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SOURCE LOCATION TECHNIQUES FOR SEISMIC ACTIVITY IN MINES

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SOURCE LOCATION TECHNIQUES FOR SEISMIC ACTIVITY IN MINES

by

J. Niewiadomski*

SUMMARY

The methods used for the location of the seismic sources in mines (i.e., rockburst) may be divided into two major groups: a) linear methods; and b) nonlinear methods.

Linear methods have a smaller area of application than the nonlinear ones. They can be used only in the presence of a relatively simple geological structure, for example, when the seismic velocity can be treated as a constant value. They are fast, normally non-iterative, relatively simple, and what is very important, allow one to use well developed mathematical methods of linear algebra.

The theory of non-linear methods for seismic source location is a subset of so called non-linear optimization. The non-linear optimization methods are used to search for the extremum of so called object function, which is constructed by using the results of measurements and assumptions related to the investigated physical phenomenon. These methods are nearly without exception iterative, and in most cases they use a linear approximation of the originally non-linear problem at every iteration step.

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LINEAR METHODS

The possibility of applying linear methods for source location is the result of simple algebraic operations on the set of basic non-linear equations, which relate the velocity of the seismic waves radiated from the source with the time needed to reach the receivers of these waves (geophones or seismometers). For example, for the i-th geophone an element of such a set may be written as follows:

$$t_i * v = \left[(x_1-a_i)^2 + (x_2-b_i)^2 + (x_3-c_i)^2\right]^2 \qquad \text{Eq 1}$$

where, x_1, x_2 and x_3 are coordinates of the source, a_i, b_i, and c_i are the coordinates of the i-th geophone, v is the seismic wave velocity, and t_i is the travel time of the impulse.

Let us assume that the velocity v is known. Then N + 2 non-linear equations may be used to construct the set of N linear equations using the method known as the USBM method (Blake et al., 1974) or N + 1 equations using Mt. Isa method (Godson et al., 1978). There exists a possibility to use a wide range of mutations of these methods. For example, one can remove the usually unused parameter t_0 - the time needed for the seismic wave to cover the distance from the source to the first geophone hit (see Mt. Isa method) - but such construction involves the time arrivals in power up to 3, which, in the case of unreliable time readings, makes the system more sensitive to errors than in the original Mt. Isa method.

All linear methods construct the system of equations, which can be written in the form:

$$A * x = b \qquad \text{Eq 2}$$

where the vector x contains components of unknown source parameters, A is the matrix of the linear system, and b is the right-hand side vector of the system. Detailed derivation of such a system may be found in the above mentioned publications.

System (2) most often is solved using conventional methods, such as Gauss-Elimination, with preceding multiplication of both sides of Equation 2 by transposed matrix A, A^T, when the system is overdetermined. This operation gives the best solution of the system in the sense of Least Squares. Despite the popularity of this method it seems that another group of methods, called General Inverse, are more attractive. An overview of the literature which covers the General Inverse (GI) methods may be found in the paper written by Ben-Israel and Charms (1963). From the many methods of GI which proved to be convenient, is a method which was discovered by Moore (1920), and evidently rediscovered by Penrose (1955). Let us assume that the Euclidean norm for vectors will be used:

$$\| x \| = (x^T * x)^{1/2} \qquad \text{Eq 3}$$

where, T stands for transposition. For system (2) the General Inverse of the matrix A, A^*, gives the solution x^*, where:

$$x^* = A^* * b \qquad \text{Eq 4}$$

This solution is the smallest (in the Euclidean norm sense) vector from the set of all vectors x, which minimize the residual vector r:

$$\| r \| = \| A * x - b \| = \min, \text{ if } x = x^* \qquad \text{Eq 5}$$

The properly constructed vector x^* represents the Least Square solution for equation 2, also in the presence of inconsistent data or/and in the case when system (2) is undetermined, as well as in the case of inconsistent data and singular matrix A.

The general inverse of the matrix A, A^*, may be easily calculated using the so called Singular Value Decomposition (SVD) technique (Lawson and Hanson, 1974). This technique allows decomposition of matrix A into the product of three elements:

$$A = U * S * V^T \qquad \text{Eq 6}$$

where, if A has dimension (m,n), then U is the matrix of dimension (m,m), matrix S: (m,n), and matrix C: (n,n). The symbol T means transposition. The matrix U consists of orthornormalized eigenvectors of the product $A*A^T$, the matrix V consists of the orthonornalized eigenvectors of the product A^T*A. The matrix S is a diagonal one with n non-negative elements on its diagonal. These elements, S_i, (i = 1,...n), are the square roots of the eigenvalues of the product A^T*A.

The Singular Value Decomposition allows one to build the inverse (GI) of the matrix A, A^*:

$$A^* = V * S^+ * U^T \qquad \text{Eq 7}$$

where S^+ is a diagonal matrix, in which diagonal elements have the inverse values of S_i, if $S_i > 0$, and 0, if $S_i = 0$. Then the solution of system (2) can be written as follows:

$$x^* = V * S^+ * U^T * b. \qquad \text{Eq 8}$$

When solving a linear system of equations for source location one may find two groups of difficulties: 1) diffulties caused by incorrect data (such as erroneous time arrival readings, or wrong coordinates of geophones); and 2) difficulties caused by ill-conditioning of the system of algebraic equations - when the matrix of the system is singular or very close to a singular one. To show how the SVD technique helps to deal with this, let us first describe one important characteristic of the matrix A, called condition number. Condition number, C(A), is defined as the ratio of the first diagonal element of the matrix S, s_1, to the last diagonal element of this matrix s_n:

$$C(A) = s_1/s_n \, . \qquad \text{Eq 9}$$

If s_n is equal to 0, matrix A is singular. Conditioning of the matrix may be worsened by rounding off and truncating the numbers in a computer.

Numerous examples of the importance of the condition number of matrix A for the solution of the linear algebraic system are published by Steward (1973), Franklin (1970), and Forsythe et al. (1977).

Some seismological applications of the SVD technique are presented by Lee and Steward (1981), Wiggins (1972), Jackson (1972), and Olson and Apsel (1982).

To demonstrate several other examples of the general inverse technique with SVD used for source location in the case of linear methods, let us start with the problem widely covered in literature (e.g., Hawley et al., 1981; Herrman, 1979), that is, find the variance of the calculated source coordinates in the presence of non-zero variances of data. Let us assume that the seismic velocity has a non-zero variance. This case gives us, at the same time, an idea on how the coordinates of the source location will behave when the velocity is known with sufficient precision, but the time readings are not exact. Let us first consider the USBM method (Blake et al., 1974). According to formula (8), the k-component of the source coordinates vector, x_k, may be expressed as follows:

$$x_k = \sum_{j=3}^{n} (V * S^+ * U^T)_{kj} \quad b_j \qquad \text{Eq 10}$$

where,

$$b_j = const_j + v^2 * (t_2 - t_j)$$

From this follows the variance of x_k coordinates:

$$var(x_k) = var(v^2) * \sum_{j=3}^{n} [(V*S^+*U^T)_{kj} * (t_2 - t_j)]^2 \qquad \text{Eq 11}$$

Two examples of the variance x_3 (elevation) of the source located by the USBM method and the Mt. Isa method are shown in Figures 1 and 2, respectively. For calculations the real geophone setting was used (Quirke Mine, Ontario, Canada). In this case the array of 32 geophones is located inside the area described by coordinates (all coordinates are in meters): $1300 < X < 3500$, $1600 < Y < 2800$, $3000 < Z < 3800$. On Figures 1 and 2 are shown the variances of x_3 for the sources located on the plane: $1000 < X,Y < 4000$, $Z = 1000$, which is a little above the array of geophones (if Z axis points downwards). The arrival times for the first 10 geophones hit from all 32 geophones were first calculated and then formulae 11 was used to calculate the variances.

From these examples of variance it follows that their maximum values in the case of the USBM method are around half of those values for Mt. Isa method. On the other hand, the area within Figure 2 covered by relatively small variance is of greater magnitude in the case of the Mt. Isa method, than when the USBM method is used. Generally, it seems that both methods give relatively similar results for some areas, but quite different results for others. The decision on which method to use must be preceded by the proper analysis of variances, because results differ for different geophone settings and different locations of the seismic sources.

Equation 2 may also be used for planning the geophone network to ensure the best precision of source locations for the expected area of seismic activity. This can be done with the aim of the co-variance matrix C (e.g., Jackson, 1972):

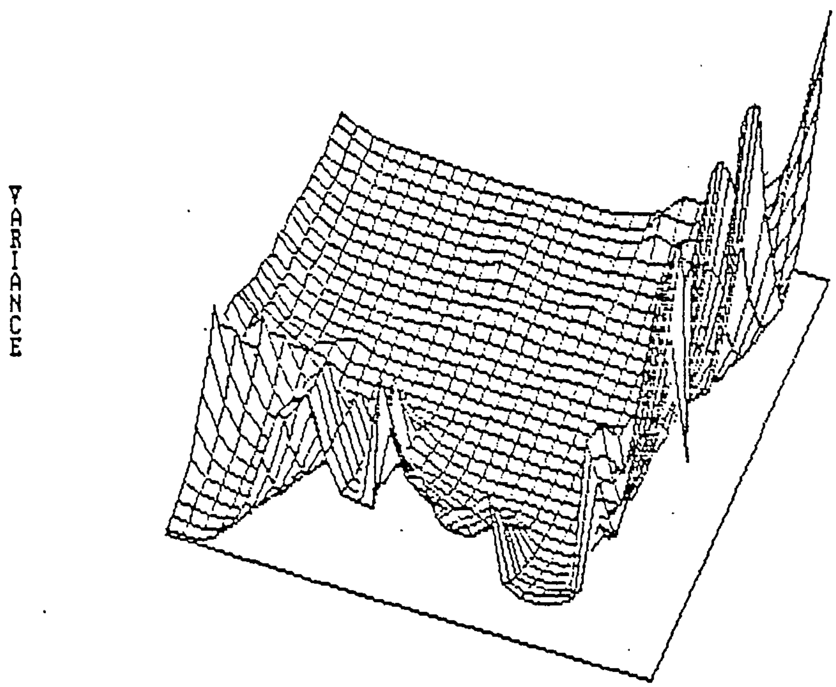

Fig. 1 - Variance of the vertical components of the source coordinates. Source coordinates were calculated by the USBM method. High values of variance indicate the area where this method may give incorrect results.

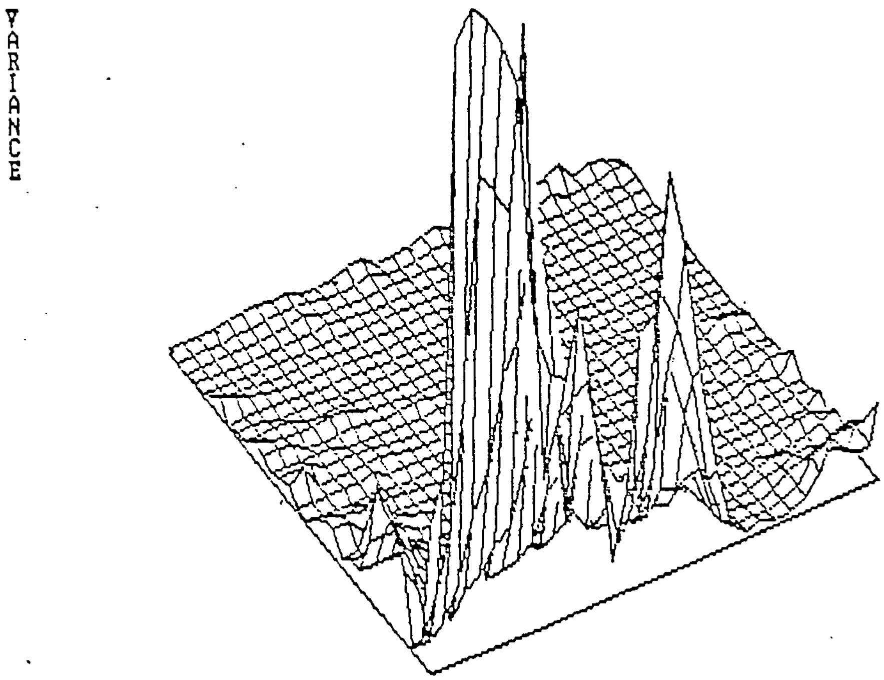

Fig. 2 - Variance of the vertical component of the source coordinates. Source coordinates were calculated by the Mt. Isa Method. The area where the variance is small, is larger than in the case of the USBM method.

$$C = \sigma^2 * (A^T * A)^{-1} \qquad \text{Eq 12}$$

where σ^2 is the variance of the data, such as time readings and/or assumed velocities. The optimal distribution of the geophone network is such a one which minimizes the determinant of the matrix C (e.g., Draper and Smith, 1966). Several algorithms for regional seismic network planning are published by Kijko (e.g., 1977, 1976), for media with constant velocities, and in the case of anisotropic media, with extensive use of the co-variance matrices.

The same problem may be formulated in terms of SVD: to find such geophone coordinates, that the co-variance matrix C, where:

$$C = \sigma^2 * (V * (S^+)^{-2} * V^T)$$

has the smallest condition number.

The SVD method may also be applied to check the set of data which are to be used for source location when some of them are expected to be incorrect. Let us assume that we have N linear equations (2) for K unknown values. Let N > K, which means that the system is overdetermined. In this case the dimension of A is (N,K). Let us construct a new matrix AB of dimension (N,K+1), which in the first K columns has the matrix A, and the additional column, K+1, consists of the right-hand side vector of the system (2), b:

$$AB = \left[(A); \quad b \right] \qquad \text{Eq 13}$$

If all the data used for construction of the matrix AB are correct, then, for N = K + 1:

$$\det (AB) = 0 \qquad \text{Eq 14}$$

The value of the determinant of the matrix AB can be treated as an indicator of correctness of the data. To use the determinant of a matrix is not convenient from a numerical point of view and it cannot be applied in the case when N - K > 1, as it is when the matrix AB is not a square one. This diffulcity may be overcome by the use of the SVD technique. In this case the condition number of the matrix AB, C(AB), may be used as the indicator. The presence of incorrect data will result in a smaller value of C(AB). Let us assume that we use N + 2 geophones to construct the system of N equations. Let M geophones register wrong (incorrect) time arrivals. If N -M > K, one may construct the matrices AB for subsets of all geophones, and calculate the condition numbers of matrices AB for all these subsets (combinations). By comparing condition numbers for different subsets of geophones it is possible to say which subsets should be used for the source location.

For illustration let us consider the same system of geophones as previously (Quirke Mine). Let a set of 8 geophones record the P-wave time arrivals from a seismic source with the coordinates outside the geophone network: (1000, 1000, 1000). Let the time arrival recorded by Geophone No. 2 be enlarged by 2%, to introduce an erroneous data.

From the set of 8, 28 subsets of geoophones are chosen, each consisting of 6 geophones. For all these subsets the values of condition numbers for matrices AB are shown on Figure 3 (Mt. Isa method), and on Figure 4 (USBM method). The values of condition numbers from the subsets of 6 geophones numbered from 1 to 15, and from 22 to 27 are much smaller than for subsets 16 to 21 and for the subset 28. The reason is, that in subsets 16 to 21 and 28 geophones, geophone No. 2 was not included.

On Figure 5 and 6 are shown the errors of the source locations calculated by using the same subsets of geophones but different methods. Figure 5 shows the errors of the source locations when the Mt. Isa method was used, and Figure 6 shows the errors when the USBM method was applied. The errors were calculated by solving the system of equations for each subset of geophones, and then calculating the differences between the calculated and real source coordinates. These differences exist because the time arrival recorded by geophone No. 2 is not correct (i.e., it is enlarged by 2%). The high precision is obtained only for the subsets for which the condition numbers are large.

To construct the errors shown on Figures 5 and 6 we needed to know where the source was really located (they are constructed only for demonstration), but the graphs shown on Figures 3 and 4 were built using only data which are used for source locations (coordinates of geophones and recorded time arrivals).

Figures 5 and 6 also allow us to compare the performance of both methods. For the previously assumed source outside the geophone array, the errors in source location calculated by the USBM method are twice as large as the errors calculated by the Mt. Isa method.

The above presented procedure works on a simple basis: it is assumed that the majority of geophones give correct data.

This method seems to be better than the one which is commonly used: to locate the source using all data, and then check which recorded time arrivals are much different to those calculated. In this case the recorded time arrivals are being compared with time arrivals calculated from a possibly incorrectly located source, which may lead to serious errors, and even an iterative procedure does not always give satisfactory results. On the contrary, the above presented method allows us to make a selection of data prior to the first attempt to locate the seismic source, and it is its main advantage.

The selection of a geophone (or geophones), which is a source of incorrect data can be accomplished by assigning a mark to all geophones in the given subset, the value of which depends on the value of the condition number for this subset, and then calculate the cumulative marks for every geophone. One such example is shown on Figure 7 where the arrival times recorded by geophones No. 3 and 4 were enlarged by 2%, to create the set of incorrect data. On this picture geophones No. 3 and 4 have the lowest marks, which indicates that these geophones whould be removed from the source location procedure.

The SVD technique may also be applied for scaling of the system of equations. All calculations performed by a computer are obtained using a final length of binary representation of the numbers, which results in their rounding off and truncation, so some time a proper scaling of the system is required prior to solving it. The scaling is accomplished by multiplication

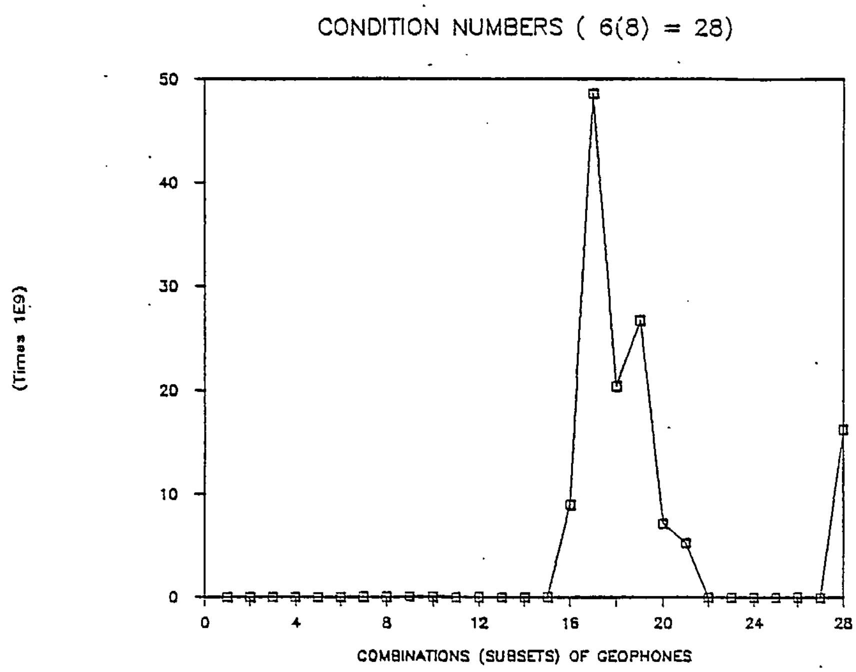

Fig. 3 - Condition numbers calculated for the matrices created by Mt. Isa method for 28 combinations of 6 geophones from the global set of 8 geophones.

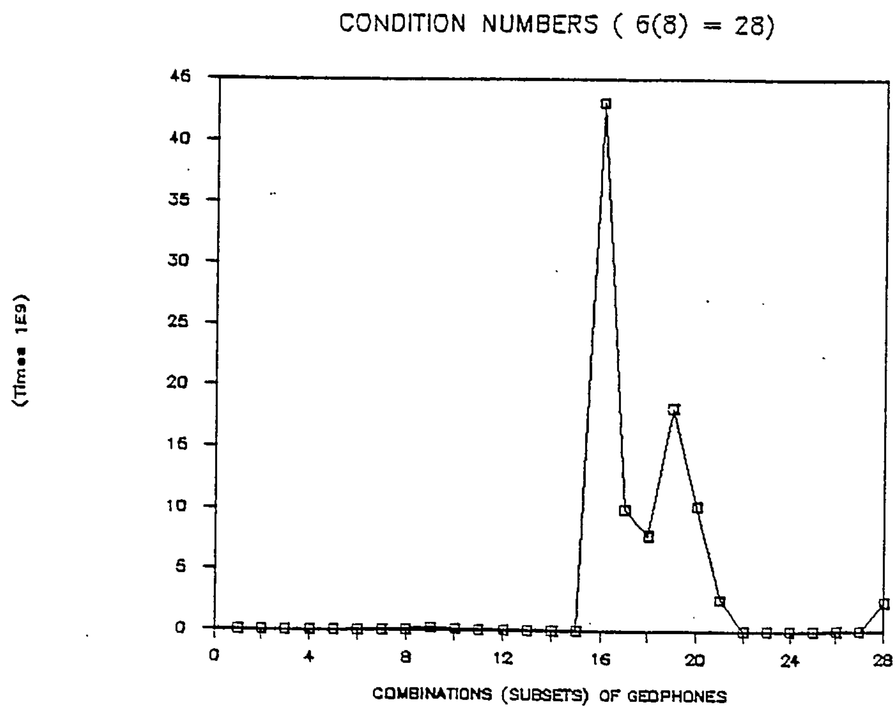

Fig. 4 - Condition numbers calculated for the matrices created by USBM method for 28 combinations of 6 geophones from the global set of 8 geophones.

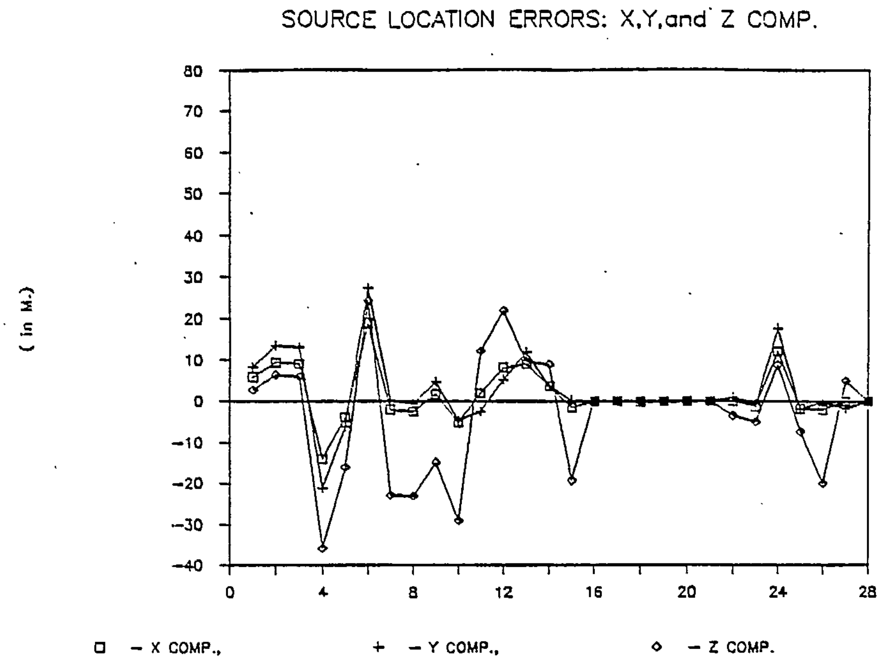

Fig. 5 - The differences between the real and calculated source coordinates caused by enlarging the time arrival recorded by geophone No. 3 by 2%. The source locations were calculated for different combinations of 6 geophones from the set of 8 (Mt. Isa method).

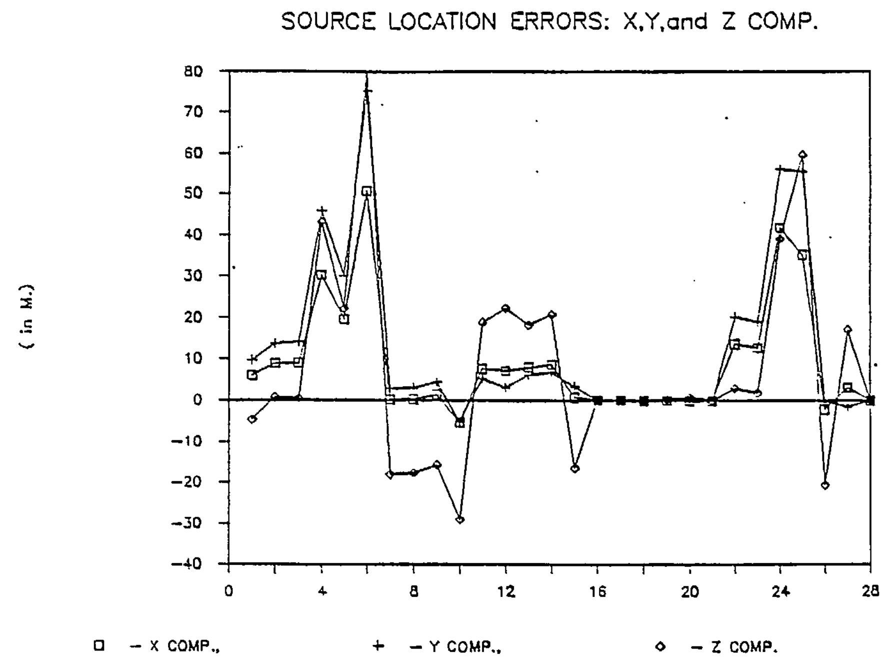

Fig. 6 - The differences between the real and calculated source coordinates caused by enlarging the time arrival recorded by geophone No. 2 by 2%. The source locations were calculated for different combinations of 6 geophones from the set of 8 (USBM method.)

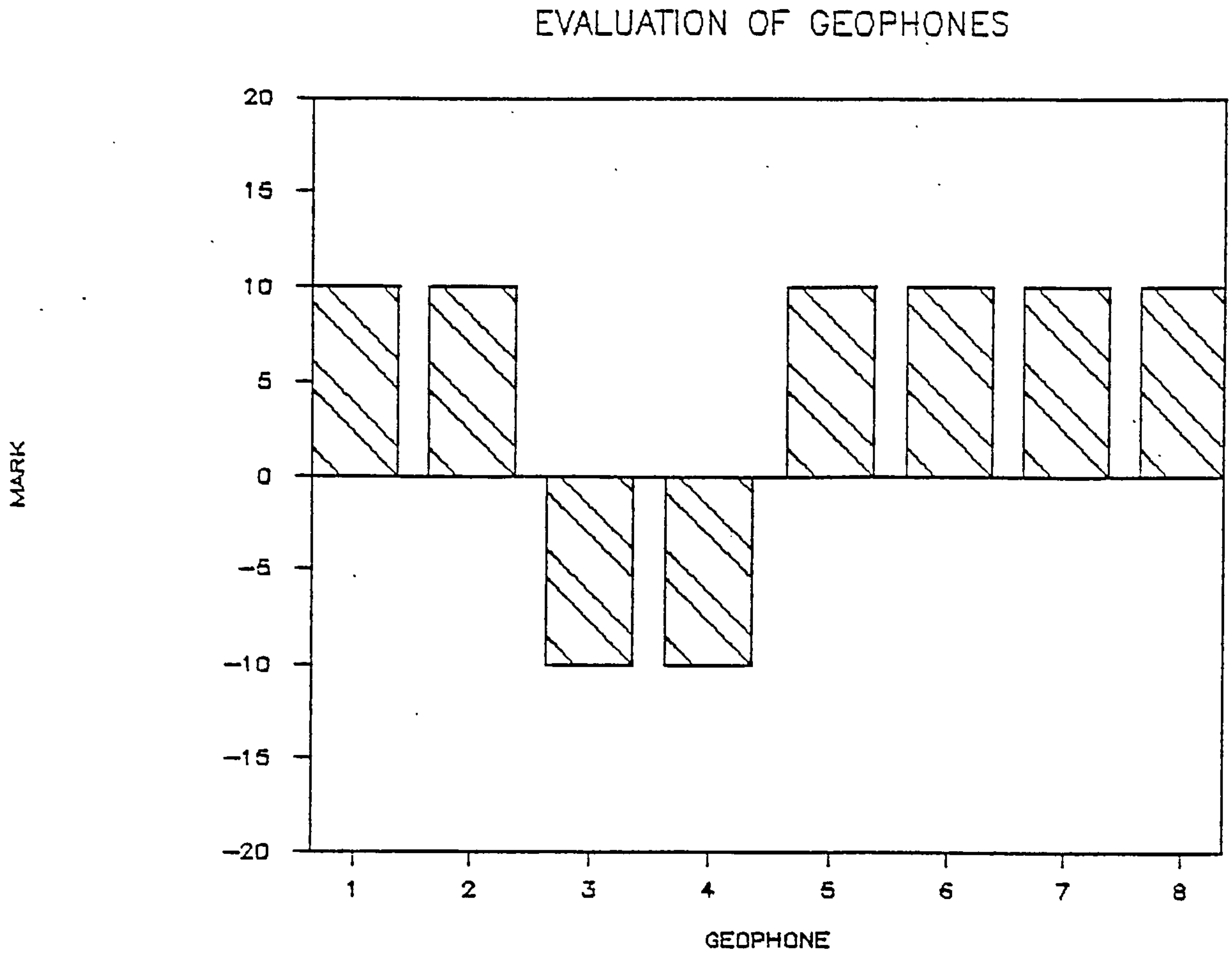

Fig. 7 - Evaluation of geophones using condition numbers of the matrices created by the Mt. Isa method. The time arrivals for geophones No. 3 and 4 were disturbed by 2%. These geophones are selected from the set of 8.

of the matrix A of system (2) by row scaling matrix S_r and column scaling matrix S_c. After scaling, the system of equations which are to be solved are the following:

$$(S_r) * A * (S_c) * (S_c^{-1} * x) = (S_r) * b \qquad \text{Eq 15}$$

The matrices S_r and S_c are diagonal. To demonstrate the effect of scaling on the solution let us consider the next system of linear equations:

$$\begin{bmatrix} 1, & 2 \\ .000101, & .000202 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3000 \\ .303 \end{bmatrix} \qquad \text{Eq 16}$$

The solution of the system is: x = 1000, y = 1000. Using Gauss Elimination method (subroutine "GELG" from System/360 Scientific Subroutine Package, IBM Application Program) and the IBM XT computer, the result is: x = 1500, y = 0. Let us multiply the second equation of the system (16) by a scaling factor s (row-scaling, when $S_r(1,1) = 1$, $S_r(2,2) = s$). For different scaling factors the solution (x,y) is different. The results are shown in Figure 8. Squares represent relative errors of the solution (x,y) calculated by Gauss-Elimination as a function of the scaling factor, where relative error, RE, is defined as:

$$RE = 100\%[(x-1000)^2 + (y-1000)^2]^{1/2}/(1000^2+1000^2)^{1/2}$$

From this Figure it follows that for only one scaling factor, s = 10000, the result is acceptable. On the same Figure are shown the relative errors of the solution (x,y) when the SVD method was used for solving the scaled system (triangles). The acceptable results are obtained for a wider range of scaling factors (from 0.01 to 1.E+8), which shows that the SVD method has a much higher performance than the Gauss-Elimination method.

Once again, the condition number may be used to find out the best value of the scaling parameters. For equation 16 the condition number as the function of the scaling factor is shown on Figure 9. The minimum value of $C(S_r * A)$ occurs for s = 10000, which demonstrates the applicability of the condition number as an indicator of proper or unproper scaling.

For every system of linear equations the diagonal parameters of the matrices S_r and S_c may be found using optimization methods. In such a case as the object function, the condition number of the scaled matrix may be used.

NON-LINEAR METHODS

Non-linear iterative methods are widely used in seismology for Earthquake location. They are also useful for rockburst location when the linear method cannot be applied. These methods include direct search methods when only the value of the object function is used (e.g., SIMPLEX method), gradient methods which use the first derivative of the object function (e.g., NEWTON LEAST SQUARE method), and methods with higher orders of the derivatives of the object functions, such as DUMPED LEAST SQUARE with the second order corrections. They belong to a well developed group of methods known as Non-linear Optimization Methods (see, for example, Himmelblau, 1972; Polak, 1971; Luenberger, 1973). Extensive literature on the non-linear methods used for source location is presented by Lee and Stewart (1981).

A non-linear method for source location was first used by Geiger (1910) for Earthquake locations, and this method with some modifications is widely used presently for the same purpose.

Let us assume that N time arrivals, t_i, (i=1,...,N), were registered by N receivers. The vector X or the source parameters, (Cartesian coordinates and the time of origin t_0), is assumed as a starting value. If the geological structure and the seismic wave velocity is known, then for all receivers the residual vector r can be calculated:

$$r_i = t_i(\text{observed}) - t_i(\text{calculated}),\ i = 1,\dots,N \qquad \text{Eq 17}$$

The square of the Euclidean norm of the residual vector r may be used as an object function F(X), which is to be minimized through an iterative procedure:

$$F(X) = r^T * r \qquad \text{Eq 18}$$

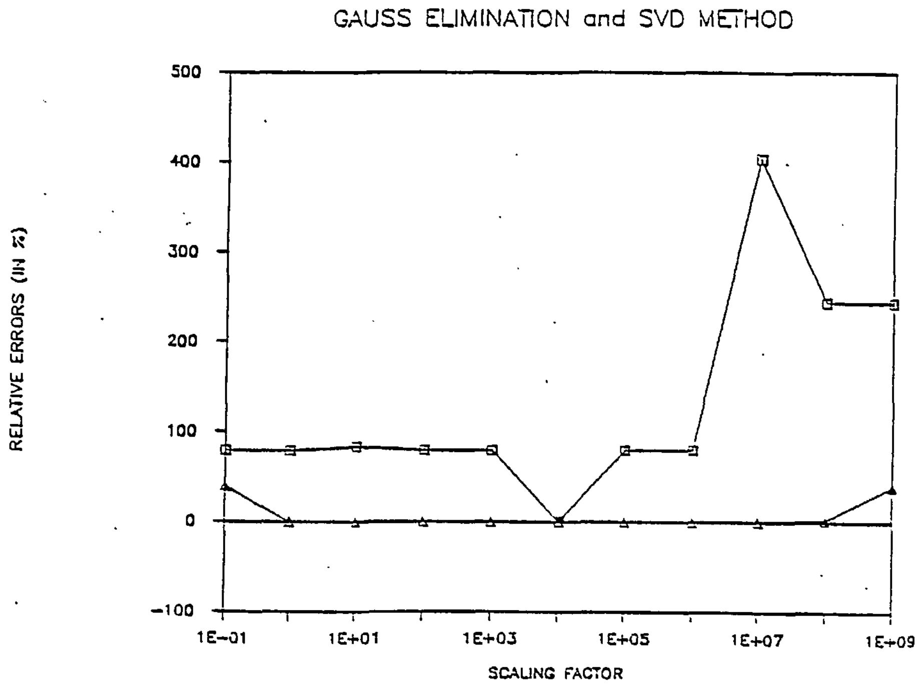

Fig. 8 - Relative errors of solution of two equations from the text. The errors can be diminished when the second equation is multiplied by proper scaling factor (row-scaling). Squares show results obtained by Gauss Elimination method, triangles - when Singular Value Decomposition was applied.

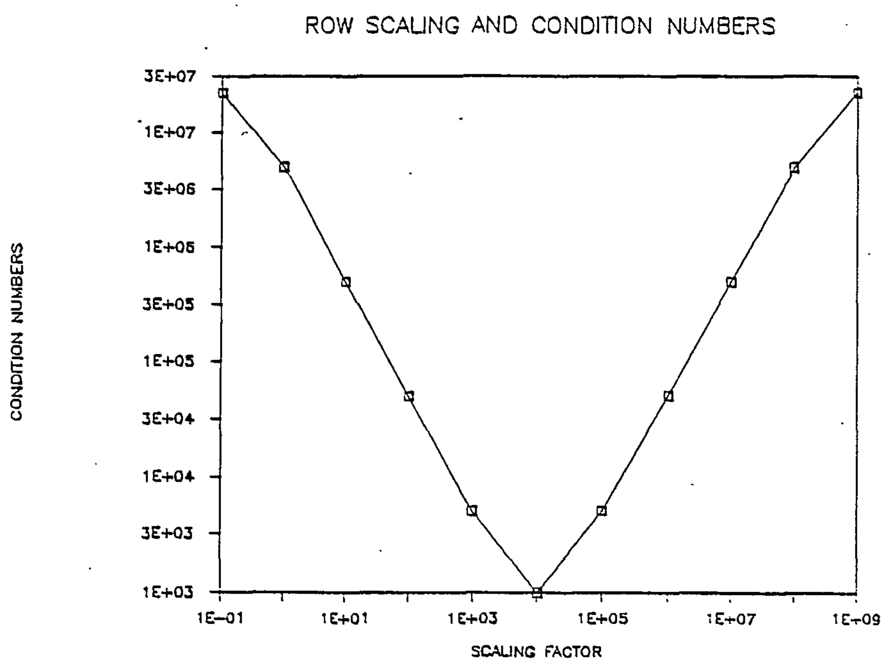

Fig. 9 - Condition numbers for the system of two linear equations. The second equation was multiplied by a scaling factor. The best scaling factor is indicated by the lowest value of the condition number.

This function is usually expanded into a Taylor series to construct a linear approximation of the function F(X), (e.g., Adby and Dempster, 1974):

$$F(X + \delta X) = F(X) + g^T * \delta X + .5 * \delta X^T * H * \delta X \qquad \text{Eq 19}$$

where, δX is the adjustment vector, g is the gradient vector, and H stands for the Hessian matrix. Let us define the new matrix A, which gives the variation of r_i due to the variation of source parameters:

$$A = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial t_i}{\partial X}, & \frac{\partial t_i}{\partial Y}, & \frac{\partial t_i}{\partial Z}, & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad \text{Eq 20}$$

The gradient vector g and Hessian H may be expressed through the matrix A:

$$g = 2 * A^T * r \qquad \text{Eq 21}$$

$$H = 2[A^T * A - (\nabla A^T) * r]$$

where, ∇ is the gradient vector operator.

The object function F is minimized when:

$$- H * \delta X = g \qquad \text{Eq 22}$$

The solution of the above equation gives the correction δX to the previously used value X.

The second component in the expression for Hessian in (22), $-2 * (\nabla A^T) * r$, is sometimes abandoned. If the system is then solved directly, it gives the Least Square (or Gauss-Newton) solution (e.g., Geiger, 1910). If the second term in Hessian is present, then the method is known as Dumped Least Square (Herrman, 1979). As the dumping factor one can use a simple number, small enough not to lose the resolution of the system (e.g., Wiggins, 1972), or thus this second term in the expression for Hessian, as suggested by Thurber (1985). A lot of other methods are also used such as QR decomposition (Buland, 1979), generalized inverse method (Bolt, 1970), or step-wise regression (Lee and Lahr, 1975). In every case the solution gives the new location of the source, and the whole process is repeated until the calculated corrections are smaller than a prescribed value.

For non-linear methods the choice of the starting value of the vector $X(x_1, x_2, x_3, t_o)$ is very important. If this vector is chosen close enough to the real source, then the selection of the data by using the Singular Value Decomposition prior to the source location itself, can also be applied.

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