaviour of the high order elements for a small number of unknowns can be attributed to the poor approximation of a circular boundary shape when using a small number of straight elements. In Figures 6.15 the error for the surface tangential stresses are compared. In the case of the quadratic element, the value used corresponds to the average of the three nodal point values.

6.6 Beam with End Shear

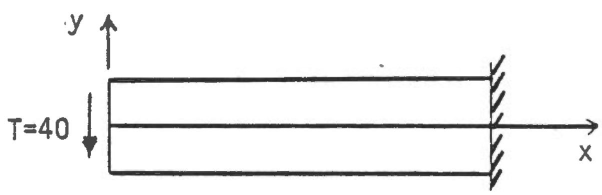


Figure 6.16 Clamped beam submitted to end shear.

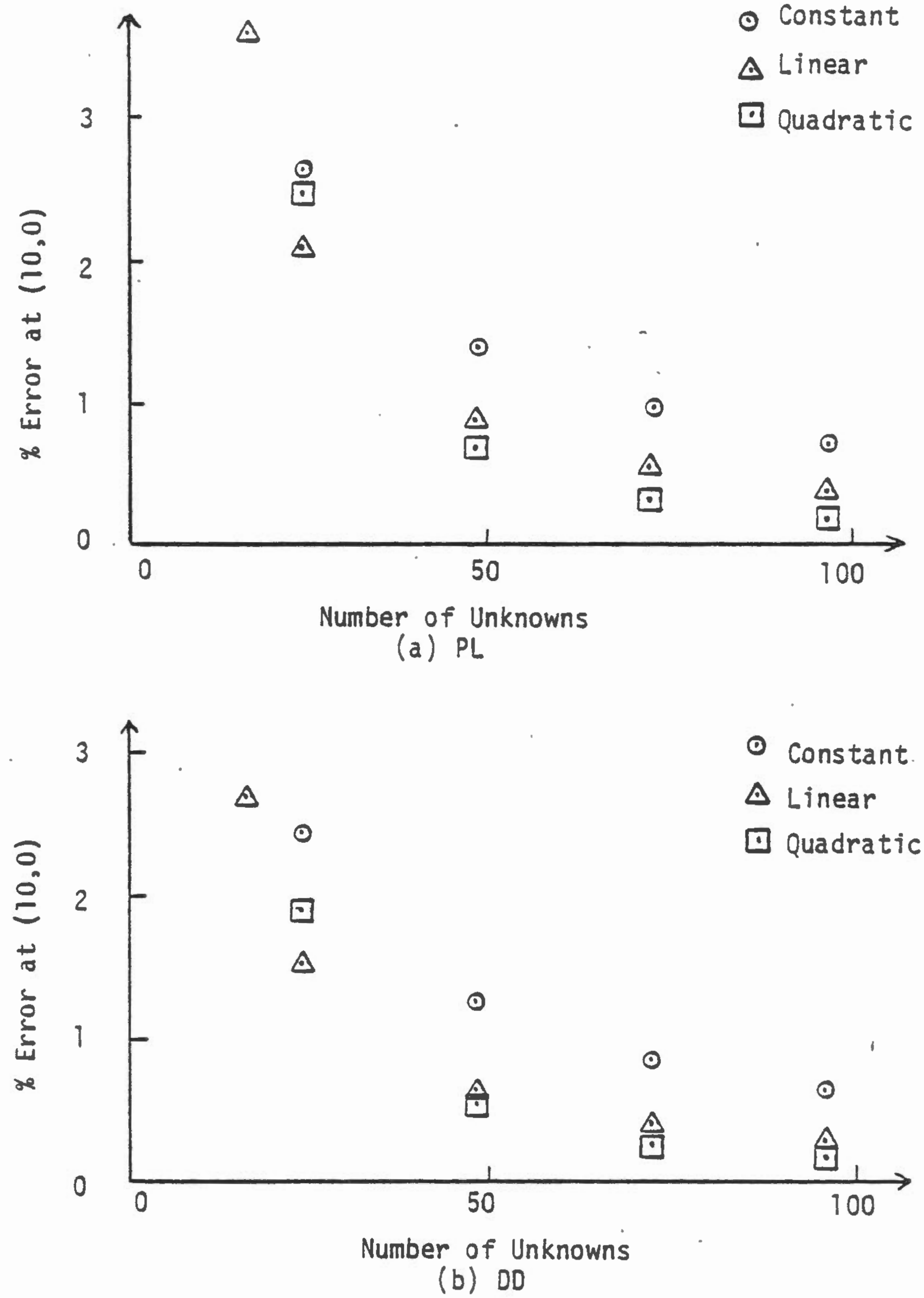


FIGURE 6.14 CIRCULAR CAVITY, ERROR IN STRESSES AND DISPLACEMENTS AT (10,0)

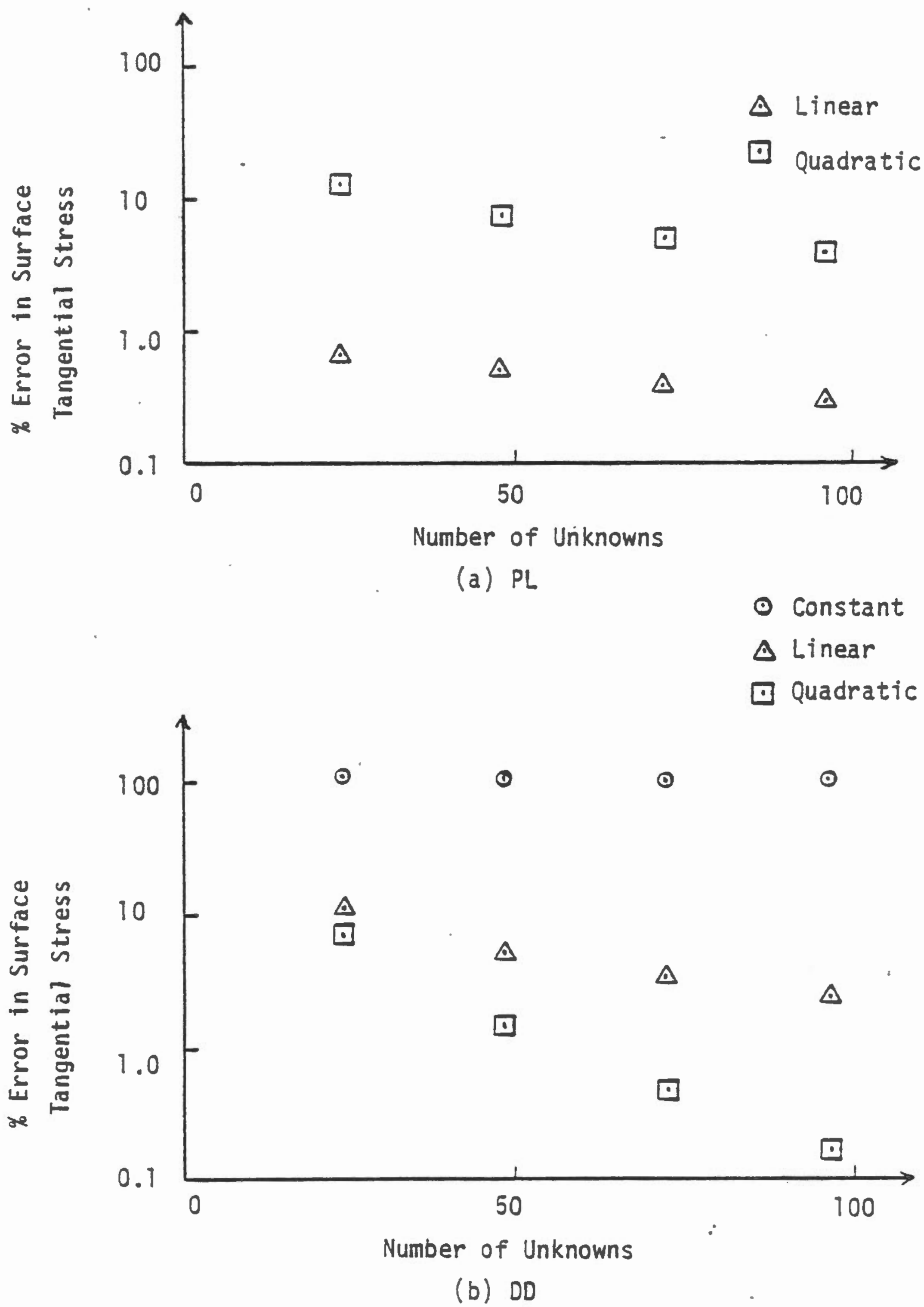


FIGURE 6.15 CIRCULAR CAVITY, ERROR IN SURFACE TANGENTIAL STRESS

A 12x48 rectangular built-in cantilever was loaded with parabolically varying end shear according to the following relation

$$p = -y^2/7.2 + 5 \quad . \tag{6.1}$$

For this particular case, Young's modulus was set at 19,200.0 while Poisson's ratio was assumed to be 0.2. These input values for the plane strain situation correspond to a plane stress analysis with the following elastic constants

$$E = 20{,}000.0, \ \nu = 1/4 \quad .$$

The theoretical tip deflection for the plane stress case is given by (Timoshenko and Goodier, 1970)

$$v = \frac{T\ell^3}{3EI} = 0.512 \quad , \tag{6.2}$$

where:

$\ell = 48$ is the beam length;

$I = 144$ is the moment of inertia of the beam.

Several examples with a variety of discretizations were tried using the PL singularity with no success. The best result obtained gave errors in displacements and stresses of approximately 20% for 20 quadratic elements (120 unknowns). This extremely poor behaviour is probably due to the unfavourable accommodation of rigid body motion by this type of singularity. The end of the beam indeed deflects a considerable amount as a rigid body. This same problem was encountered by Massonnet (1965).

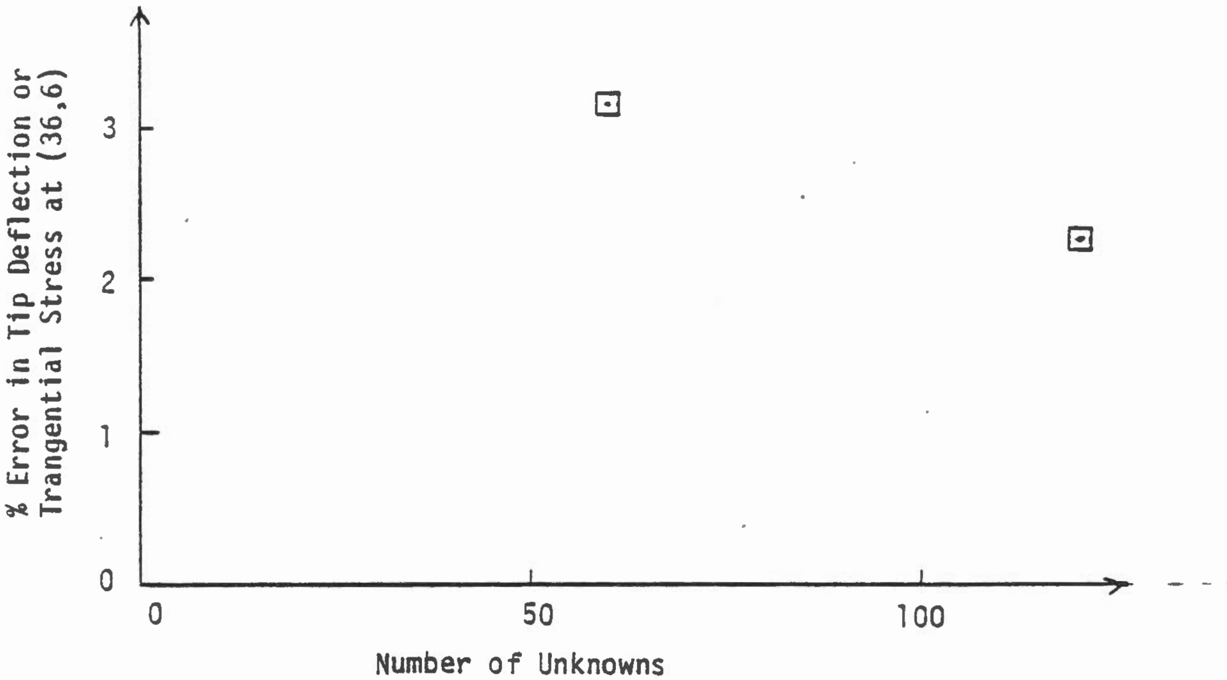


FIGURE 6.17 BEAM WITH END SHEAR (DD)

The DD-singularity showed equally poor correlation for both the constant and linear elements, however for the quadratic element, reasonably accurate results were found (Refer to Figure 6.17). Both the tip deflection and stress at point (36,0) had the same error magnitude. These results can be compared to the ones presented by Brebbia and Connor (1974) who treated a similar situation using the finite element approach.

In view of the fact that the variation in deflection is a function of the cube of x, it can be expected that cubic DD-elements should provide very accurate answers with only a few unknowns considered. For this type of problem the use of boundary integral techniques may prove to be exceedingly more economic than finite element approaches.

6.7 Pressurized Crack

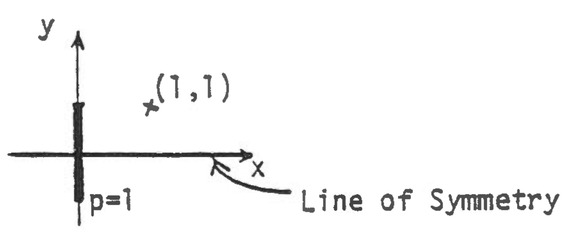
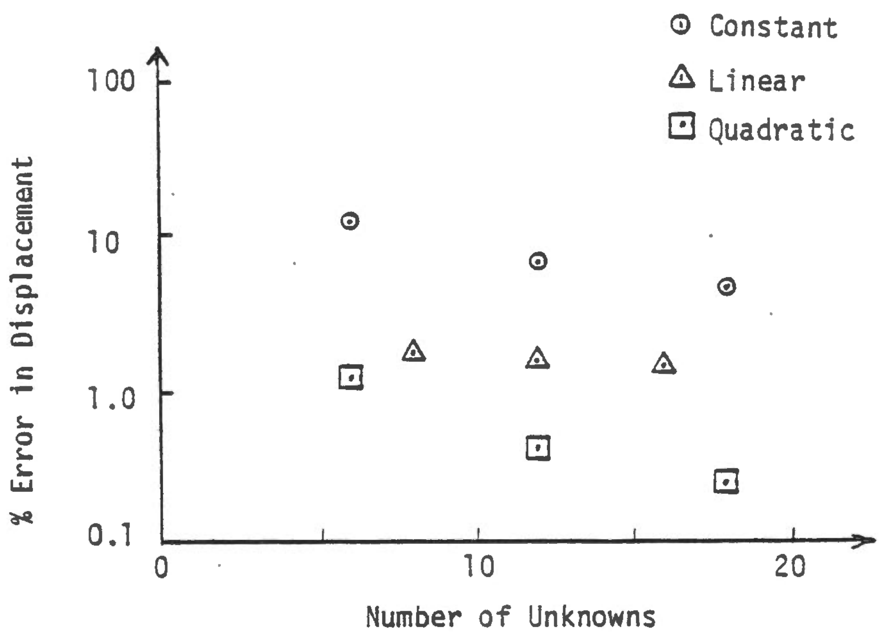


Figure 6.18 Pressurized Crack

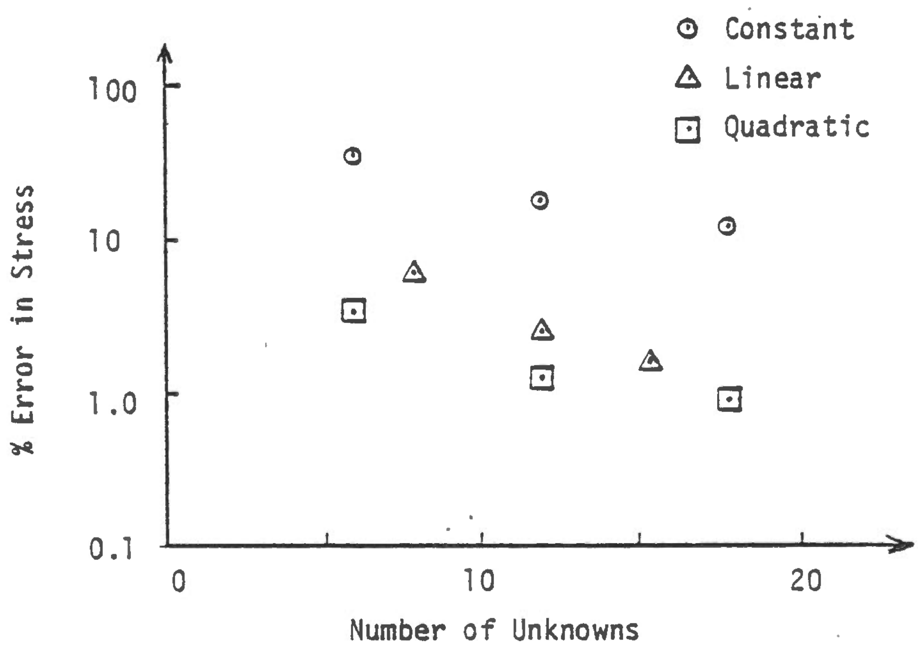
Consider a classical Sneddon flat elliptic crack, uniformly pressurized as shown in figure 6.18.

The higher order elements show a drastic improvement in accuracy in this case. In all cases the surface tangential stress is equal to the pressure in the crack in agreement with theory.

The mid-side surface normal displacement is compared to the analytically obtained value in Figure 6.19a. In Figure 6.19b the average of the error in the stresses is presented.



(a) Mid-side Normal Displacement



(b) Average of Error in Stresses at (1,1)

FIGURE 6.19 PRESSURIZED CRACK

CHAPTER VII

A BIE PROGRAM

7.1 Generalities

The computer programs used to conduct the analyses included in this report are described and discussed in the following sections. The restrictions imposed in setting up the programs reflect the present needs of the authors rather than difficulties in providing more general coding. Programs permitting larger degrees of variability and solving capabilities have however been developed.

7.2 Program Documentation

(i) Input

INPUT	FORMAT	DESCRIPTION
1	20A4	Title
2	4I5, 4F10.0,I5	Control information: NNOD is the total number of element corner points to be read-in; NORD is the element order to be used in analysis (0 is constant, 1 is linear, 2 is quadratic); NLS is the number of lines of symmetry to be considered (1 for just x-axis, 2 for both x and y axes); MSOL is the solver type to be used (0 for Gaussian elimination, 1 for Gauss-Siedel iteration with relaxation); E is Young's modulus; PR is Poisson's ratio; CP is the permissable error in the value of stress (CPxE for displacement) estimated by the iterative solver; RP is the relaxation parameter (>1 for overrelaxation, <1 for under-relaxation); MIT is the maximum number of iterations permitted.

3	4F10.0,2I5	Mesh data and boundary condition information: X is the x-coordinate; Y is the y-coordinate; N is the known normal boundary value; S is the known shear boundary value; NCODE, SCODE indicate respectively, whether N and S are stresses (code 0), displacements (code 1) or discontinuities (code -1).
4	3F10.0	Field stresses; SX is the stress in the x-direction SY is the stress in the y-direction; SXY is the shear stress in the x-y plane.
5	I5	Number of interior points at which stresses and displacements are to be determined (0 ends the progam)
6	2F10.0	Coordinates of interior points: X is the x-coordinate; Y is the y-coordinate.

Table 7.1 Program Input

(ii) Discussion

The mesh data cards are input in order of increasing corner point as well as element numbering (i.e. corner points 1 and 2 define element 1, corner points 2 and 3 definte element 2, and so on). The last corner point input combined with point 1 define the last element.

As far as the boundary conditions are concerned, each input associated with a particular corner point automatically defines the boundary conditions for the element.

If any lines of symmetry are present it is assumed that at least one of the lines intersects the boundary surface. The number of elements is then assumed to be one less than the number of corner points. This may not always be true but can be easily be overridden through the use of the subroutine SETUP.

If the boundary surface is input in clockwise order, a finite body is analysed while counterclockwise input corresponds to a cavity in an infinite medium.

The backsubstitution program is set-up to accept additional load vectors.

The iterative solver calculates its final approximation of the input boundary conditions and substitutes this approximation for the actual values. The converged permissible error in this approximation is equal to the parameter CP for stresses and CPxE for displacements. Any desired convergence criterion can however be substituted.

(iii) Sign Conventions

Tensile stresses are taken as positive. As far as displacements are concerned, they are positive in the direction of the axes. The boundary displacements are defined as shown in Figure 7.1

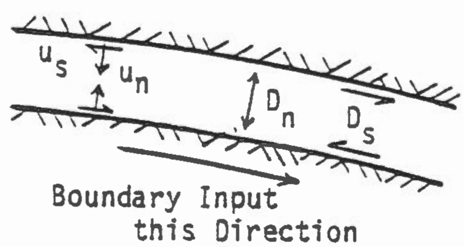


Figure 7.1 Positive Normal and Shear Displacements

7.2 Program Listing

MAIN

This main program controls the program execution.

The dimensions required are

X(ND/2+1), Y(ND/2+1), PS(ND), P(ND), CS(ND,ND),
CU(ND,ND), CT(ND12,ND), MT(ND/2)

where ND is equal to the total number of unknowns,

which is twice the number of elements times the number of nodes for each

element (1 for constant, 2 for linear, 3 for quadratic).

```
      IMPLICIT INTEGER*2(I-M),INTEGER*4(N),REAL*8(A-H,O-Z)
      COMMON/CONS/E,PR,PI,GP(3),SX,SY,SXY,NNOD,NORD,NLS,NORD2,NOR,MSOL
      COMMON/ITSOL/CP,RP,MIT
      COMMON X(16),Y(16),PS(30),P(30),CS(30,30),CU(30,30),CT(15,30),
     *MT(30)
      ND=30
C BIE - DISPLACEMENT DISCONTINUITY
      WRITE(6,1000)
      CALL SETUP
      CALL ASEMB
      IF(MSOL)4,5,6
    5 CALL GAUSS(CS,NOR,ND)
      CALL BSUB(CS,P,NOR,ND)
      GOTO 7
    6 CALL GSI(CS,PS,P,NOR,ND)
    7 CALL STRESS
      WRITE(6,1001)
    4 RETURN
 1000 FORMAT(1H1,33H BIE - DISPLACEMENT DISCONTINUITY,4H  DP)
 1001 FORMAT(/12H END PROGRAM)
      END
```

SETUP

This subroutine reads in all information, sets all constants and initializes all variables. All input stresses and displacements are assumed constant on any element. This is not necessary and can easily be changed. The field stresses are substracted at this point.

```
      SUBROUTINE SETUP
      IMPLICIT INTEGER*2(I-M),INTEGER*4(N),REAL*8(A-H,O-Z)
      DIMENSION NHED(20)
      COMMON/CELEM/C(3,6,2),CC,CE,C1,C2,C3,C4,C5,C6,C7,C8,RX,RY
      COMMON/ITSCL/CP,RP,MIT
      COMMON/CONS/E,PR,PI,GP(3),SX,SY,SXY,NNOD,NORD,NLS,NORD2,NDR,MSOL
      COMMON X(16),Y(16),PS(30),P(30),CS(30,30),CU(30,30),CT(15,30),
     .MT(30)
C NNOD  - NUMBER OF CORNER NODES
C NORD  - ORDER OF ELEMENT:  0 CONSTANT; 1 LINEAR; 2 QUADRATIC
C NLS   - SYMMETRY:  1 X-AXIS;  2 X&Y-AXES
C MSOL  - SOLVER TYPE:  0 ELIMINATION; 1 ITERATION
C E     - YOUNG'S MODULUS
C PR    - POISSON'S RATIO
C CP    - CONVERGED ABSOLUTE ERROR (TIMES E FOR DISPL)
C RP    - RELAXATION FACTOR
C MIT   - MAXIMUM NO. ITERATIONS
C MT    - BOUNDARY KNOWNS:  0 STRESS; 1 DISPLACEMENTS; -1 DISCONTINUITY
C SX,SY,SXY - FIELD STRESSES
      READ(5,1000)NHED,NNOD,NORD,NLS,MSOL,E,PR,CP,RP,MIT
      WRITE(6,1001)NHED,NNOD,NORD,NLS,MSOL,E,PR
      IF(MSOL.EQ.1)WRITE(6,1002)CP,RP,MIT
      IF(MSOL.EQ.0)WRITE(6,1003)
      WRITE(6,1004)
      NORD=NORD+1
      NORD2=NORD*2
      DO 1 N=1,NNOD
      NN=(N-1)*NORD2
      READ(5,2000)X(N),Y(N),PS(NN+1),PS(NN+2),MT(NN+1),MT(NN+2)
    1 WRITE(6,2001)N,X(N),Y(N),PS(NN+1),PS(NN+2),MT(NN+1),MT(NN+2)
      X(NNOD+1)=X(1)
      Y(NNOD+1)=Y(1)
      IF(NLS.GE.1)NNOD=NNOD-1
      READ(5,2002)SX,SY,SXY
      WRITE(6,2003)SX,SY,SXY
      DO 4 N=1,NNOD
      NN=(N-1)*NORD2
      IF(MT(NN+1).NE.0.AND.MT(NN+2).NE.0)GOTO 5
      XD=X(N+1)-X(N)
      YD=Y(N+1)-Y(N)
      R=DSQRT(XD**2+YD**2)
      CA=XD/R
      SA=YD/R
      SN=SX*SA**2-2.D0*SXY*CA*SA+SY*CA**2
      SS=(SY-SX)*CA*SA+SXY*(CA**2-SA**2)
C REMOVE FIELD STRESSES
      IF(MT(NN+1).EQ.0)PS(NN+1)=PS(NN+1)-SN
      IF(MT(NN+2).EQ.0)PS(NN+2)=PS(NN+2)-SS
    5 IF(NORD.EQ.1)GOTO 4
      DO 2 J=3,NORD2,2
      MT(NN+J)=MT(NN+1)
      MT(NN+J+1)=MT(NN+2)
      PS(NN+J)=PS(NN+1)
    2 PS(NN+J+1)=PS(NN+2)
    4 CONTINUE
      NDR=NNOD*NORD*2
      NDR2=NDR/2
      PI=3.141592653589793
```

Constants for DD singularity

```
      CC=E/(4.D0*PI*(1.D0-PR**2))
      CE=.25D0/(PI*(1.D0-PR))
      C1=2.D0*(1.D0-PR)
      C2=1.D0-2.D0*PR
      C3=3.D0-2.D0*PR
      C4=1.D0+2.D0*PR
      C5=4.D0-2.D0*PR
      DO 3 J=1,NDR
      DO 6 K=1,NDR
      CS(J,K)=0.D0
    6 CU(J,K)=0.D0
      DO 3 K=1,NDR2
    3 CT(K,J)=0.D0
      CALL QUAD
      RETURN
 1000 FORMAT(20A4/4I5,4F10.0,I5)
 1001 FORMAT(////1X,20A4//T7,4HNNOD,T16,5HORDER,T28,3HNLS,T37,4HMSOL,
     .T49,1HE,T64,2HPR/4I10,2F15.5)
 1002 FORMAT(//17H ITERATIVE SOLVER,T33,2HCP,T48,2HRP,T63,3HMIT/
     .25X,2F15.5,I10)
 1003 FORMAT(//21H GAUSSIAN ELIMINATION)
 1004 FORMAT(//T10,1HN,T19,1HX,T34,1HY,T49,1HN,T64,1HS,T76,5HNCODE,
     .T86,5HSCODE)
 2000 FORMAT(4F10.0,2I5)
 2001 FORMAT(I10,2F15.5/T41,2F15.5,2I10)
 2002 FORMAT(3F10.0)
 2003 FORMAT(//15H FIELD STRESSES//T9,2HSX,T24,2HSY,T39,3HSXY/3F15.5)
      END
```

Constants for PL singularity

```
      CC=.25D0/(PI*(1.D0-PR))
      CE=CC/E
      C1=1.D0+PR
      C2=1.D0-PR
      C3=1.D0+2.D0*PR
      C4=1.D0-2.D0*PR
      C5=3.D0-2.D0*PR
      C6=5.D0-2.D0*PR
      C7=2.D0-PR
      C8=3.D0-PR-4.D0*PR**2
```

ASEMB

This subroutine assembles the coefficient matrices F_σ and F_u (see equations 3.4 and 3.5) and reorders them, ready to be solved.

The symmetry loop considers each reflected portion of the boundary consecutively (1 input boundary, 2 reflected in x-axis, 3 reflected in y-axis, 4 reflected in x and y axes).

The loaded element loop sets the location of the distributed load or discontinuity. The affected loop sets the location of the point at which the effect of the loaded element is to be determined.

During assembly the normal and shear stress coefficients are put into CS, the normal and shear displacement coefficients are put into CU and the tangential stress coefficients are put into CT.

Portions of CS and CU are the only parts which are used to solve the problem. Consequently, only one complete matrix of values needs to be stored at this time. The coefficients required to determine the unknowns could be recalculated or recalled from disk in subroutine stress to save storage space.

The columns correspond to the location of the interpolation load point value and the rows correspond to the location where the load has an effect (see figure 7.4). The rows of the vectors P and PS correspond to the interpolation point values.

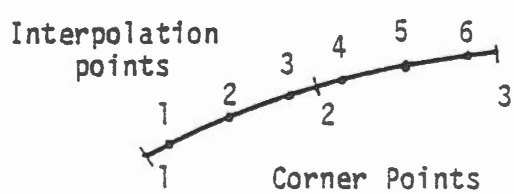


Figure 7.4 Coefficient Matrix Set-up

All knowns are then put into CS and P ready to be solved. Any displacement coefficients are multiplied by Young's modulus to avoid numerical errors.

```
      SUBROUTINE ASEMB
      IMPLICIT INTEGER*2(I-M),INTEGER*4(N),REAL*8(A-H,O-Z)
      COMMON/CELEM/C(3,6,2),CC,CE,C1,C2,C3,C4,C5,C6,C7,C8,RX,RY
      COMMON/CONS/E,PR,PI,GP(3),SX,SY,SXY,NNCD,NORD,NLS,NORD2,NDR,MSOL
      COMMON X(16),Y(16),PS(30),P(30),CS(30,30),CU(30,30),CT(15,30),
     .MT(30)
C ASSEMBLE COEFFICIENT MATRICIES
C CS   - NORMAL & SHEAR STRESS COEF
C CU   - NORMAL & SHEAR DISPL COEF
C CT   - TANGENTIAL STRESS COEF
      RX=1.D0
      RY=1.D0
      NLSP=2**(NLS+1)/2
C SYMMETRY LOOP
      DO 1 NO=1,NLSP
      IF(NO.EQ.1)GOTO 7
      IF(NO-3)3,4,3
    4 RY=-RY
    3 RX=-RX
    7 RXY=RX*RY
      SN=RXY
C LOADED ELEMENT LOOP
      DO 1 N12=1,NNCD
      NJ=(N12-1)*NORD2
C AFFECTED LOOP
      DO 1 N34=1,NNCD
      NL=(N34-1)*NORD
      NK=NL*2
      NT=N34
      XM=(X(N34+1)+X(N34))/2.D0
      XD=(X(N34+1)-X(N34))/2.D0
      YM=(Y(N34+1)+Y(N34))/2.D0
      YD=(Y(N34+1)-Y(N34))/2.D0
      IF(N34.EQ.N12.AND.NO.EQ.1)NT=-1
      DO 1 J=1,NORD
      JK=NK+(J-1)*2
      XP=XM+XD*GP(J)
      YP=YM+YD*GP(J)
      CALL ELEM(XP,YP,X(N34),Y(N34),X(N12),Y(N12),NT)
      DO 1 M=1,NORD2
      SN=SN*RXY
      DO 2 K=1,2
      CS(JK+K,NJ+M)=C(K+1,M,1)*SN+CS(JK+K,NJ+M)
    2 CU(JK+K,NJ+M)=C(K+1,M,2)*SN+CU(JK+K,NJ+M)
    1 CT(NL+J,NJ+M)=C(1,M,1)*SN+CT(NL+J,NJ+M)
C PUT ALL KNOWNS IN CS & P.
      DO 10 N=1,NDR
      IF(MT(N).NE.1)GOTO 9
      DO 12 J=1,NDR
      CSS=CS(N,J)
      CS(N,J)=CU(N,J)*E
   12 CU(N,J)=CSS
      PS(N)=PS(N)*E
    9 P(N)=PS(N)
   10 IF(MT(N).EQ.-1)P(N)=0.D0
C ELIMINATE KNOWN P'S
      DO 5 N=1,NDR
      IF(MT(N).NE.-1)GOTO 5
      DO 8 J=1,NDR
      P(J)=P(J)-CS(J,N)*PS(N)
    8 CS(J,N)=0.D0
      CS(N,N)=-1.D0
    5 CONTINUE
      RETURN
      END
```

QUAD

This subroutine sets the location of the interpolation points given by equation (5.4).

```
      SUBROUTINE QUAD
      IMPLICIT INTEGER*2(I-M),INTEGER*4(N),REAL*8(A-H,O-Z)
      COMMON/CONS/E,PR,PI,GP(3),SX,SY,SXY,NNOD,NORD,NLS,NORD2,NOR,MSOL
C LOCATION OF INTERPOLATION POINTS
      N=NORD
      IF(N-2)1,2,3
    1 GP(1)=0.D0
      GOTO 4
    2 GP(2)=DCOS(PI/4.D0)
      GP(1)=-GP(2)
      GOTO 4
    3 GP(3)=DCOS(PI/6.D0)
      GP(2)=0.D0
      GP(1)=-GP(3)
    4 RETURN
      END
```

ELEM

This subroutine calculates the coefficients called for by subroutine ASEMB and STRESS. The loaded element and affected point are first redefined such that the loaded element is centered at the origin. The coefficients are then calculated in the matrix S such that: column 1 respectively refers to the tangential, normal and shear components; column 2 respectively refers to the normal and shear loads; column 3 refers to the stress and displacements; and column 4 refers to the element order.

The matrix W contains the interpolation formula (equations 5.1 and 5.2).

PL Singularity

```
      SUBROUTINE ELEM(XP,YP,X34,Y34,X12,Y12,NT)
      IMPLICIT INTEGER*2(I-M),INTEGER*4(N),REAL*8(A-H,O-Z)
      DIMENSION X12(2),Y12(2),X34(2),Y34(2),S(3,2,2,3),W(3,3)
      COMMON/CELEM/C(3,6,2),CC,CE,C1,C2,C3,C4,C5,C6,C7,C8,RX,RY
      COMMON/CONS/E,PR,PI,GP(3),SX,SY,SXY,NNOD,NORD,NLS,NORD2,NDR,MSOL
C CALCULATE COEF.
      IF(NT-1)5,5,6
    5 X0=(X12(2)+X12(1))/2.D0*RY
      XD=(X12(2)-X12(1))/2.D0*RX
      Y0=(Y12(2)+Y12(1))/2.D0*RX
      YD=(Y12(2)-Y12(1))/2.D0*RY
      A=DSQRT(XD**2+YD**2)
      B=-A*GP(1)*RX*RY
      CA=XD/A
      SA=YD/A
    6 X=(XP-X0)*CA+(YP-Y0)*SA
      Y=(YP-Y0)*CA-(XP-X0)*SA
      Y2=Y**2
      DO 15 J=1,3
      DO 15 K=1,NORD2
      DO 15 L=1,2
   15 C(J,K,L)=0.D0
      AMX=A-X
      APX=A+X
      R1=DSQRT(APX**2+Y2)
      R2=DSQRT(AMX**2+Y2)
      IF(DABS(Y).GT.1.E-5)GOTO 14
      TD=0.D0
      IF(DABS(X).LE.A)TD=PI
      GOTO 16
   14 TD=DATAN(-AMX/Y)-DATAN(APX/Y)
   16 AL1=DLOG(R1)
      AL2=DLOG(R2)
      RI=1.D0/R2**2-1.D0/R1**2
      XRI=AMX/R2**2+APX/R1**2
      ALD=AL2-AL1
```

```
C CONSTANT VARIATION
      S(1,1,1,1)=-Y*XRI-2.D0*PR*TD
      S(1,2,1,1)=-Y2*RI-C5*ALD
      S(2,1,1,1)=Y*XRI-2.D0*C2*TD
      S(2,2,1,1)=Y2*RI+C4*ALD
      S(3,1,1,1)=Y2*RI-C4*ALD
      S(3,2,1,1)=-Y*XRI-2.D0*C2*TD
      XALD=AMX*AL2+APX*AL1
      S(2,1,2,1)=-2.D0*C1*C4*(2.D0*A+Y*TD)+C8*XALD
      S(2,2,2,1)=C1*Y*ALD
      S(3,1,2,1)=S(2,2,2,1)
      S(3,2,2,1)=C8*(XALD-2.D0*A)-4.D0*C1*C2*Y*TD
      W(1,1)=1.D0
      IF(NORD.EQ.1)GOTO 10
      Y3=Y**3
C LINEAR VARIATION
      S(1,1,1,2)=Y3*RI+C3*Y*ALD
      S(1,2,1,2)=-Y2*XRI-2.D0*C5*A-2.D0*C7*Y*TD
      S(2,1,1,2)=-Y3*RI+C4*Y*ALD
      S(2,2,1,2)=Y2*XRI+2.D0*C2*Y*TD+2.D0*C4*A
      S(3,1,1,2)=Y2*XRI-2.D0*C4*A+2.D0*PR*Y*TD
      S(3,2,1,2)=Y3*RI+C5*Y*ALD
      X2ALD=AMX**2*AL2-APX**2*AL1
      RALD=X2ALD+Y2*ALD
      X2D=AMX**2-APX**2
      D1=C1*Y2*ALD
      D2=C8/2.D0*(RALD-X2D/2.D0)
      S(2,1,2,2)=D2-D1+C1*X2D/2.D0
      S(2,2,2,2)=C1*Y*(2.D0*A+Y*TD)
      S(3,1,2,2)=S(2,2,2,2)
      S(3,2,2,2)=D1+D2
      IF(NORD-2)10,11,12
   11 B2=2.D0*B
      W(1,1)=(B-X)/B2
      W(2,1)=(B+X)/B2
      W(1,2)=-1.D0/B2
      W(2,2)=-W(1,2)
      GOTO 10
   12 Y4=Y**4
C QUADRATIC VARIATION
      S(1,1,1,3)=Y3*XRI+2.D0*C1*Y2*TD+2.D0*C3*Y*A
      S(1,2,1,3)=Y4*RI+C6*Y2*ALD-C5*X2D/2.D0
      S(2,1,1,3)=-Y3*XRI-2.D0*PR*Y2*TD+2.D0*C4*Y*A
      S(2,2,1,3)=-Y4*RI-C5*Y2*ALD+C4*X2D/2.D0
      S(3,1,1,3)=-Y4*RI-C3*Y2*ALD-C4*X2D/2.D0
      S(3,2,1,3)=Y3*XRI+2.D0*C7*Y2*TD+2.D0*C5*Y*A
      X3ALD=AMX**3*AL2+APX**3*AL1
      X3D=AMX**3+APX**3
      D1=C1*Y2*(2.D0*A+Y*TD)
      D2=C8/3.D0*(X3ALD-X3D/3.D0+2.D0*Y2*A+Y3*TD)
      S(2,1,2,3)=D2-D1+C1*X3D/3.D0
      S(2,2,2,3)=C1*Y*(X2D/2.D0-Y2*ALD)
      S(3,1,2,3)=S(2,2,2,3)
      S(3,2,2,3)=D1+D2
      W(1,1)=X*(X-B)
      W(2,1)=2.D0*(B+X)*(B-X)
      W(3,1)=X*(X+B)
      W(1,2)=2.D0*X-B
      W(2,2)=-4.D0*X
      W(3,2)=2.D0*X+B
      W(1,3)=1.D0
      W(2,3)=-2.D0
      W(3,3)=1.D0
```

```
      BS=2.D0*B**2
      DO 13 J=1,3
      DO 13 K=1,3
   13 W(J,K)=W(J,K)/BS
   10 DO 7 L=1,NCRD
      LL=(L-1)*2
      DO 7 K=1,2
      LK=LL+K
      DO 7 N=1,NGRD
      DO 8 J=1,3
    8 C(J,LK,1)=C(J,LK,1)+S(J,K,1,N)*CC*W(L,N)
      DO 7 J=2,3
    7 C(J,LK,2)=C(J,LK,2)+S(J,K,2,N)*CE*W(L,N)
      IF(NT)2,4,3
    4 CALL ROT(-SA,CA)
      GOTO 2
    3 R34=DSQRT((X34(2)-X34(1))**2+(Y34(2)-Y34(1))**2)
      CG=(X34(2)-X34(1))/R34
      SG=(Y34(2)-Y34(1))/R34
      SB=SG*CA-CG*SA
      CB=CG*CA+SG*SA
      CALL ROT(SB,CB)
    2 RETURN
      END
```

DD Singularity

```
      SUBROUTINE ELEM(XP,YP,X34,Y34,X12,Y12,NT)
      IMPLICIT INTEGER*2(I-M),INTEGER*4(N),REAL*8(A-H,O-Z)
      DIMENSION X12(2),Y12(2),X34(2),Y34(2),S(3,2,2,3),W(3,3)
      COMMON/CELEM/C(3,6,2),CC,CE,C1,C2,C3,C4,C5,C6,C7,C8,RX,RY
      COMMON/CONS/E,PR,PI,GP(3),SX,SY,SXY,NNOD,NORD,NLS,NORD2,NDR,MSOL
C CALCULATE COEF.
      IF(NT-1)5,5,6
    5 X0=(X12(2)+X12(1))/2.D0*RY
      XD=(X12(2)-X12(1))/2.D0*RX
      Y0=(Y12(2)+Y12(1))/2.D0*RX
      YD=(Y12(2)-Y12(1))/2.D0*RY
      A=DSQRT(XD**2+YD**2)
      B=-A*GP(1)*RX*RY
      CA=XD/A
      SA=YD/A
    6 X=(XP-X0)*CA+(YP-Y0)*SA
      Y=(YP-Y0)*CA-(XP-X0)*SA
      Y2=Y**2
      DO 15 J=1,3
      DO 15 K=1,NORD2
      DO 15 L=1,2
   15 C(J,K,L)=0.D0
      AMX=A-X
      APX=A+X
      R1=DSQRT(APX**2+Y2)
      R2=DSQRT(AMX**2+Y2)
      IF(DABS(Y).GT.1.E-5)GOTO 14
      TD=0.D0
      IF(DABS(X).LE.A)TD=-PI
      GOTO 16
   14 TD=DATAN(AMX/Y)-DATAN(-APX/Y)
   16 ALD=DLOG(R2)-DLOG(R1)
C CONSTANT VARIATION
      Y3=Y**3
      R2I=1.D0/R2**2-1.D0/R1**2
      R4I=1.D0/R2**4-1.D0/R1**4
      X1R2=AMX/R2**2+APX/R1**2
      X1R4=AMX/R2**4+APX/R1**4
      X2R4=AMX**2/R2**4-APX**2/R1**4
      X3R4=AMX**3/R2**4+APX**3/R1**4
      S(1,1,1,1)=-X3R4+Y2*X1R4
      S(1,2,1,1)=-Y3*R4I-3.D0*Y*X2R4
```

```
      S(2,1,1,1)=-X3R4-3.D0*Y2*X1R4
      S(2,2,1,1)=-Y3*R4I+Y*X2R4
      S(2,1,2,1)=Y*X1R2+C1*TD
      S(2,2,2,1)=Y2*R2I-C2*ALD
      S(3,1,2,1)=Y2*R2I+C2*ALD
      S(3,2,2,1)=-Y*X1R2+C1*TD
      W(1,1)=1.D0
      IF(NCRD.EQ.1)GOTO 10
C LINEAR VARIATION
      Y4=Y**4
      S(1,1,1,2)=4.D0*Y2*R2I-2.D0*Y4*R4I+ALD
      S(1,2,1,2)=-4.D0*Y*X1R2+2.D0*Y3*X1R4+2.D0*TD
      S(2,1,1,2)=-2.D0*Y2*R2I+2.D0*Y4*R4I+ALD
      S(2,2,1,2)=2.D0*Y*X1R2-2.D0*Y3*X1R4
      S(2,1,2,2)=-Y3*R2I+C2*Y*ALD
      S(2,2,2,2)=Y2*X1R2-2.D0*PR*Y*TD-2.D0*C2*A
      S(3,1,2,2)=Y2*X1R2-C1*Y*TD+2.D0*C2*A
      S(3,2,2,2)=Y3*R2I+C3*Y*ALD
      IF(NCRD-2)10,11,12
   11 B2=2.D0*B
      W(1,1)=(B-X)/B2
      W(2,1)=(B+X)/B2
      W(1,2)=-1.D0/B2
      W(2,2)=-W(1,2)
      GOTO 10
C QUADRATIC VARIATION
   12 Y5=Y**5
      S(1,1,1,3)=5.D0*Y2*X1R2-2.D0*Y4*X1R4-4.D0*Y*TD+2.D0*A
      S(1,2,1,3)=-2.D0*Y5*R4I+7.D0*Y3*R2I+6.D0*Y*ALD
      S(2,1,1,3)=-3.D0*Y2*X1R2+2.D0*Y4*X1R4+2.D0*A
      S(2,2,1,3)=-5.D0*Y3*R2I+2.D0*Y5*R4I-2.D0*Y*ALD
      X2D=AMX**2-APX**2
      S(2,1,2,3)=-Y3*X1R2+2.D0*C2*Y*A+2.D0*PR*Y2*TD
      S(2,2,2,3)=-Y4*R2I-C2*X2D/2.D0-C4*Y2*ALD
      S(3,1,2,3)=-Y4*R2I+C2*X2D/2.D0-C3*Y2*ALD
      S(3,2,2,3)=Y3*X1R2+2.D0*C3*Y*A-C5*Y2*TD
      W(1,1)=X*(X-B)
      W(2,1)=2.D0*(B+X)*(B-X)
      W(3,1)=X*(X+B)
      W(1,2)=2.D0*X-B
      W(2,2)=-4.D0*X
      W(3,2)=2.D0*X+B
      W(1,3)=1.D0
      W(2,3)=-2.D0
      W(3,3)=1.D0
      BS=2.D0*B**2
      DO 13 J=1,3
      DO 13 K=1,3
   13 W(J,K)=W(J,K)/BS
   10 DO 9 L=1,NCRD
      LL=(L-1)*2
      DO 7 K=1,2
      LK=LL+K
      DO 7 N=1,NCRD
      DO 7 J=1,2
      JP=J+1
      C(J,LK,1)=C(J,LK,1)+S(J,K,1,N)*CC*W(L,N)
    7 C(JP,LK,2)=C(JP,LK,2)+S(JP,K,2,N)*CE*W(L,N)
      C(3,LL+1,1)=C(2,LL+2,1)
    9 C(3,LL+2,1)=C(1,LL+1,1)
      IF(NT)2,4,3
    4 CALL ROT(-SA,CA)
      GOTO 2
    3 R34=DSQRT((X34(2)-X34(1))**2+(Y34(2)-Y34(1))**2)
      CG=(X34(2)-X34(1))/R34
      SG=(Y34(2)-Y34(1))/R34
      SB=SG*CA-CG*SA
      CB=CG*CA+SG*SA
      CALL ROT(SB,CB)
    2 RETURN
      END
```

ROT

This subroutine rotates any orthogonal set of stresses and displacements to a new axis frame.

```
      SUBROUTINE ROT(SB,CB)
      IMPLICIT INTEGER*2(I-M),INTEGER*4(N),REAL*8(A-H,O-Z)
      DIMENSION S(3,6,2)
      COMMON/CELEM/C(3,6,2),CC,CE,C1,C2,C3,C4,C5,C6,C7,C8,RX,RY
      COMMON/CONS/E,PR,PI,GP(3),SX,SY,SXY,NNOD,NORD,NLS,NORD2,NOR,MSOL
C ROTATE TO REQUIRED ANGLE
      DO 1 J=1,3
      DO 1 K=1,NORD2
      DO 1 N=1,2
    1 S(J,K,N)=C(J,K,N)
      DO 4 K=1,NORD2
      C(2,K,2)=S(2,K,2)*CB-S(3,K,2)*SB
      C(3,K,2)=S(2,K,2)*SB+S(3,K,2)*CB
      C(1,K,1)=S(1,K,1)*CB**2+S(2,K,1)*SB**2+S(3,K,1)*2.D0*SB*CB
      C(2,K,1)=S(1,K,1)*SB**2+S(2,K,1)*CB**2-S(3,K,1)*2.D0*SB*CB
    4 C(3,K,1)=(S(2,K,1)-S(1,K,1))*SB*CB+S(3,K,1)*(CB**2-SB**2)
      RETURN
      END
```

STRESS

This subroutine first calculates the boundary unknowns, and then any interior point values. The field stresses are added back in at this stage.

```
      SUBROUTINE STRESS
      IMPLICIT INTEGER*2(I-M),INTEGER*4(N),REAL*8(A-H,O-Z)
      DIMENSION S(3,2)
      COMMON/CELEM/C(3,6,2),CC,CE,C1,C2,C3,C4,C5,C6,C7,C8,RX,RY
      COMMON/CONS/E,PR,PI,GP(3),SX,SY,SXY,NNOD,NORD,NLS,NORD2,NDR,MSOL
      COMMON X(16),Y(16),PS(30),P(30),CS(30,30),CU(30,30),CT(15,30),
     .MT(30)
C CALCULATE BOUNDARY UNKNOWNS
      WRITE(6,2000)
      DO 12 N=1,NOR
      IF(MT(N).NE.-1)GOTO 12
      DD=PS(N)
      PS(N)=P(N)
      P(N)=DD
   12 CONTINUE
      DO 1 N=1,NNOD
      XD=X(N+1)-X(N)
      YD=Y(N+1)-Y(N)
      R=DSQRT(XD**2+YD**2)
      CA=XD/R
      SA=YD/R
      ST=SX*CA**2+2.D0*SXY*CA*SA+SY*SA**2
      SN=SX*SA**2-2.D0*SXY*CA*SA+SY*CA**2
      SS=(SY-SX)*CA*SA+SXY*(CA**2-SA**2)
      JJ=(N-1)*NORD
      NN=JJ*2
      DO 1 K=1,NORD
      NS=NN+(K-1)*2
      NT=JJ+K
      N1=2
      N2=2
      IF(MT(NS+1).EQ.1)N1=1
      IF(MT(NS+2).EQ.1)N2=1
C ADD FIELD STRESSES BACK IN
      S(1,1)=ST
      S(2,1)=SN
      S(3,1)=SS
      S(2,2)=0.D0
      S(3,2)=0.D0
      IF(N1.EQ.1)GOTO 20
      S(2,1)=S(2,1)+PS(NS+1)
      GOTO 21
   20 S(2,2)=PS(NS+1)/E
   21 IF(N2.EQ.1)GOTO 22
      S(3,1)=S(3,1)+PS(NS+2)
      GOTO 10
   22 S(3,2)=PS(NS+2)/E
   10 DO 3 J=1,NOR
      S(1,1)=S(1,1)+CT(NT,J)*P(J)
      S(2,N1)=S(2,N1)+CU(NS+1,J)*P(J)
    3 S(3,N2)=S(3,N2)+CU(NS+2,J)*P(J)
    1 WRITE(6,2001)N,P(NS+1),P(NS+2),(S(J,2),J=2,3),(S(J,1),J=1,3)
```

```
C CALCULATE NC INTERNAL STRESSES & DISPLACEMENTS AT (XP,YP)
   11 READ(5,3000)NC
      IF(NC.LT.1)RETURN
      WRITE(6,3001)
      DO 4 J=1,NC
      DO 9 K=2,3
    9 S(K,2)=0.D0
C ADD FIELD STRESSES
      S(1,1)=SX
      S(2,1)=SY
      S(3,1)=SXY
      READ(5,3002)XP,YP
      RX=1.D0
      RY=1.D0
      NLSP=2**(NLS+1)/2
      DO 7 NO=1,NLSP
      IF(NO.EQ.1)GOTO 8
      IF(NO-3)5,6,5
    6 RY=-RY
    5 RX=-RX
    8 RXY=RX*RY
      SN=RXY
      DO 7 N=1,NNOD
      NK=(N-1)*NORD2
      CALL ELEM(XP,YP,X(1),Y(1),X(N),Y(N),0)
      DO 7 L=1,NORD2
      SN=SN*RXY
      DO 2 K=2,3
    2 S(K,2)=S(K,2)+C(K,L,2)*P(NK+L)*SN
      DO 7 K=1,3
    7 S(K,1)=S(K,1)+C(K,L,1)*P(NK+L)*SN
    4 WRITE(6,3003)XP,YP,(S(4-K,2),K=1,2),(S(K,1),K=1,3)
      GOTO 11
 2000 FORMAT(1H1,36H BOUNDARY DISPLACEMENTS AND STRESSES/
     .T10,1HN,T19,2HDN,T34,2HDS,T49,2HUN,T64,2HUS,T79,2HST,T94,2HSN,T109
     .,2HSS)
 2001 FORMAT(I10,4D15.5,3F15.5)
 3000 FORMAT(I5)
 3001 FORMAT(///36H INTERNAL DISPLACEMENTS AND STRESSES/
     .T20,1HX,T35,1HY,T49,2HUX,T64,2HUY,T79,2HSX,T94,2HSY,T109,3HSXY)
 3002 FORMAT(2F10.0)
 3003 FORMAT(10X,2F15.5,2D15.5,3F15.5)
      END
```

GAUSS

This subroutine conducts Gaussian elimination on the matrix C.

```
      SUBROUTINE GAUSS(C,N,ND)
      IMPLICIT INTEGER*2(I-M),INTEGER*4(N),REAL*8(A-H,O-Z)
      DIMENSION C(ND,ND)
C GAUSSIAN ELIMINATION
      NM=N-1
      DO 2 J=1,NM
      B=C(J,J)
      JP=J+1
      DO 2 K=JP,N
      A=C(K,J)/B
      DO 1 L=JP,N
    1 C(K,L)=C(K,L)-C(J,L)*A
    2 CONTINUE
      RETURN
      END
```

BSUB

This subroutine first reduces the vector of knowns P, then back substitutes replacing the knowns in P.

```
      SUBROUTINE BSUB(C,P,N,ND)
      IMPLICIT INTEGER*2(I-M),INTEGER*4(N),REAL*8(A-H,O-Z)
      DIMENSION C(ND,ND),P(ND)
C REDUCTION
      NM=N-1
      DO 1 J=1,NM
      A=C(J,J)
      JP=J+1
      DO 1 K=JP,N
    1 P(K)=P(K)-P(J)*C(K,J)/A
C BACK SUBSTITUTION
      P(N)=P(N)/C(N,N)
      DO 2 J=1,NM
      K=N-J
      S=0.
      KP=K+1
      DO 3 L=KP,N
    3 S=S+C(K,L)*P(L)
    2 P(K)=(P(K)-S)/C(K,K)
      RETURN
      END
```

GSI

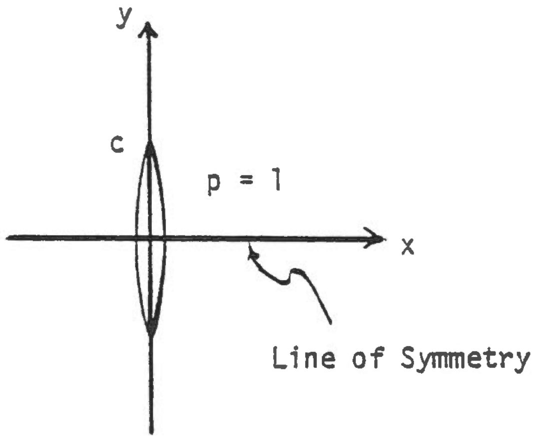
This subroutine conducts Gauss-Siedel iteration with relaxation on the matrix C such that R contains the knowns, and P is filled with the determined values.

```
      SUBROUTINE GSI(C,R,P,N,ND)
      IMPLICIT INTEGER*2(I-M),INTEGER*4(N),REAL*8(A-H,O-Z)
      DIMENSION C(ND,ND),R(ND),P(ND)
      COMMON/ITSCL/CP,RP,MIT
C GAUSS-SIEDEL ITERATION WITH RELAXATION
      NIT=0
    1 M=0
      IF(NIT.EQ.MIT)GOTO 7
      NIT=NIT+1
      DO 6 I=1,N
      T=R(I)
      IF(I.EQ.1)GOTO 2
      IM=I-1
      DO 4 J=1,IM
    4 T=T-P(J)*C(I,J)
    2 IF(I.EQ.N)GOTO 3
      IP=I+1
      DO 5 J=IP,N
    5 T=T-P(J)*C(I,J)
    3 ER=DABS(P(I)*C(I,I)-T)
C ERROR IN ESTIMATE OF KNOWN VALUE
      IF(ER.GT.CP)M=1
      T=T/C(I,I)-P(I)
    6 P(I)=P(I)+RP*T
      IF(M.NE.0)GOTO 1
    7 DO 8 I=1,N
      R(I)=0.
      DO 8 J=1,N
    8 R(I)=R(I)+C(I,J)*P(J)
      RETURN
      END
```

7.3 Sample Problem - Pressurized Crack

(i) Problem

Consider a mathematically flat crack, modelled using one quadratic DD element and exhibiting one line of symmetry (Refer to figure 7.2).



$E = 10^6$
$\nu = 0.25$
$c = 1$

Figure 7.2 Pressurized Crack

(ii) Input

The required program input is shown below.

Card Number		Column Number 5	10	15	20	25	30	35	40	45	50
1	Title	Pressurized Crack									
2	Control	2	2	1	0		1000000.		0.25		
3	Mesh and B.C.		0.		1.0		1.0		0.0	0	0
4	Mesh		0.		0.0						
5	Field Stresses		0.		0.0		0.0				
6	Internal Points	1									
7	Coordinates		1.		1.0						
8	End Program	0									

Table 7.2 Sample Problem Input

```
 - DISPLACEMENT DISCONTINUITY  DP

SSURIZED CRACK
  NNOD      ORDER          NLS        MSCL           E                   PR
   2          3              1           0   1000000.00000          0.25000

SIAN ELIMINATION

    N          X                Y                  N               S              NCODE      SCODE
    1        0.00000          1.00000
                                                1.00000         0.00000           0          0
    2        0.00000          0.00000
                                                0.00000         0.00000           0          0

D STRESSES

   SX               SY                SXY
 0.00000          0.00000           0.00000

NDARY DISPLACEMENTS AND STRESSES
   N       DN              DS               UN              US                ST              SN              SS
   1   -0.15369D-05   0.00000D 00   0.76844D-06   -0.54853D-06       1.00000         1.00000         0.00000
   1   -0.32638D-05   0.00000D 00   0.16344D-05   -0.32509D-06       1.00000         1.00000         0.00000
   1   -0.37810D-05   0.00000D 00   0.18940D-05   -0.37462D-07       1.00000         1.00000         0.00000

RNAL DISPLACEMENTS AND STRESSES
              X                Y              UX              UY                SX              SY              SXY
           1.00000          1.00000    -0.62164D-06    -0.13924D-06       0.05734         0.10622         0.30022
PROGRAM

 USAGE       OBJECT CODE=   28432 BYTES,ARRAY AREA=   20040 BYTES,TOTAL AREA AVAILABLE=   73672  BYTES
NOSTICS        NUMBER OF ERRORS=        0, NUMBER OF WARNINGS=        0, NUMBER OF EXTENSIONS=        0
ILE TIME=     0.31 SEC,EXECUTION TIME=     0.06 SEC,
```

(iv) Results

Jaeger and Cook (1969) give the normal displacement on the surface of this crack as

$$v = \frac{p(1-2\nu)(1+\nu)(c^2-y^2)^{\frac{1}{2}}}{E}$$

$$= (1-y^2)^{\frac{1}{2}} \times 6.25 \times 10^{-7} \quad . \qquad (7.1)$$

The stresses at the point (1,1) can also be determined from the equations given by Jaeger and Cook (1969) as

$$\sigma_x = 0.0527 \ , \ \sigma_y = 0.1064 \ , \ \tau_{xy} = 0.2979 \ . \qquad (7.2)$$

It can be observed that the results from the program agree very well with this closed form solution.

CHAPTER VIII

CONCLUSIONS

The boundary element method offers considerable advantages over finite element analyses when the ratio of surface area to total volume of the body considered is small. This is particularly evident in problems such as the pressurized crack sample problem described previously.

The indirect approach is most desirable when modelling cracks or joints using the singularity corresponding to a discontinuity in displacement since the value of the discontinuity will be required. On the other hand, the direct approach should be used when modelling closed boundaries since the value of the discontinuity or boundary force is of no interest and fewer unknowns need be considered.

The singularity corresponding to a discontinuity in displacement offers considerable advantages in situations where rigid body motion cannot be accounted for by independent means (bending).

Higher order elements give consistently more accurate results than lower order elements for the same number of unknowns considered. In many of the examples considered, quadratic elements gave an order of magnitude accuracy increase.

Errors due to kinks in boundaries are limited to the immediate vicinity of the kink or corner and do not appear to affect the accuracy of the soltuion at other distant points. The accuracy increases as one moves away from the boundary. The unknown boundary stress and displacement values determined in the analyses are reasonably accurate however, particularly away from any kinks or corners. Bending is poorly accommodated by the point load singularity. High order displacement discontinuity elements show promise for effectiveness in this situation.

IX. BIBLIOGRAPHY

Benjumea, R. and Sikarskie, D.L., "On the Solution of Plane, Orthotropic Elasticity Problems by an Integral Method", J. of Applied Mech., Sept., p. 801, 1972.

Brebbia, C.A. and Dominguez, J., "Boundary Element Methods for Potential Problems", Appl. Math. Modelling, Vol. 1, Dec., p. 372, 1977.

Brebbia, C.A., The Boundary Element Method for Engineers, Pentech Press, Plymouth, Devon, England, 1978.

Crouch, S.L., Analysis of Stresses and Displacement Around Underground Excavations: An Application of the Displacement Discontinuity Method, Dept. of Civil and Mineral Engineering, U. of Minnesota, Minneapolis, Minn., 1976.

Cruse, T.A., "Numerical Solutions in Three Dimensional Elastostatics", Int. J. Solids Structures, Vol. 5, p. 1259, 1969.

Hornbeck, R.W., Numerical Methods, Quantum Publishers, New York, 1975.

Jaeger, J.C. and Cook, N.G.W., Fundamentals of Rock Mechanics, Chapman and Hall, London, 1976.

Lachat, J.C. and Watson, J.O., "Effective Numerical Treatment of Boundary Integral Equations: A Formulation for Three-Dimensional Elastostatics", Int. J. for Num. Methods in Eng., Vol. 10., p. 991, 1976.

Love, A.E.H., The Mathematical Theory of Elasticity, Cambridge U. Press, 1927.

Massonnet, C.E., "Numercial Use of Integral Procedures", A paper presented in Stress Analysis, ed. Zienkiewicz, O.C. and Holister, G.S., J. Wiley and Sons, New York, p. 198, 1965.

Rizzo, F.J., "An Integral Equation Approach to Boundary Value Problems of Classical Elastostatics", Quarterly of Applied Math., Vol. 25, p. 83, 1967.

Rizzo, F.J. and Shippy, D.J., "A Formulation and Solution Procedure for the General Non-Homogeneous Elastic Inclusion Problem", Int. J. Solids Structures, Vol. 4, p. 1161, 1968.

Rizzo, F.J., "The Boundary-Integral Equation: A Modern Computational Procedure in Applied Mechanics", A paper presented in Boundary-Integral Equation Method: Computational Applications in Applied Mechanics, Applied Mechanics Conference, ed. Cruse, T.A. and Rizzo, F.J., Am. Soc. Mech. Eng., New York, pl, 1975.

Timeshenko, S.P. and Goodier, J.N., Theory of Elasticity, McGraw-Hill, New York, 1970.

Appendix I

A Point Load Singularity

A.1 Constant Variation

a is half the element length

$$r^2 = (x'-x)^2 + y^2$$

$$\sigma_x = \frac{X}{4\pi(1-\nu)}\left[\frac{y^2}{r^2} + (3-2\nu)\ln r\right]_{-a}^{a}$$

$$+ \frac{Y}{4\pi(1-\nu)}\left[\frac{y(x'-x)}{r^2} + 2\nu \tan^{-1}\left(\frac{x'-x}{-y}\right)\right]_{-a}^{a}$$

$$\sigma_y = \frac{X}{4\pi(1-\nu)}\left[-\frac{y^2}{r^2} - (1-2\nu)\ln r\right]_{-a}^{a}$$

$$+ \frac{Y}{4\pi(1-\nu)}\left[-\frac{y(x'-x)}{r^2} + 2(1-\nu)\tan^{-1}\left(\frac{x'-x}{-y}\right)\right]_{-a}^{a}$$

$$\tau_{xy} = \frac{X}{4\pi(1-\nu)}\left[\frac{y(x'-x)}{r^2} + 2(1-\nu)\tan^{-1}\left(\frac{x'-x}{-y}\right)\right]_{-a}^{a}$$

$$+ \frac{Y}{4\pi(1-\nu)}\left[-\frac{y^2}{r^2} + (1-2\nu)\ln r\right]_{-a}^{a}$$

$$u = \frac{X}{4\pi E(1-\nu)}\left\{\begin{matrix} -(3-\nu-4\nu^2)(x'-x)(\ln r - 1) \\ + 4(1-\nu^2)\, y \tan^{-1}\left(\frac{x'-x}{-y}\right)\end{matrix}\right\}_{-a}^{a}$$

$$+ \frac{Y}{4\pi E(1-\nu)}\left[-(1+\nu)\, y \ln r\right]_{-a}^{a}$$

$$v = \frac{X}{4\pi E(1-\nu)}\left[-(1+\nu)\, y \ln r\right]_{-a}^{a}$$

$$+ \frac{Y}{4\pi E(1-\nu)}\left\{\begin{matrix} 2(1-\nu-2\nu^2)\left[(x'-x) + y\tan^{-1}\left(\frac{x'-x}{-y}\right)\right] \\ -(3-\nu-4\nu^2)(x'-x)\ln r\end{matrix}\right\}_{-a}^{a}$$

A.2 Linear Variation

$$\sigma_x = \frac{X}{4\pi(1-\nu)}\left[\frac{y^2(x'-x)}{r^2} + (3-2\nu)(x'-x) + 2(1-\nu)\, y \tan^{-1}\left(\frac{x'-x}{-y}\right)\right]_{-a}^{a}$$

$$+ \frac{Y}{4\pi(1-\nu)}\left[-\frac{y^3}{r^2} - (1+2\nu)\, y \ln r\right]_{-a}^{a}$$

$$\sigma_y = \frac{X}{4\pi(1-\nu)}\left[-\frac{y^2(x'-x)}{r^2} - (1-2\nu)(x'-x) - 2(1-\nu)\, y \tan^{-1}\left(\frac{x'-x}{-y}\right)\right]_{-a}^{a}$$

$$+ \frac{Y}{4\pi(1-\nu)}\left[\frac{y^3}{r^2} - (1-2\nu)\, y \ln r\right]_{-a}^{a}$$

$$\tau_{xy} = \frac{X}{4\pi(1-\nu)}\left[-\frac{y^3}{r^2} - (3-2\nu)\, y \ln r\right]_{-a}^{a}$$

$$+ \frac{Y}{4\pi(1-\nu)}\left[-\frac{y^2(x'-x)}{r^2} + (1-2\nu)(x'-x) - 2\nu\, y \tan^{-1}\left(\frac{x'-x}{-y}\right)\right]_{-a}^{a}$$

$$u = \frac{X}{4\pi E(1-\nu)}\left\{-(1+\nu)\, y^2 \ln r - (3-\nu-4\nu^2)\left[\frac{r^2}{2}\ln r - \frac{(x'-x)^2}{4}\right]\right\}_{-a}^{a}$$

$$+ \frac{Y}{4\pi E(1-\nu)}\left[-(1+\nu)\, y(x'-x) - (1+\nu)\, y^2 \tan^{-1}\left(\frac{x'-x}{-y}\right)\right]_{-a}^{a}$$

$$v = \frac{X}{4\pi E(1-\nu)}\left[-(1+\nu)\, y(x'-x) - (1+\nu)\, y^2 \tan^{-1}\left(\frac{x'-x}{-y}\right)\right]_{-a}^{a}$$

$$+ \frac{Y}{4\pi E(1-\nu)}\left\{-(1+\nu)\left[\frac{(x'-x)^2}{2} - y^2 \ln r\right] - (3-\nu-4\nu^2)\left[\frac{r^2}{2}\ln r - \frac{(x'-x)^2}{4}\right]\right\}_{-a}^{a}$$

A.3 <u>Quadratic Variation</u>

$$\sigma_x = \frac{x}{4\pi(1-\nu)}\left[-\frac{y^4}{r^2} - (5-2\nu)\,y^2 \ln r + (3-2\nu)\frac{(x'-x)^2}{2}\right]_{-a}^{a}$$

$$+ \frac{y}{4\pi(1-\nu)}\left[-\frac{y^3(x'-x)}{r^2} - 2(1+\nu)y^2 \tan^{-1}\left(\frac{x'-x}{-y}\right) - (1+2\nu)\,y\,(x'-x)\right]_{-a}^{a}$$

$$\sigma_y = \frac{x}{4\pi(1-\nu)}\left[\frac{y^4}{r^2} + (3-2\nu)\,y^2 \ln r - (1-2\nu)\frac{(x'-x)^2}{2}\right]_{-a}^{a}$$

$$+ \frac{y}{4\pi(1-\nu)}\left[\frac{y^3(x'-x)}{r^2} + 2\nu y^2 \tan^{-1}\left(\frac{x'-x}{-y}\right) - (1-2\nu)\,y\,(x'-x)\right]_{-a}^{a}$$

$$\tau_{xy} = \frac{x}{4\pi(1-\nu)}\left[-\frac{y^3(x'-x)}{r^2} - 2(2-\nu)y^2 \tan^{-1}\left(\frac{x'-x}{-y}\right) - (3-2\nu)\,y\,(x'-x)\right]_{-a}^{a}$$

$$+ \frac{y}{4\pi(1-\nu)}\left[\frac{y^4}{r^2} + (1+2\nu)\,y^2 \ln r + (1-2\nu)\frac{(x'-x)^2}{2}\right]_{-a}^{a}$$

$$u = \frac{x}{4\pi E(1-\nu)}\left\{-(1+\nu)\left[y^2(x'-x) + y^3 \tan^{-1}\left(\frac{x'-x}{-y}\right)\right]\right.$$

$$\left. - (3-\nu-4\nu^2)\left[\frac{(x'-x)^3}{3}\ln r - \frac{(x'-x)^3}{9} + \frac{y^2(x'-x)}{3} + \frac{y^3}{3}\tan^{-1}\left(\frac{x'-x}{-y}\right)\right]\right\}_{-a}^{a}$$

$$+ \frac{y}{4\pi E(1-\nu)}\left\{-(1+\nu)\left[\frac{y(x'-x)^2}{2} - y^3 \ln r\right]\right\}_{-a}^{a}$$

$$v = \frac{x}{4\pi E(1-\nu)}\left\{-(1+\nu)\left[\frac{y(x'-x)^2}{2} - y^3 \ln r\right]\right\}_{-a}^{a}$$

$$+ \frac{y}{4\pi E(1-\nu)}\left\{-(1+\nu)\left[\frac{(x'-x)^3}{3} - y^2(x'-x) - y^3 \tan^{-1}\left(\frac{x'-x}{-y}\right)\right]\right.$$

$$\left. - (3-\nu-4\nu^2)\left[\frac{(x'-x)^3}{3}\ln r - \frac{(x'-x)^3}{9} + \frac{y^2(x'-x)}{3} + \frac{y^3}{3}\tan^{-1}\left(\frac{x'-x}{-y}\right)\right]\right\}_{-a}^{a}$$

B Displacement Discontinuity Singularity

B.1 Constant Variation

$$\sigma_x = \frac{E X}{4\pi(1-\nu^2)} \left[\frac{-3y(x'-x)^2 - y^3}{r^4} \right]_{-a}^{a}$$

$$+ \frac{E Y}{4\pi(1-\nu^2)} \left[\frac{-(x'-x)^3 + (x'-x)y^2}{r^4} \right]_{-a}^{a}$$

$$\sigma_y = \frac{E X}{4\pi(1-\nu^2)} \left[\frac{y(x'-x)^2 - y^3}{r^4} \right]_{-a}^{a}$$

$$+ \frac{E Y}{4\pi(1-\nu^2)} \left[\frac{-(x'-x)^3 - 3y^2(x'-x)}{r^4} \right]_{-a}^{a}$$

$$\tau_{xy} = \frac{E X}{4\pi(1-\nu^2)} \left[\frac{-(x'-x)^3 + y^2(x'-x)}{r^4} \right]_{-a}^{a}$$

$$+ \frac{E Y}{4\pi(1-\nu^2)} \left[\frac{y(x'-x)^2 - y^3}{r^4} \right]_{-a}^{a}$$

$$u = \frac{X}{4\pi(1-\nu)} \left[-\frac{y(x'-x)}{r^2} + 2(1-\nu)\tan^{-1}\left(\frac{x'-x}{y}\right) \right]_{-a}^{a}$$

$$+ \frac{Y}{4\pi(1-\nu)} \left[(1-2\nu)\ln r + \frac{y^2}{r^2} \right]_{-a}^{a}$$

$$v = \frac{X}{4\pi(1-\nu)} \left[-(1-2\nu)\ln r + \frac{y^2}{r^2} \right]_{-a}^{a}$$

$$+ \frac{Y}{4\pi(1-\nu)} \left[\frac{y(x'-x)}{r^2} + 2(1-\nu)\tan^{-1}\left(\frac{x'-x}{y}\right) \right]_{-a}^{a}$$

B. 2 Linear Variation

$$\sigma_x = \frac{E X}{4\pi(1-\nu^2)}\left[-\frac{4y(x'-x)}{r^2} + \frac{2y^3(x'-x)}{r^4} + 2\tan^{-1}\left(\frac{x'-x}{y}\right)\right]_{-a}^{a}$$

$$+ \frac{E Y}{4\pi(1-\nu^2)}\left[\frac{4y^2}{r^2} - \frac{2y^4}{r^4} + \ln r\right]_{-a}^{a}$$

$$\sigma_y = \frac{E X}{4\pi(1-\nu^2)}\left[\frac{2y(x'-x)}{r^2} - \frac{2y^3(x'-x)}{r^4}\right]_{-a}^{a}$$

$$+ \frac{E Y}{4\pi(1-\nu^2)}\left[-\frac{2y^2}{r^2} + \frac{2y^4}{r^4} + \ln r\right]_{-a}^{a}$$

$$\tau_{xy} = \frac{E X}{4\pi(1-\nu^2)}\left[\frac{4y^2}{r^2} - \frac{2y^4}{r^4} + \ln r\right]_{-a}^{a}$$

$$+ \frac{E Y}{4\pi(1-\nu^2)}\left[\frac{2y(x'-x)}{r^2} - \frac{2y^3(x'-x)}{r^4}\right]_{-a}^{a}$$

$$u = \frac{X}{4\pi(1-\nu)}\left[\frac{y^3}{r^2} + (3-2\nu)\, y \ln r\right]_{-a}^{a}$$

$$+ \frac{Y}{4\pi(1-\nu)}\left[(1-2\nu)(x'-x) + \frac{y^2(x'-x)}{r^2} - 2(1-\nu)\, y \tan^{-1}\left(\frac{x'-x}{y}\right)\right]_{-a}^{a}$$

$$v = \frac{X}{4\pi(1-\nu)}\left[-(1-2\nu)(x'-x) - 2\nu\, y \tan^{-1}\left(\frac{x'-x}{y}\right) + \frac{y^2(x'-x)}{r^2}\right]_{-a}^{a}$$

$$+ \frac{Y}{4\pi(1-\nu)}\left[(1-2\nu)\, y \ln r - \frac{y^3}{r^2}\right]_{-a}^{a}$$

8.3 Quadratic Variation

$$\sigma_x = \frac{EX}{4\pi(1-\nu^2)}\left[\frac{7y^3}{r^2} - \frac{2y^5}{r^4} + 6y\ln r\right]_{-a}^{a}$$

$$+\frac{EY}{4\pi(1-\nu^2)}\left[(x'-x) - 4y\tan^{-1}\left(\frac{x'-x}{y}\right) + \frac{5y^2(x'-x)}{r^2} - \frac{2y^4(x'-x)}{r^4}\right]_{-a}^{a}$$

$$\sigma_y = \frac{EX}{4\pi(1-\nu^2)}\left[-\frac{5y^3}{r^2} + \frac{2y^5}{r^4} - 2y\ln r\right]_{-a}^{a}$$

$$+\frac{EY}{4\pi(1-\nu^2)}\left[(x'-x) - 3y^2\frac{(x'-x)}{r^2} + 2y^4\frac{(x'-x)}{r^4}\right]_{-a}^{a}$$

$$\tau_{xy} = \frac{EX}{4\pi(1-\nu^2)}\left[(x'-x) - 4y\tan^{-1}\left(\frac{x'-x}{y}\right) + \frac{5y^2(x'-x)}{r^2} - \frac{2y^4(x'-x)}{r^4}\right]_{-a}^{a}$$

$$+\frac{EY}{4\pi(1-\nu^2)}\left[-\frac{5y^3}{r^2} + \frac{2y^5}{r^4} - 2y\ln r\right]_{-a}^{a}$$

$$u = \frac{X}{4\pi(1-\nu)}\left[\frac{y^3(x'-x)}{r^2} - 2y^2(2-\nu)\tan^{-1}\left(\frac{x'-x}{y}\right) + (3-2\nu)\,y\,(x'-x)\right]_{-a}^{a}$$

$$+\frac{Y}{4\pi(1-\nu)}\left[(1-2\nu)\frac{(x'-x)^2}{2} - \frac{y^4}{r^2} - (3-2\nu)\,y^2\ln r\right]_{-a}^{a}$$

$$v = \frac{X}{4\pi(1-\nu)}\left[-(1-2\nu)\frac{(x'-x)^2}{2} - \frac{y^4}{r^2} - (1+2\nu)\,y^2\ln r\right]_{-a}^{a}$$

$$+\frac{Y}{4\pi(1-\nu)}\left[(1-2\nu)y(x'-x) - y\frac{3(x'-x)}{r^2} + 2\nu y^2\tan^{-1}\left(\frac{x'-x}{y}\right)\right]_{-a}^{a}$$