SOME ANALYTIC EXPRESSIONS CONCERNING ERRORS ASSOCIATED WITH THEODOLITES USED FOR ABSOLUTE MAGNETIC MEASUREMENTS

R.L. Coles

DIVISION OF SEISMOLOGY AND GEOMAGNETISM

Earth Physics Branch
Energy, Mines and Resources Canada
Ottawa, Ontario
K1A 0Y3

Internal Report

1985-3 (G)

CONTENTS

ANALYTICAL EXPRESSIONS FOR THE ERRORS INTRODUCED BY AN OFF-LEVEL ADJUSTMENT OF A THEODOLITE USED IN THE DETERMINATION OF MAGNETIC DECLINATION AND INCLINATION

One of the major contributing factors to errors in the measurement of declination and inclination of the magnetic field vector is imprecise levelling of the theodolite. This study develops the analytical expressions for the apparent declination and inclination in terms of the amount of tilt applied to the theodolite.

We will consider a reference frame (I, J, K) fixed in geographic terms, with the K axis vertical (Fig 1). Since the azimuth reference is the fixed meridian that is set initially in operating the D & I theodolite, we will define this plane to contain the I axis. Whether or not the instrument is level, this vertical plane will remain fixed. It will define scale zero for 'horizontal' angles. D is determined in the true horizontal plane, relative to the vertical plane containing I and K.

When the instrument is tilted, θ and ϕ define the tilt of the k axis (the 'vertical' instrument axis), and ψ defines the rotation required about the k-axis to bring the instrument i-axis back to the vertical reference plane containing the I-axis.

Let direction cosines of (i, j, k) instrument axes, relative to (I, J, K) be

$$\lambda_1 \ , \ \mu_1 \ , \ \nu_1$$
$$\lambda_2 \ , \ \mu_2 \ , \ \nu_2$$
$$\lambda_3 \ , \ \mu_3 \ , \ \nu_3$$

Let the magnetic field vector F have direction cosines (l, m, n) relative to (I, J, K). The plane perpendicular to F through the origin O will be detected by a null in the fluxgate sensor. The intersection of this plane with the plane perpendicular to the k-axis of the theodolite defines the orthogonal to the 'apparent H', denoted by H'.

These planes are defined by

$$lx + my + nz = 0$$
$$\lambda_3 x + \mu_3 y + \nu_3 z = 0$$

The direction of the intersection, H', is

$$(m\nu_3 - n\mu_3 \ , \ n\lambda_3 - l\nu_3 \ , \ l\mu_3 - m\lambda_3)$$

The angle between this direction $H'_{\perp}$ and the i-axis (which lies in the reference meridian plane) leads to the apparent declination D' in the tilted frame.

The direction cosines of the i-axis relative to (I, J, K) are (Synge and Griffith, 1959, p. 261)

$$(-\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi,\ \cos\phi\sin\psi + \cos\theta\sin\phi\cos\psi,\ -\sin\theta\cos\psi)$$
$$\text{or} \quad (\lambda_1, \mu_1, \nu_1)$$

So, the angle between H' and the i-axis, in the tilted frame, is given by

$$\cos D'_{\perp} = \frac{\lambda_1(m\nu_3 - n\mu_3) + \mu_1(n\lambda_3 - l\nu_3) + \nu_1(l\mu_3 - m\lambda_3)}{\sqrt{(\lambda_1^2 + \mu_1^2 + \nu_1^2)\left((m\nu_3 - n\mu_3)^2 + (n\lambda_3 - l\nu_3)^2 + (l\mu_3 - m\lambda_3)^2\right)}}$$

This is the angle read off the 'horizontal' circle relative to the azimuth reference reading.

The theodolite head will then be rotated by 90° about the k-axis, intending to place the fluxgate sensor in the magnetic meridian for the I measurement. In general, however, this new direction, H', will not be in the magnetic meridian, only the apparent meridian.

The direction of $H'_{\perp}$ is known above. We have set the instrument sensor in the plane perpendicular to $H'_{\perp}$. This plane is given by

$$(m\nu_3 - n\mu_3)x + (n\lambda_3 - l\nu_3)y + (l\mu_3 - m\lambda_3)z = 0$$

The intersection of this plane with the plane perpendicular to F gives a direction in the apparent magnetic meridian and perpendicular to the vector F. This direction will be detected by a fluxgate null. The direction of this line comes from the direction cosines of F and of the plane above as

$$\left\{[m(l\mu_3 - m\lambda_3) - n(n\lambda_3 - l\nu_3)],\ [n(m\nu_3 - n\mu_3) - l(l\mu_3 - m\lambda_3)],\ [l(n\lambda_3 - l\nu_3) - m(m\nu_3 - n\mu_3)]\right\}$$
$$\text{or} \quad \{a, b, c\}$$

The angle between this dircetion and the -k-axis gives I', the apparent inclination on the 'vertical' circle.

The direction of the -k-axis is $(-\lambda_3, -\mu_3 - \nu_3)$

So,

$$\cos I' = \frac{(l\mu_3 - m\lambda_3)^2 + (l\nu_3 - n\lambda_3)^2 + (m\nu_3 - n\mu_3)^2}{\sqrt{\lambda_3^2 + \mu_3^2 + \nu_3^3} \cdot \sqrt{a^2 + b^2 + c^2}}$$

Given a K-axis tilt defined by θ, ϕ, as in Synge and Griffith p.259, we need to determine ψ necessary to return the i-axis into the I meridian.

In the spherical triangles, Fig. 2, we need to determine $\zeta = -\psi$.

$$\tan \omega = \frac{\tan u}{\cos V}$$

$$\tan \omega = \frac{\tan(\pi/2 + \phi)}{\cos \theta} = \tan(\zeta + \pi/2)$$

$$\text{or} \quad \psi = -\zeta = \frac{\pi}{2} - \tan^{-1}\left(\frac{\tan(\pi/2 + \phi)}{\cos\theta}\right)$$

Thus, given the angles θ, ϕ, defining the tilt of the theodolite k-axis, we can then determine ψ, and thence the direction cosines of the (i, j, k) axes relative to (I, J, K).

	I	J	K
i	$(-\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi)$	$(\cos\phi\sin\psi + \cos\theta\sin\phi\cos\psi)$	$(-\sin\theta\cos\psi)$
j	$(-\sin\phi\cos\psi - \cos\theta\cos\phi\sin\psi)$	$(\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi)$	$(\sin\theta\sin\psi)$
k	$(\sin\theta\cos\phi)$	$(\sin\theta\sin\phi)$	$(\cos\theta)$

The angle ϕ is the angle from the I-axis to the plane containing the K and k axes.

The angle θ is the angle from the K-axis to the k-axis.

It remains to express the relationship between the conventional geomagnetic angles D and I and the direction cosines (l, m, n) of the geomagnetic field F.

$$(l, m, n) \equiv (\cos I \cos D, \cos I \sin D, \sin I)$$

$$\cos D'_{\perp} = \frac{\cos\psi\cos I\sin(\phi - D) - \sin\psi\sin I\sin\theta - \cos\theta\sin\psi\cos I\cos(\phi - D)}{\sqrt{\cos^2 I(\cos^2\theta + \sin^2\theta . \sin^2(\phi - D)) + \sin^2 I\sin^2\theta - 2\cos I\sin I\cos\theta\sin\theta\cos(\phi - D)}}$$

$$D' = D'_{\perp} - 90^{\circ}$$

$$\cos I' = \sqrt{\cos^2 I \left(\cos^2\theta + \sin^2\theta \sin^2(\phi - D)\right) + \sin^2 I \sin^2\theta - 2\cos I \sin I \cos\theta \sin\theta \cos(\phi - D)}$$

Thus it is possible to calculate the apparent declination and apparent inclination measured by a fluxgate theodolite that is off-level by a predefined amount. Given the limiting sensitivities of levelling bubbles, the limiting accuracy obtainable from the instrument can be determined, assuming that all other aspects of the instrument are perfect.

R. L. Coles

Feb. 20 /1985

REFERENCE

Synge, J.L and Griffith, B.A. 1959 Principles of Mechanics (3rd ed.), McGraw-Hill Book Company, London, 552p.

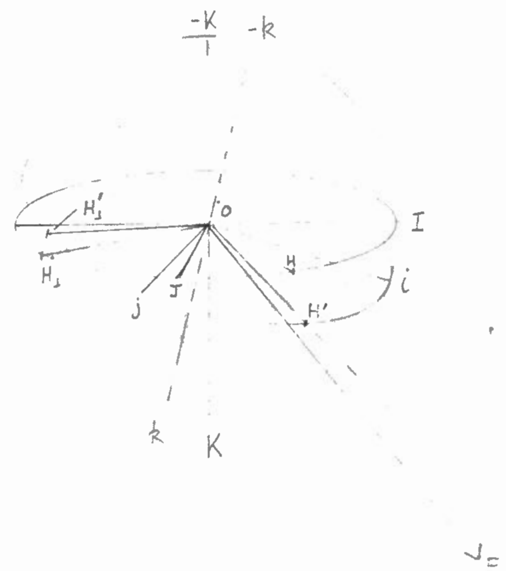
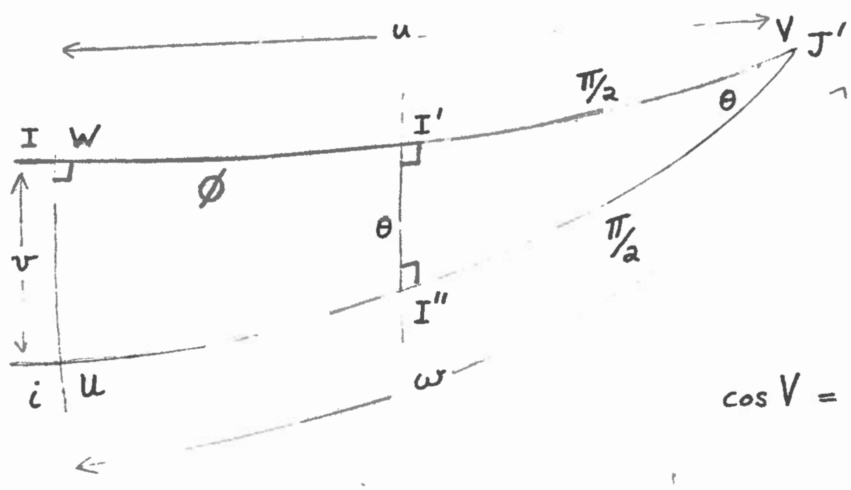


FIG. 1



$\cos V = \tan u \cot \omega$

Fig. 2.

A WORKED EXAMPLE FOR SOUTHERN CANADA

The Jena 020A theodolite has level bubble sensitivity of 30" per 2mm. In practice, the limitation is 15" levelling accuracy, at best.

Consider then the example where the theodolite vertical axis is tilted to the east by 0.25 ′ of arc, equivalent to the above. Take the Ottawa situation as an example. At Ottawa, the azimuth reference is essentially north, so that the maximum effect is caused by an east or west tilt.

Then, $\phi = 90^\circ$, $\theta = 0.25'$, $\psi = -90^\circ$

For Ottawa, take $D = -15^\circ$, $I = 75^\circ$, $H = 17000\,nT$, $F = 58000\,nT$

$$\phi - D = 105^\circ$$

$$\cos D'_\perp = \frac{\cos\psi\cos I\sin(\phi-D) - \sin\psi\sin I\sin\theta - \cos\theta\sin\psi\cos I\cos(\phi-D)}{\sqrt{\cos^2 I\left(\cos^2\theta + \sin^2\theta\sin^2(\phi-D)\right) + \sin^2 I\sin^2\theta - 2\cos I\sin I\cos\theta\sin\theta\cos(\phi-D)}}$$

$$D' = D'_\perp - 90^\circ \qquad \underline{D' = -14^\circ 58.7'}$$

$$X = H\cos D = 16420 \qquad X' = H\cos D' = 16422 \qquad \underline{\text{So, } \Delta X = 2nT}$$

$$Y = H\sin D = -4400 \qquad Y' = H\sin D' = -4395 \qquad \underline{\text{So, } \Delta Y = 5nT}$$

$$\cos I' = \sqrt{\cos^2 I\left(\cos^2\theta + \sin^2\theta\sin^2(\phi-D)\right) + \sin^2 I\sin^2\theta - 2\cos I\sin I\cos\theta\sin\theta\cos(\phi-D)}$$

$$I' = 74^\circ 59.9'$$

$$Z = F\sin I = 56023.7$$

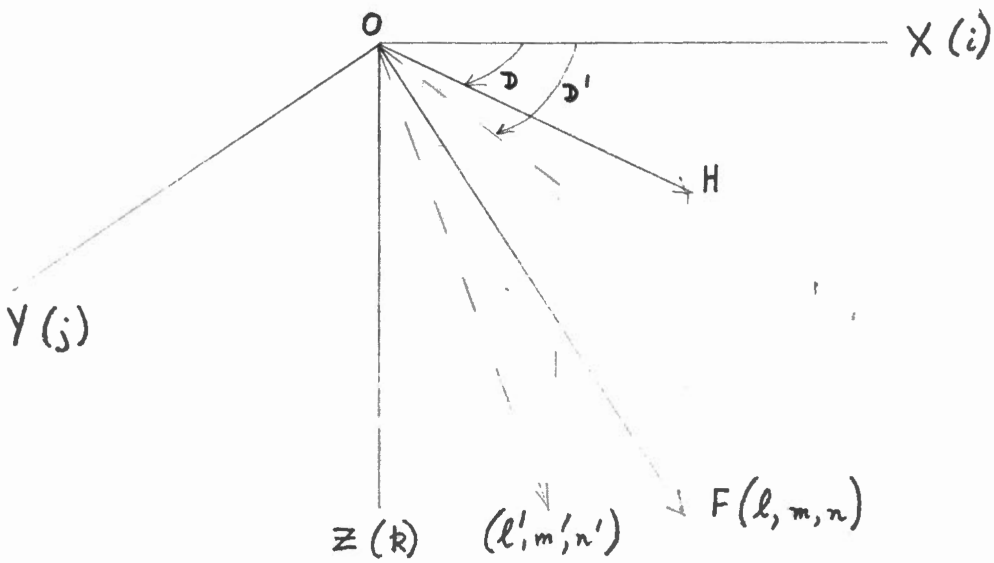
$$Z' = F\sin I' = 56023.4 \qquad \underline{\Delta Z < 1nT}$$

TO DETERMINE THE EXPRESSION FOR CORRECTING I READINGS WHEN THE THEODOLITE WAS NOT SET IN THE TRUE MAGNETIC MERIDIAN

This is a common operator error.

1. Assume correct levelling.
2. Assume correct determination of nulls for the four D readings and correct reading of scales.
3. FAULT - incorrect computation of mean D as D′.
4. Assume correct setting of theodolite in INCORRECT meridian using the faulty D′.
5. Using the incorrect meridian, the four I′ readings are taken.
6. Assume these I′ readings are correctly nulled and correctly read from the scales.
7. Assume that the mean of the four readings is correctly calculated to determine I′.

Solution:



1. Define direction of magnetic field F as (l, m, n)

2. Define plane(1) perpendicular to F through O as

$$lx + my + nz = 0$$

3. The angle between plane(1) and the horizontal plane gives (90 - I).

4. Horizontal plane(2) is z = 0.

5. The angle between these two planes (1) and (2) is given by

$$\cos\theta = \sin I = \frac{l.0 + m.0 + n.1}{\sqrt{l^2+m^2+n^2}} = \frac{n}{\sqrt{l^2+m^2+n^2}}$$

This angle I is measured in the vertical plane perpendicular to both planes (1) and (2), and is the true inclination I.

6. Now consider the fault to have occurred.

Define a new plane (3) parallel to the direction (l′,m′,n′) and to the z-axis (0,0,k), through O.

$$\begin{vmatrix} x & y & z \\ l' & m' & n' \\ 0 & 0 & k \end{vmatrix} = 0$$

or $$m'x - l'y = 0$$

This is then the faulty meridian.

7. The intersection of this plane (3) with plane (1), perpendicular to F, will define a fluxgate null, for a sensor capable only of rotating in plane (3).

8. Direction of intersection of planes (1) and (3)

i.e. $$\begin{aligned} lx + my + nz &= 0 \\ m'x - l'y + 0 &= 0 \end{aligned}$$

is $$[\,nl',\ nm',\ -(ll' + mm')\,]$$

9. Angle between this direction and −k axis gives I′.

$$\cos I' = \frac{mm' + ll'}{\sqrt{n^2 l'^2 + n^2 m'^2 + (ll' + mm')^2}}$$

$$\begin{aligned} l &= \cos I \cos D & \qquad l' &= \cos I' \cos D \\ m &= \cos I \sin D & \qquad m' &= \cos I' \sin D' \\ n &= \sin I & \qquad n' &= \sin I' \end{aligned}$$

We know D′, I′, and D from the readings by the operator; we need to find I.

Substituting in the equation for cosI′, and expanding, we obtain

$$\cos^2 I = \frac{\cos^2 I'}{\cos^2(D-D') + \cos^2 I' \cdot \sin^2(D-D')}$$

10. So, the correct I value can be retrieved, under the stated assumptions.

R. L. Coles

20 Feb 1985