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Compact polarimetric equations under backward scattering alignment convention

F. Charbonneau

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F. Charbonneau

Canada Centre for Mapping and Earth Observation, 580 Booth Street, Ottawa, Ontario

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Abstract

Inconsistencies in scientific literature related to radar compact polarimetry (CP) has raised issues in analysis tool development and in derived results comparison. These inconsistencies primarily result from confusion between the Forward Scattering Alignment (FSA) and the Backward Scattering Alignment (BSA) conventions. This can lead to the misinterpretation of results when the circular direction of rotation of the transmitted signal is not properly considered. This technical note summarizes and demonstrates the relationships between coherently received wave under a linear or a circular configuration for right circular transmitted signals under the BSA convention. As well, it defines the received Stokes vector, for right circular transmitted signals, in terms of a linear or a circular receiving bases and in terms of fully polarimetric data.

Résumé

L'incohérence de la littérature relativement à la polarimétrie radar compacte (PC) a entrainé des problèmes dans le développement des outils d'analyse et dans l'interprétation des résultats dérivés. Cette incohérence résulte principalement de la confusion entre les conventions « Forward Scattering Alignment (FSA) » et « Backward Scattering Alignment (BSA) ». L'interprétation des résultats est également parfois erronée lorsque la direction circulaire de rotation du signal transmis n'est pas prise en compte. Cette note technique résume et démontre les relations entre l'onde reçue de manière cohérente dans une configuration linéaire ou circulaire, pour des signaux transmis circulaires droits selon la convention BSA. De même, elle définit le vecteur de Stokes reçu, à partir d'un signal émis circulaire droit, en fonction d'une base de réception linéaire ou circulaire et en fonction de données entièrement polarimétriques.

Introduction

With the broader availability of circular transmitted and coherent linear received SAR data, aka CP (Compact Polarimetric) within the Earth Observation family, users have rediscovered the advantage of using polarimetric data with the help of the Stokes vector. The Stokes vector follows two different conventions depending on if the SAR system configuration is monostatic or bistatic. For the monostatic case, the Stokes vector will use the Backward Scattering Alignment (BSA) convention, while the Forward Scattering Alignment (FSA) convention is used for the bistatic configuration (or frequently called the "optical" convention). Unfortunately, since the Stokes vector was initially developed for optical studies under bistatic configuration, the FSA convention is the most frequent form seen in the general literature and from online search results. It's worth mentioning that using the Stokes vector under the FSA convention, with monostatic SAR configuration, is not wrong, but the interpretation of its derived parameters must be adapted accordingly. This is the main point that has raised confusion and inconsistency in CP literature around the expression of this vector. There is even more confusion when a publication doesn't specify which convention is used. This short note has as its objective to simply present the **received Stokes vector for a right circular Transmitted wave under a monostatic configuration (BSA convention)**, like the RADARSAT Constellation Mission SAR antenna. Section 1 lists common relationships between right circular transmitted - linear received with circular received CP configurations, with Stokes equations, and with fully polarimetric configuration. Section 2 demonstrates the origin of the equations presented in Section 1 and shows the consistency of those relationships moving from one base to another.

1. Base Equations under BSA Convention

This section presents the common relationships between polarizations under different bases for the complex form, for the Stokes vector and for the intensity components. Note: All backscattering parameters (RH, RR, HH, VV, …) mentioned below are complex amplitudes.

1.1. Complex amplitude elements of the scattering matrix

The complex amplitude scattering for a circular (R: Right or L: Left) transmitted signal can be expressed in terms of fully polarimetric data (HH, HV, VH, VV) or, alternatively, for different received configurations (linear H: Horizontal and V: Vertical or circular L: Left and R: Right).

$$\mathbf{RH} = \frac{1}{\sqrt{2}}(HH - iVH) \tag{1}$$

$$\mathbf{RV} = \frac{1}{\sqrt{2}}(HV - iVV) \tag{2}$$

$$\mathbf{RL} = \frac{1}{2}i(HH + VV) = \frac{1}{\sqrt{2}}(iRH - RV) = \mathbf{LR} = \frac{1}{\sqrt{2}}(iLV - LH) \tag{3}$$

$$\mathbf{RR} = \frac{1}{2}(VV - HH + iHV + iVH) = \frac{1}{\sqrt{2}}(iRV - RH) \tag{4}$$

$$\mathbf{LL} = \frac{1}{2}(HH - VV + iHV + iVH) = \frac{1}{\sqrt{2}}(iLH - LV) \tag{5}$$

1.2 Received Stokes Vector

The received Stokes vector for a right circular transmitted signal can be expressed as a function of the linear received (H and V) or circular received (R and L) configurations.

$$\begin{bmatrix} \langle \boldsymbol{S_0} \rangle \\ \langle \boldsymbol{S_1} \rangle \\ \langle \boldsymbol{S_2} \rangle \\ \langle \boldsymbol{S_3} \rangle \end{bmatrix} = \begin{bmatrix} \langle |RH|^2 \rangle + \langle |RV|^2 \rangle \\ \langle |RH|^2 \rangle - \langle |RV|^2 \rangle \\ 2 \cdot Re\langle RH \cdot RV^* \rangle \\ +2 \cdot Im\langle RH \cdot RV^* \rangle \end{bmatrix} = \begin{bmatrix} \langle |RL|^2 \rangle + \langle |RR|^2 \rangle \\ 2 \cdot Im\langle RR \cdot RL^* \rangle \\ 2 \cdot Re\langle RR \cdot RL^* \rangle \\ +(\langle |RL|^2 \rangle - \langle |RR|^2 \rangle) \end{bmatrix} = S_{pol} + S_{unpol} = S_0 \cdot \left(m \cdot \begin{bmatrix} 1 \\ \cos(2\psi) \cdot \cos(2\chi) \\ \sin(2\psi) \cdot \cos(2\chi) \\ \sin(2\chi) \end{bmatrix} + \begin{bmatrix} 1-m \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \quad \begin{matrix} (6.a) \\ (6.b) \\ (6.c) \\ (6.d) \end{matrix}$$

It is important to note the sign of the element <S_3>, for linear and for circular received configuration, which is negative under FSA convention. <> symbol refers to averaging parameter function and | | is the modulus of the complex number. The right side of equation (6) expresses the polarized (S_{pol}) and unpolarized (S_{unpol}) components of the Stokes vector. Where m is the degree of polarization, defined as

$m = \sqrt{\sum_{i=1}^{3} S_i^2} / S_0$, and ψ and χ are respectively the orientation and the ellipticity angles of the polarized component S_{pol}.

From fully polarimetric configuration, under the Linear-Linear base, the received Stokes vector for a transmitted right circular wave is expressed as:

$$\begin{bmatrix} \langle \boldsymbol{S_0} \rangle \\ \langle \boldsymbol{S_1} \rangle \\ \langle \boldsymbol{S_2} \rangle \\ \langle \boldsymbol{S_3} \rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot \langle |HH|^2 \rangle + \frac{1}{2} \cdot \langle |VV|^2 \rangle + \langle |HV|^2 \rangle - Im\langle HH \cdot HV^* \rangle - Im\langle HV \cdot VV^* \rangle \\ \frac{1}{2} \cdot \langle |HH|^2 \rangle - \frac{1}{2} \cdot \langle |VV|^2 \rangle - Im\langle HH \cdot HV^* \rangle + Im\langle HV \cdot VV^* \rangle \\ Re\langle HH \cdot HV^* \rangle - Im\langle HH \cdot VV^* \rangle + Re\langle HV \cdot VV^* \rangle \\ Im\langle HH \cdot HV^* \rangle + Re\langle HH \cdot VV^* \rangle + Im\langle HV \cdot VV^* \rangle - \langle |HV|^2 \rangle \end{bmatrix} \quad \begin{matrix} (7.a) \\ (7.b) \\ (7.c) \\ (7.d) \end{matrix}$$

Note: For easier reading, the <> is omitted for the following sections

1.3 Intensity form

Similarly, received intensities can be estimated from fully polarimetric configurations or for other received configurations.

$$|\boldsymbol{RH}|^2 = \frac{1}{2}\left(|HH|^2 - 2 \cdot Im\big(HH \cdot conj(VH)\big) + |VH|^2\right) \quad (8)$$

$$|\boldsymbol{RV}|^2 = \frac{1}{2}\left(|HV|^2 - 2 \cdot Im\big(HV \cdot conj(VV)\big) + |VV|^2\right) \quad (9)$$

$$|\boldsymbol{RL}|^2 = \frac{1}{2}\left(|RH|^2 + 2 \cdot Im(RH \cdot conj(RV)) + |RV|^2\right) = \frac{1}{4}\left(|HH|^2 + |VV|^2 + 2 \cdot Re(HH \cdot conj(VV))\right) \quad (10)$$

$$\begin{aligned} |\boldsymbol{RR}|^2 &= \frac{1}{2}\left(|RV|^2 + 2 \cdot Im(RV \cdot conj(RH)) + |RH|^2\right) \\ &= \frac{1}{4}\left(|VV|^2 + |HH|^2 + 4 \cdot |HV|^2 - 2 \cdot Re\big(HH \cdot conj(VV)\big) - 4 \cdot Im\big(HH \cdot conj(HV)\big) - 4 \cdot Im(HV \cdot conj(VV))\right) \end{aligned} \quad (11)$$

2. Equation Development

This section shows the derivation of relationships presented in the previous section in order to confirm their overall consistency.

2.1 Complex Amplitudes

In order to transform the Sinclair backscattering matrix from the linear base (S_{linear_base}) to the circular base ($S_{circular\_base}$), a unitary rotation matrix (R_{circ}) needs to be applied for transmission (R_{T_circ}) and reception (R_{R_circ}) configurations.

$$S_{linear\_base} = \begin{bmatrix} HH & HV \\ VH & VV \end{bmatrix} \tag{12}$$

$$S_{circ\_base} = R_{T\_circ} * S_{linear_{base}} * \left(R^*_{R\_circ}\right)^{-1} \qquad \text{where } R_{circ} = \begin{bmatrix} \cos(\tau) & \mathrm{i} \cdot \sin(\tau) \\ \mathrm{i} \cdot \sin(\tau) & \cos(\tau) \end{bmatrix} \tag{13}$$

Note: Under the BSA convention, the complex conjugate is applied on the rotation matrix before its inversion [Cloude, 2010, p. 56 and Lopez, 2021, p. 16].

The right circular (τ = -45°) unitary rotation matrix is defined as

$$\boldsymbol{R_{Right\_circ}} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \tag{14}$$

From equations (12), (13) and (14), the circular Sinclair matrix can be derived for the linear base:

$$\begin{bmatrix} \boldsymbol{RR} & \boldsymbol{RL} \\ \boldsymbol{LR} & \boldsymbol{LL} \end{bmatrix} = \frac{\sqrt{2}}{2} \cdot \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \cdot \begin{bmatrix} HH & HV \\ VH & VV \end{bmatrix} \cdot \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} \boldsymbol{RR} & \boldsymbol{RL} \\ \boldsymbol{LR} & \boldsymbol{LL} \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} HH - iVH & HV - iVV \\ VH - iHH & VV - iHV \end{bmatrix} \cdot \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = (-) \frac{1}{2} \begin{bmatrix} VV - HH + 2 \cdot iHV & i(HH + VV) \\ i(HH + VV) & HH + 2 \cdot iHV - VV \end{bmatrix} \qquad (16)$$

The negative sign (-) on the right side of equation (16) can be ignored since it is applied to all matrix elements (i.e. preservation of the intensities and relative polarimetric phases). One can also note the **right** and **left** circular transmitted-linear received complex elements (colored highlighted) in equation (16) are explicitly identified in equation (17), which also confirms equations (1) and (2).

$$\begin{bmatrix} \boldsymbol{RR} & \boldsymbol{RL} \\ \boldsymbol{LR} & \boldsymbol{LL} \end{bmatrix} = \begin{bmatrix} \boldsymbol{RH} & \boldsymbol{RV} \\ \boldsymbol{LH} & \boldsymbol{LV} \end{bmatrix} \cdot \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \qquad (17)$$

Since literature sometimes shows the circular base Sinclair matrix from a left circular (τ = +45°) point of view, the previous steps can also be done with the left circular rotation matrix.

$$\boldsymbol{R}_{\boldsymbol{Left\_circ}} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \qquad (18)$$

$$\begin{bmatrix} \boldsymbol{LL} & \boldsymbol{LR} \\ \boldsymbol{RL} & \boldsymbol{RR} \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \cdot \begin{bmatrix} HH & HV \\ VH & VV \end{bmatrix} \cdot \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \qquad (19)$$

$$\begin{bmatrix} \boldsymbol{LL} & \boldsymbol{LR} \\ \boldsymbol{RL} & \boldsymbol{RR} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} HH + iVH & HV + iVV \\ VH + iHH & VV + iHV \end{bmatrix} \cdot \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} HH + 2 \cdot iHV - VV & i(HH + VV) \\ i(HH + VV) & -HH + 2 \cdot iHV + VV \end{bmatrix} \qquad (20)$$

As shown, equations (16) and (20) are identical in terms of the complex amplitudes of the circular base Sinclair matrix derived from its linear base. This also proves the validity of equations (3-5).

2.2 Intensities

RH and RV intensities express by equations (8) and (9) can be derived from equations (1) and (2) as shown in equations (21-23) and (24-26)

$$|\boldsymbol{RH}|^2 = \frac{1}{\sqrt{2}}(HH - iVH) \cdot conj\left(\frac{1}{\sqrt{2}}(HH - iVH)\right) \quad (21)$$

$$|\boldsymbol{RH}|^2 = \frac{1}{2}(|HH|^2 + iHH \cdot conj(VH) - iVH \cdot conj(HH) + |VH|^2) \quad (22)$$

$$|\boldsymbol{RH}|^2 = \frac{\mathbf{1}}{\mathbf{2}}(|\boldsymbol{HH}|^2 - \mathbf{2} \cdot \boldsymbol{Im}(\boldsymbol{HH} \cdot \boldsymbol{conj}(\boldsymbol{VH})) + |\boldsymbol{VH}|^2) \quad \mathbf{(23)}$$

$$|\boldsymbol{RV}|^2 = \frac{1}{\sqrt{2}}(HV - iVV) \cdot conj\left(\frac{1}{\sqrt{2}}(HV - iVV)\right) \quad (24)$$

$$|\boldsymbol{RV}|^2 = \frac{1}{2}(|HV|^2 + iHV \cdot conj(VV) - iVV \cdot conj(HV) + |VV|^2) \quad (25)$$

$$|\boldsymbol{RV}|^2 = \frac{\mathbf{1}}{\mathbf{2}}(|\boldsymbol{HV}|^2 - \mathbf{2} \cdot \boldsymbol{Im}(\boldsymbol{HV} \cdot \boldsymbol{conj}(\boldsymbol{VV})) + |\boldsymbol{VV}|^2) \quad \mathbf{(26)}$$

2.3 Cross-correlation term used in Circular-Linear base Stokes vector

Still using the equations (1) and (2), we can derive the cross-correlation terms used to define elements $\mathbf{S_2}$ and $\mathbf{S_3}$ of the Stokes vector

$$\boldsymbol{RH} \cdot \boldsymbol{conj}(\boldsymbol{RV}) = \frac{1}{\sqrt{2}}(HH - iVH) \cdot conj\left(\frac{1}{\sqrt{2}}(HV - iVV)\right) \quad (27)$$

$$\boldsymbol{RH} \cdot \boldsymbol{conj}(\boldsymbol{RV}) = \frac{1}{2}(HH \cdot conj(HV) + iHH \cdot conj(VV) - iVH \cdot conj(HV) + VH \cdot conj(VV)) \quad (28)$$

$$\boldsymbol{RH} \cdot \boldsymbol{conj}(\boldsymbol{RV}) = \frac{1}{2}(HH \cdot conj(HV) + iHH \cdot conj(VV) - i|HV|^2 + VH \cdot conj(VV)) \quad (29)$$

By taking the real and the imaginary components of equation (29), we get elements $\mathbf{S_2}$ and $\mathbf{S_3}$ of the Stokes vector, respectively equations (30) and (31) as expressed in the linear base, as initially stated with equations (7c) and (7d)

$$2 \cdot \boldsymbol{Re}(\boldsymbol{RH} \cdot \boldsymbol{conj}(\boldsymbol{RV})) = \boldsymbol{Re}\big(\boldsymbol{HH} \cdot \boldsymbol{conj}(\boldsymbol{HV})\big) - \boldsymbol{Im}\big(\boldsymbol{HH} \cdot \boldsymbol{conj}(\boldsymbol{VV})\big) + \boldsymbol{Re}(\boldsymbol{VH} \cdot \boldsymbol{conj}(\boldsymbol{VV})) = \boldsymbol{S_2} \tag{30}$$

$$2 \cdot \boldsymbol{Im}\big(\boldsymbol{RH} \cdot \boldsymbol{conj}(\boldsymbol{RV})\big) = \boldsymbol{Im}\big(\boldsymbol{HH} \cdot \boldsymbol{conj}(\boldsymbol{HV})\big) + \boldsymbol{Re}\big(\boldsymbol{HH} \cdot \boldsymbol{conj}(\boldsymbol{VV})\big) + \boldsymbol{Im}\big(\boldsymbol{VH} \cdot \boldsymbol{conj}(\boldsymbol{VV})\big) - |\boldsymbol{HV}|^2 = \boldsymbol{S_3} \tag{31}$$

2.3 Transform to circular-circular base

2.3.1 Circular receive Intensities from right circular transmit and linear receive configurations

It also worth converting the linear received CP data into circular received intensities, which has a greater discriminating power then using RH and RV intensities. These circular received intensities, presented in equations (10) and (11), can be respectively calculated, from CP equations (3) and (4), as shown with equations (32-34) and (35-37).

$$|\boldsymbol{RL}|^2 = \frac{1}{\sqrt{2}}(iRH - RV) \cdot conj\left(\frac{1}{\sqrt{2}}iRH - RV\right) \tag{32}$$

$$|\boldsymbol{RL}|^2 = \frac{1}{2}(|RH|^2 - iRH \cdot conj(RV) + iRV \cdot conj(RH) + |RV|^2) \tag{33}$$

$$|\boldsymbol{RL}|^2 = \frac{1}{2}(|\boldsymbol{RH}|^2 + 2 \cdot \boldsymbol{Im}(\boldsymbol{RH} \cdot \boldsymbol{conj}(\boldsymbol{RV})) + |\boldsymbol{RV}|^2) \tag{34}$$

$$|\boldsymbol{RR}|^2 = \frac{1}{\sqrt{2}}(iRV - RH) \cdot conj\left(\frac{1}{\sqrt{2}}iRV - RH\right) \tag{35}$$

$$|\boldsymbol{RR}|^2 = \frac{1}{2}(|RV|^2 - iRV \cdot conj(RH) + iRH \cdot conj(RV) + |RH|^2) \tag{36}$$

$$|\boldsymbol{RR}|^2 = \frac{1}{2}(|\boldsymbol{RV}|^2 - 2 \cdot \boldsymbol{Im}(\boldsymbol{RH} \cdot \boldsymbol{conj}(\boldsymbol{RV})) + |\boldsymbol{RH}|^2) \tag{37}$$

2.3.2 Circular Intensities from fully polarimetric configuration

In the same manner, these circular intensities (10) and (11) can be respectively calculated from the fully polarimetric Sinclair linear base, as shown with equations (38-40) and (41-43).

$$|\boldsymbol{RL}|^2 = \frac{1}{2} i(HH + VV) \cdot conj\left(\frac{1}{2} i(HH + VV)\right) \tag{38}$$

$$|\boldsymbol{RL}|^2 = \frac{1}{4}(|HH|^2 + HH \cdot conj(VV) + VV \cdot conj(HH) \ + |VV|^2) \tag{39}$$

$$|\boldsymbol{RL}|^2 = \frac{\mathbf{1}}{\mathbf{4}}(|\boldsymbol{HH}|^2 + \ |\boldsymbol{VV}|^2 + 2 \cdot \boldsymbol{Re}(\boldsymbol{HH} \cdot \boldsymbol{conj}(\boldsymbol{VV}))) \tag{40}$$

and

$$|\boldsymbol{RR}|^2 = \frac{1}{2}(VV - HH + 2 \cdot iHV) \cdot conj(\frac{1}{2}(VV - HH + 2 \cdot iHV)) \tag{41}$$

$$\begin{aligned} |\boldsymbol{RR}|^2 = \frac{1}{4}(&|VV|^2 + |HH|^2 + 4 \cdot |HV|^2 - VV \cdot conj(HH) - 2 \cdot iVVconj(HV) - HH \cdot conj(VV) \ldots \\ &+2 \cdot iHH \cdot conj(HV) + 2 \cdot iHV \cdot conj(VV) - 2 \cdot iHV \cdot conj(HH)) \end{aligned} \tag{42}$$

$$|\boldsymbol{RR}|^2 = \frac{\mathbf{1}}{\mathbf{4}}\left(|\boldsymbol{VV}|^2 + |\boldsymbol{HH}|^2 + 4|\boldsymbol{HV}|^2 - 2 \cdot \boldsymbol{Re}\big(\boldsymbol{HH} \cdot \boldsymbol{conj}(\boldsymbol{VV})\big) - 4 \cdot \boldsymbol{Im}\big(\boldsymbol{HH} \cdot \boldsymbol{conj}(\boldsymbol{HV})\big) - 4 \cdot \boldsymbol{Im}(\boldsymbol{HV} \cdot \boldsymbol{conj}(\boldsymbol{VV}))\right) \tag{43}$$

2.3.3 Cross-correlation term used in Circular-Circular base Stokes vector (elements S_1 and S_2)

Using equations (3) and (4), which express the RL and RR complex amplitudes as functions of RH and RV, we can derive the cross-correlation term used to define elements $\mathbf{S_1}$ and $\mathbf{S_2}$ of the Stokes vector.

$$\boldsymbol{RR} \cdot \boldsymbol{conj}(\boldsymbol{RL}) = \frac{1}{\sqrt{2}}(iRV - RH) \cdot conj\left(\frac{1}{\sqrt{2}}(iRH - RV) \ \right) \tag{44}$$

$$\boldsymbol{RR} \cdot \boldsymbol{conj}(\boldsymbol{RL}) = \frac{1}{2}(RV \cdot conj(RH) - i|RV|^2 + i|RH|^2 + RH \ \cdot conj(RV) \ ) \tag{45}$$

$$\mathbf{2} \cdot \boldsymbol{RR} \cdot \boldsymbol{conj}(\boldsymbol{RL}) = -i|RV|^2 + i|RH|^2 + 2 \cdot Re(RH \ \cdot conj(RV) \tag{46}$$

By taking the real and the imaginary components of equation (46), we get elements $\mathbf{S_2}$ and $\mathbf{S_1}$ of the Stokes vector, respectively equations (48) and (47) as expressed in the linear base, as initially stated with equations (6c) and (6b).

$$2 \cdot Re(RR \cdot conj(RL)) = 2 \cdot Re(RH \cdot conj(RV) = S_2 \quad (47)$$

$$2 \cdot Im(RR \cdot conj(RL)) = |RH|^2 - |RV|^2 = S_1 \quad (48)$$

2.3.4 Circular-Circular base Stokes vector elements S_0 and S_3

Even though we know that $\mathbf{S_0}$ is the total intensity independent of the polarization base, and it is the sum of the received intensities, we can show that from equations (34) and (37) we get the relationship equation (6a).

$$S_0 = |RL|^2 + |RR|^2 = \frac{1}{2}\left(|RH|^2 + 2 \cdot Im\left(RH \cdot conj(RV)\right) + |RV|^2\right) + \frac{1}{2}(|RV|^2 - 2 \cdot Im(RH \cdot conj(RV)) + |RH|^2) \quad (49)$$

$$S_0 = |RL|^2 + |RR|^2 = |RH|^2 + |RV|^2 \quad (50)$$

Under the circular-circular base, the $\mathbf{S_3}$ term is the intensity difference between RL and RR (for BSA convention), consequently from equations (34) and (37) we get the relationship equation (6d).

$$S_3 = |RL|^2 - |RR|^2 = \frac{1}{2}\left(|RH|^2 + 2 \cdot Im\left(RH \cdot conj(RV)\right) + |RV|^2\right) - \frac{1}{2}(|RV|^2 - 2 \cdot Im(RH \cdot conj(RV)) + |RH|^2) \quad (51)$$

$$S_3 = |RL|^2 - |RR|^2 = +2 \cdot Im\left(RH \cdot conj(RV)\right) \quad (52)$$

Equation (52) is very important, it confirms the positive sign of “ $+2 \cdot Im\left(RH \cdot conj(RV)\right)$ ” which is a significant source of confusion in the CP literature.

2.3.5 Stokes vector elements under linear base derived from right circular transmit and linear or circular received

This last sub-section completes the exercise in defining the received Stokes vector, under the linear base (equations 7), for a right circular transmitted wave and received under a linear (RH & RV) or circular (RR & RL) base.

S_0 defined from RH (1) and RV (2) and their corresponding intensities (8) and (9) detailed the equation (7.a) can be expressed as

$$\boldsymbol{S_0} = |\boldsymbol{RH}|^2 + |\boldsymbol{RV}|^2 = \frac{1}{2}(|HH|^2 - 2 \cdot Im\big(HH \cdot conj(VH)\big) + |VH|^2) + \frac{1}{2}(|HV|^2 - 2 \cdot Im\big(HV \cdot conj(VV)\big) + |VV|^2) \tag{53}$$

$$\boldsymbol{S_0} = |\boldsymbol{RH}|^2 + |\boldsymbol{RV}|^2 = \frac{\mathbf{1}}{\mathbf{2}}\Big(|\boldsymbol{HH}|^2 + |\boldsymbol{VV}|^2 + \mathbf{2} \cdot |\boldsymbol{HV}|^2 - \mathbf{2} \cdot \boldsymbol{Im}(\boldsymbol{HH} \cdot \boldsymbol{HV}^*) - \mathbf{2} \cdot \boldsymbol{Im}(\boldsymbol{HV} \cdot \boldsymbol{VV}^*)\Big) \tag{54}$$

In the same manner, **S_0** defined from RL (3) and RR (4) and their corresponding intensities (10) and (11) is

$$\begin{aligned}\boldsymbol{S_0} = |\boldsymbol{RL}|^2 + |\boldsymbol{RR}|^2 = &\frac{1}{4}(|HH|^2 + |VV|^2 + 2 \cdot Re(HH \cdot conj(VV))) \ldots \\ &+ \frac{1}{4}(|VV|^2 + |HH|^2 + 4 \cdot |HV|^2 - 2 \cdot Re\big(HH \cdot conj(VV)\big) - 4 \cdot Im\big(HH \cdot conj(HV)\big) - 4 \cdot Im(HV \cdot conj(VV)))\end{aligned} \tag{55}$$

$$\boldsymbol{S_0} = |\boldsymbol{RL}|^2 + |\boldsymbol{RR}|^2 = \frac{\mathbf{1}}{\mathbf{2}}\Big(|\boldsymbol{HH}|^2 + |\boldsymbol{VV}|^2 + \mathbf{2} \cdot |\boldsymbol{HV}|^2 - \mathbf{2} \cdot \boldsymbol{Im}(\boldsymbol{HH} \cdot \boldsymbol{HV}^*) - \mathbf{2} \cdot \boldsymbol{Im}(\boldsymbol{HV} \cdot \boldsymbol{VV}^*)\Big) \tag{56}$$

S_1 defined from RH (1) and RV (2) and their corresponding intensities (8) and (9) detailed the equation (7.b) is

$$\boldsymbol{S_1} = |\boldsymbol{RH}|^2 - |\boldsymbol{RV}|^2 = \frac{1}{2}(|HH|^2 - 2 \cdot Im\big(HH \cdot conj(VH)\big) + |VH|^2) - \frac{1}{2}(|HV|^2 - 2 \cdot Im\big(HV \cdot conj(VV)\big) + |VV|^2) \tag{57}$$

$$\boldsymbol{S_1} = |\boldsymbol{RH}|^2 - |\boldsymbol{RV}|^2 = \frac{\mathbf{1}}{\mathbf{2}}\Big(|\boldsymbol{HH}|^2 - |\boldsymbol{VV}|^2 - \mathbf{2} \cdot \boldsymbol{Im}(\boldsymbol{HH} \cdot \boldsymbol{HV}^*) + \mathbf{2} \cdot \boldsymbol{Im}(\boldsymbol{HV} \cdot \boldsymbol{VV}^*)\Big) \tag{58}$$

The **S_2** element was derived in equation (7.c) from the relations for RH and RV presented in equations (1) and (2) and was shown to correspond to the real part of their correlation product as shown by equation (30).

$$S_2 = 2 \cdot Re(RH \cdot conj(RV) = Re\big(HH \cdot conj(HV)\big) - Im\big(HH \cdot conj(VV)\big) + Re(VH \cdot conj(VV)) \tag{59}$$

Similarly, $\mathbf{S_1}$ and $\mathbf{S_2}$ can be derived from the circular base from the cross-product between RR and RL

$$RR \cdot conj(RL) = \frac{1}{2}(VV - HH + 2iHV) \cdot conj(\frac{1}{2}i(HH + VV)) \tag{60}$$

$$RR \cdot conj(RL) = \frac{1}{4}\big(-iVV \cdot conj\ (HH) - i|VV|^2 + i|HH|^2 + iHH \cdot conj\ (VV) + 2HV \cdot conj(HH) + 2HV \cdot conj(VV)\big) \tag{61}$$

$$RR \cdot conj(RL) = \frac{1}{4}\big(-i|VV|^2 + i|HH|^2 + 2 \cdot Im(HH \cdot conj\ (VV)) + 2HV \cdot conj(HH) + 2HV \cdot conj(VV)\big) \tag{62}$$

By taking the **imaginary part** of equation (62), we obtain equation (63) which is the same as equation (58).

$$S_1 = 2 \cdot Im(RR \cdot conj(RL)) = \frac{1}{2}\big(|HH|^2 - |VV|^2 - 2 \cdot Im(HH \cdot HV^*) + 2 \cdot Im(HV \cdot VV^*)\big) \tag{63}$$

By taking the **real part** of equation (62), we get equation (64) which is the same as equation (59)

$$S_2 = 2 \cdot Re(RR \cdot conj(RL)) = Re\big(HH \cdot conj(HV)\big) - Im\big(HH \cdot conj(VV)\big) + Re(VH \cdot conj(VV)) \tag{64}$$

Finally, the $\mathbf{S_3}$ element (equation 7.d) was derived from the relations for RH and RV presented in equations (1) and (2) and was shown to correspond to the **imaginary part** of their correlation product (equation 31). Same demonstration can be done from the circular-circular base through equations (40) and (43) to show

$$\begin{aligned} S_3 = |RL|^2 - |RR|^2 &= \tfrac{1}{4}(|HH|^2 + |VV|^2 + 2 \cdot Re\big(HH \cdot conj(VV)\big))\ldots \\ &-\tfrac{1}{4}(|VV|^2 + |HH|^2 + 4 \cdot |HV|^2 - 2 \cdot Re\big(HH \cdot conj(VV)\big) - 4 \cdot Im\big(HH \cdot conj(HV)\big) - 4 \cdot Im(HV \cdot conj(VV))) \end{aligned} \tag{65}$$

$$S_3 = |RL|^2 - |RR|^2 = Im(HH \cdot conj(HV)) + Re(HH \cdot conj(VV)) + Im(VH \cdot conj(VV)) - |HV|^2 \quad (66)$$

3. Conclusion

The literature on compact radar polarimetry can be inconsistent and confusing. This results from the different CP configurations and the scattering alignment convention used. This technical note summarizes the fundamental relationships for right circular compact polarimetric configurations derived from linear or circular base, under Backward Scattering Alignment convention. The relationship integrity among the different bases is shown through the Stokes vector. The adoption of the BSA convention for CP data and the clarity of the above equations should contribute to the harmonization of CP tool development and the physical interpretation of the derived parameters.

4. References

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