Some Mathematical Investigations of the Relationship Between Changes in Gravity and Ground Displacements Due to Sub-surface Mass Movements.

by

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Introduction

Perceptible subsurface movements of mass are known to occur from time to time over different parts of the earth. They manifest themselves in various forms such as earthquakes, volcanic magma migration, postglacial uplift, etc. In general, a re-distribution of mass causes displacements of individual mass-elements which constitute the whole Earth. These displacements change the distances of these mass-elements from any particular ground station resulting in a change in gravity at the station from before to after the event. In many practical cases, such changes of gravity as well as displacements of the free-surface can be observed. The purpose of this study is to investigate mathematically the possible relationships between these two kinds of observable data.

Earlier work in this field includes papers by Walsh and Rice (1979) and Savage (1984) in which the changes in gravity are expressed in terms of dislocations at depth and Walsh (1982) where changes in gravity due to arbitrary tractions on an elastic half-space have been evaluated.

In our investigation we first develop the expression for the change in gravity at an observation point taking into account not only the deformation of the gravitating body but also any possible movement of the point of observation. Then, applying Betti's reciprocal theorem (in the absence of body forces) to Love's solution for the displacement field throughout a semi-infinite elastic medium in which there is a center of dilatation, we arrive at an expression for the change in gravity at a particular point on the surface of the half-space. The expression only requires a knowledge of the horizontal displacements observable at the surface. Although not completely general, these displacements can be due to any cause not requiring body forces for their maintenance. The important result derived here is that the expression does not require a knowledge of the displacement field throughout the medium and allows two different sets of observations (gravity and position) made at the surface to be related through a simple surface integration.

Volume Integral for the Change in Gravity

With reference to a cartesian system fixed in space, let (ξ,η,ζ) be the initial position of a mass-element with density $\rho(\xi,\eta,\zeta)$ and (X,Y,Z) be the initial position of a station of observation. After the occurence of mass movement, let their respective positions be $(\xi+u,\eta+v,\zeta+w)$ with the density $\rho'(\xi+u,\eta+v,\zeta+w)$ and $(X+U,Y+V,Z+W)$, where displacements (u,v,w) are functions of (ξ,η,ζ) and (U,V,W) are functions of (X,Y,Z). Throughout our analysis these displacements and the density differential $(\rho'-\rho)$ are considered as small quantities whose second and higher orders are negligible.

From the consideration of the conservation of mass on an initial unit volume and taking account of its dilatation after the event, we have

$$\rho(\xi,\eta,\zeta).\ 1 = \rho'(\xi+u,\ \eta+v,\ \zeta+w).\ (1 + \partial u/\partial\xi + \partial v/\partial\eta + \partial w/\partial\zeta)$$

or $$\rho'(\xi,\eta,\zeta) = \rho(\xi,\eta,\zeta) - \{\partial(\rho' u)/\partial\xi + \partial(\rho' v)/\partial\eta + \partial(\rho' w)/\partial\zeta\}$$

$$= \rho(\xi,\eta,\zeta) - \{\partial(\rho u)/\partial\xi + \partial(\rho v)/\partial\eta + \partial(\rho w)/\partial\zeta\} \qquad (1)$$

Denoting R and R' as the distances of a point (ξ,η,ζ) from the station of observation before and after the event respectively, we can write

$$1/R' = 1/R - U\partial(1/R)/\partial\xi - V\partial(1/R)/\partial\eta - W\partial(1/R)/\partial\zeta \qquad (2)$$

where $R^2 = (X-\xi)^2+(Y-\eta)^2+(Z-\zeta)^2$, $\quad R'^2 = (X'-\xi)^2+(Y'-\eta)^2+(Z'-\zeta)^2$

and $\quad X'=X+U,\ Y'=Y+V,\ Z'=Z+W$

From equations (1) and (2) we obtain

$$\rho'/R' = \rho/R - \rho U\partial(1/R)/\partial\xi - \rho V\partial(1/R)/\partial\eta - \rho W\partial(1/R)/\partial\zeta - (1/R)\partial(\rho u)/\partial\xi - (1/R)\partial(\rho v)/\partial\eta - (1/R)\partial(\rho w)/\partial\zeta \quad (3)$$

The change in gravitational potential at a station can be expressed as

$$\Delta P(X,Y,Z) = P'(X',Y',Z') - P(X,Y,Z)$$

$$= \gamma \int_{T'} (\rho'/R')\, d\tau - \gamma \int_{T} (\rho/R) d\tau \quad (4)$$

where γ is the universal constant of gravitation and T and T' are the volumes of contribution before and after the event. Obviously, the difference between these two volumes is of the same order of magnitude as the displacement involved so that second and higher orders of the difference can be ignored. Then substituting (3) in (4) we get

$$\Delta P = \gamma \int_{T'-T} (\rho/R)\, d\tau - \gamma \int_{T} \rho\, \{U\partial(1/R)/\partial\xi + V\partial(1/R)\partial\eta + W\partial(1/R)\partial\zeta\}\, d\tau - \gamma \int_{T} (1/R)\{\partial(\rho u)/\partial\xi + \partial(\rho v)/\partial\eta + \partial(\rho w)/\partial\zeta\}\, d\tau \quad (5)$$

Now the volume (T'-T) can be approximated as a thin shell so that $d\tau \simeq u_n dS$ where u_n is the component of the displacement along the outward normal direction at a point on S, the bounding surface of the volume T. Then

$$\gamma \int_{T'-T} (\rho/R)\, d\tau = \gamma \int_{S} (\rho/R)\, u_n dS = \gamma \int_{T} \{\partial(\rho u/R)/\partial\xi + \partial(\rho v/R)/\partial\eta + \partial(\rho w/R)\partial\zeta\}\, d\tau \quad (6)$$

the second step following from the application of Gauss' well known theorem. Hence, combining (5) and (6) we have:

$$\Delta P = \gamma \int_T \rho\{(u-U)\partial(1/R)/\partial\xi + (v-V)\partial(1/R)/\partial\eta + (w-W)\partial(1/R)/\partial\zeta\}\, d\tau \qquad (7)$$

We can work out the change in the vertical component of the gravitational attraction, Δg, in a similar manner starting from the relation

$$\Delta g = g'(X',Y',Z') - g(X,Y,Z)$$
$$= -\partial\{P'(X',Y',Z')\}/\partial Z' + \partial\{P(X,Y,Z)\}/\partial Z$$

to obtain

$$\Delta g = \gamma \int_T \rho[(u-U)\partial\{(Z-\zeta)/R^3\}/\partial\xi + (v-V)\partial\{(Z-\zeta)/R^3\}/\partial\eta + (w-W)\partial\{(Z-\zeta)/R^3\}/\partial\zeta]\, d\tau \quad (8)$$

It is interesting to note that, in general, the change in gravity is not derivable from the change in potential simply by a single differentiation with respect to Z, that is:

$$\Delta g \neq -\partial(\Delta P)/\partial Z$$

where $-\partial(\Delta P)/\partial Z$ is given by

$$\gamma \int_T \rho\, [(u-U)\partial\{(Z-\zeta)/R^3\}/\partial\xi + (v-V)\partial\{(Z-\zeta)/R^3\}/\partial\eta + (w-W)\partial\{(Z-\zeta)/R^3\}/\partial\zeta]\, d\tau$$
$$+ \gamma \int_T (\rho/R^3)\{(X-\xi)\partial U/\partial Z + (Y-\eta)\partial V/\partial Z + (Z-\zeta)\partial W/\partial Z\}d\tau$$

This is because what is measured is, by definition

$$\Delta g = -\partial \int_{T'} \rho'\ \partial(1/R')/\partial Z'\ d\tau + \partial \int_T \rho\ \partial(1/R)/\partial Z\ d\tau$$

while

$$-\partial(\Delta P)/\partial Z = -\partial \int_{T'} \rho'\ \partial(1/R')/\partial Z\ d\tau + \partial \int_T \rho\ \partial(1/R)/\partial Z\ d\tau$$

and $\partial(1/R')/\partial Z$ and $\partial(1/R')/\partial Z'$ are not equal but are related as follows:

$$\partial(1/R')/\partial Z = \{(\xi-X')/R'^3\}\partial U/\partial Z + \{(\eta-Y')/R'^3\}\partial V/\partial Z + \{(\zeta-Z')/R'^3\}(1+\partial W/\partial Z)$$
$$= \partial(1/R')/\partial Z' + \{(\xi-X)\partial U/\partial Z + (\eta-Y)\partial V/\partial Z + (\zeta-Z)\partial W/\partial Z\}/R'^3$$

Later on, however, we will need to use equation (8) to evaluate the change in gravity at any point when the displacements are given at every point within the volume of the attracting mass. The practical use of the formula will greatly increase if the volume integral can be converted to one or more surface integrals since observations for displacements are normally available only on the bounding surface. The following theorem enables us to do this.

Betti's Reciprocal Theorem and Its Application

This theorem states that when an elastic body is subjected to two different sets of deformations under the application of two different systems of forces, then the work done by the first system of forces acting through the second set of displacements equals the work done by the second system of forces acting through the first set of displacements.

Thus, if $(X,Y,Z,X_\nu,Y_\nu,Z_\nu)$ form a system of body forces and surface tractions acting on a body in a volume T bounded by the surface S to produce displacement components (u,v,w) and $(X',Y',Z',X_\nu',Y_\nu',Z_\nu')$ and (u',v',w') form a similar second system involving the same body, then we have

$$\int_T \rho\,(Xu'+Yv'+Zw')\,d\tau + \int_S (X_\nu u'+Y_\nu v'+Z_\nu w')\,dS$$

$$= \int_T \rho\,(X'u+Y'v+Z'w)\,d\tau + \int_S (X_\nu' u+Y_\nu' v+Z_\nu' w)\,dS \qquad (9)$$

In our analysis, we assume the first system to be the real one acting on the body while the second system is a suitably chosen hypothetical model such that (u',v',w') are the displacement components appropriate to a unit centre of dilatation at (ξ,η,ζ) in an infinite medium. Then we can write

$$(u',v',w') = (\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)\,(1/r) \qquad (10)$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 \qquad (11)$

In applying Betti's theorem, body forces are ignored and the point (ξ,η,ζ) is excluded by a sphere Σ of radius ε $(\varepsilon\to 0)$. Equation (9) then reduces to

$$\int_\Sigma \{(X_\nu u'-X'_\nu u)+(Y_\nu v'-Y_\nu' v)+(Z_\nu w'-Z'_\nu w)\}\, d\Sigma$$

$$= -\int_S \{(X_\nu u'-X_\nu' u)+(Y_\nu v'-Y_\nu' v)+(Z_\nu w'-Z_\nu' w)\}\, dS \qquad (12)$$

The left hand side of (12) has been worked out in detail by Love (1927, p. 234) as

$$4\pi(\lambda+2\mu)(\partial u/\partial\xi+\partial v/\partial\eta+\partial w/\partial\zeta)$$

$$= - \int_S \{(X_\nu u'-X_\nu' u)+(Y_\nu v'-Y_\nu' v)+(Z_\nu w'-Z_\nu' w)\}\, dS \qquad (13)$$

Now we consider another similar model system $(u'', \ldots . Z_\nu'')$ with the restriction that $u'=u''$, $v'=v''$ and $w'=w''$ on S. Hence

$$\int_S (X_\nu u'+Y_\nu v'Z_\nu w')\, dS = \int_S (X_\nu u''+Y_\nu v''+Z_\nu w'')\, dS$$

$$= \int_S (X_\nu'' u+Y_\nu'' v+Z_\nu'' w)\, dS \qquad (14)$$

Eliminating (X_ν,Y_ν,Z_ν) between (13) and (14), we then find

$$4\pi(\lambda+2\mu)(\partial u/\partial\xi+\partial v/\partial\eta+\partial w/\partial\zeta)$$

$$= \int_S \{(X_\nu'-X_\nu'')u+(Y_\nu'-Y_\nu'')v+(Z_\nu'-Z_\nu'')w\}\, dS \qquad (15)$$

In this equation, $(X_\nu{}'-X_\nu{}'')$ etc. can be looked upon as surface tractions corresponding to displacements (u'-u") etc. such that these displacement components vanish on the bounding surface S.

Half-Space Solutions for Displacements

In this case, S is the plane boundary z=0 and T is the volume z≥0 and we have, following Love (p. 237-240)

$$(u'',v'',w'') = (\partial/\partial x,\ \partial/\partial y,\ -\partial/\partial z)\ (1/r_i) + 2z\ \{(\lambda+\mu)/(\lambda+3\mu)\}\ (\partial^2/\partial x\partial z,\ \partial^2/\partial y\partial z,\ \partial^2/\partial z^2)\ (1/r_i) \qquad (16)$$

where $r_i^2 = (x-\xi)^2+(y-\eta)^2+(z+\zeta)^2$ (17)

which implies inclusion of an image centre of dilatation at $(\xi,\ \eta,\ -\zeta)$.

$(X_\nu{}'-X_\nu{}'')$ etc are then worked out on S using (10) and (16) and then substituted

in (15). Then in terms of:

$$L(\xi,\eta,\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{u(x,y,0)/r_0\}\ dxdy$$

$$M(\xi,\eta,\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{v(x,y,0)/r_0\}\ dxdy \qquad (18)$$

$$N(\xi,\eta,\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{w(x,y,0)/r_0\}\ dxdy$$

and

$$\phi(\xi,\eta,\zeta) = \partial L/\partial\xi+\partial M/\partial\eta+\partial N/\partial\zeta$$

where $r_0{}^2 = (x-\xi)^2+(y-\eta)^2+\zeta^2$ (19)

we can write

$$\Delta(\xi,\eta,\zeta) = \partial u/\partial\xi+\partial v/\partial\eta+\partial w/\partial\zeta = -\{\mu/\pi(\lambda+3\mu)\}\partial\phi/\partial\zeta$$

$$u(\xi,\eta,\zeta) = -(1/2\pi)\partial L/\partial\zeta + \{(\lambda+\mu)/2\pi(\lambda+3\mu)\}\zeta\partial\phi/\partial\xi$$

$$v(\xi,\eta,\zeta) = -(1/2\pi)\partial M/\partial\zeta + \{(\lambda+\mu)/2\pi(\lambda+3\mu)\}\zeta\partial\phi/\partial\eta \qquad (20)$$

$$w(\xi,\eta,\zeta) = -(1/2\pi)\partial N/\partial\zeta + \{(\lambda+\mu)/2\pi(\lambda+3\mu)\}\zeta\partial\phi/\partial\zeta$$

Further substitution of (18) and (19) in (20) and subsequent simplification reduce the formulae for displacements to:

$$u(\xi,\eta,\zeta)$$
$$= \{\mu\zeta/\pi(\lambda+3\mu)\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{u(x,y,0)/r_0^{3}\}\,dxdy$$
$$+ \{3(\lambda+\mu)\zeta/2\pi(\lambda+3\mu)\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(\xi-x)u(x,y,0)+(\eta-y)v(x,y,0)+\zeta w(x,y,0)\}\{(\xi-x)/r_0^{5}\}\,dxdy$$

$$v(\xi,\eta,\zeta)$$
$$= \{\mu\zeta/\pi(\lambda+3\mu)\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{v(x,y,0)/r_0^{3}\}\,dxdy \qquad (21)$$
$$+ \{3(\lambda+\mu)\zeta/2\pi(\lambda+3\mu)\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(\xi-x)u(x,y,0)+(\eta-y)v(x,y,0)+\zeta w(x,y,0)\}\{(\eta-y)/r_0^{5}\}\,dxdy$$

$$w(\xi,\eta,\zeta)$$
$$= \{\mu\zeta/\pi(\lambda+3\mu)\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{w(x,y,0)/r_0^{3}\}\,dxdy$$
$$+ \{3(\lambda+\mu)\zeta^2/2\pi(\lambda+3\mu)\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(\xi-x)u(x,y,0)+(\eta-y)v(x,y,0)+\zeta w(x,y,0)\}(1/r_0^{5})\,dxdy$$

These equations express the displacement components at any point inside the half space in terms of the displacement on the plane boundary.

Surface Integral for the Change in Gravity

For the change in gravity on the bounding plane of a homogeneous half-space, equation (8) can be modified to

$$\Delta g(X,Y,0)$$

$$= 3\gamma\rho \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{(\xi-X)u(\xi,\eta,\zeta)+(\eta-Y)v(\xi,\eta,\zeta)+\zeta w(\xi,\eta,\zeta)\}(\zeta/R_0^5)\, d\xi d\eta d\zeta$$

$$- \gamma\rho \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{w(\xi,\eta,\zeta)/R_0^3\}\, d\xi d\eta d\zeta - 3\gamma\rho U(X,Y,0) \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty (\xi-X)(\zeta/R_0^5)\, d\xi d\eta d\zeta$$

$$- 3\gamma\rho V(X,Y,0) \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty (\eta-Y)(\zeta/R_0^5)\, d\xi d\eta d\zeta - 3\gamma\rho W(X,Y,0) \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty (\zeta^2/R_0^5)\, d\xi d\eta d\zeta$$

$$+ \gamma\rho W(X,Y,0) \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty (1/R_0^3)\, d\xi d\eta d\zeta \qquad (22)$$

where $R_0^2 = (\xi-X)^2 + (\eta-Y)^2 + \zeta^2$ (23)

Substitution of (21) in (22) results in

$$\{2\pi(\lambda+3\mu)/\gamma\rho\}\Delta g(X,Y,0)$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty u(x,y,0)\, dxdy \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty [6\mu\zeta^2(\xi-X)/(R_0^5 r_0^3) + 9(\lambda+\mu)\zeta^2(\xi-x)$$

$$\{(\xi-x)(\xi-X)+(\eta-y)(\eta-Y)+\zeta^2\}/(R_0^5 r_0^5) - 3(\lambda+\mu)\zeta^2(\xi-x)/(R_0^3 r_0^5)]\, d\xi d\eta d\zeta$$

$$+ \int_{-\infty}^\infty \int_{-\infty}^\infty v(x,y,0)\, dxdy \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty [6\mu\zeta^2(\eta-Y)/(R_0^5 r_0^3) + 9(\lambda+\mu)\zeta^2(\eta-y)$$

$$\{(\xi-x)(\xi-X)+(\eta-y)(\eta-Y)+\zeta^2\}/(R_0^5 r_0^5) - 3(\lambda+\mu)\zeta^2(\eta-y)/(R_0^3 r_0^5)]\, d\xi d\eta d\zeta$$

$$+ \int_{-\infty}^\infty \int_{-\infty}^\infty w(x,y,0)\, dxdy \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty [6\mu\zeta^3/(R_0^5 r_0^3) + 9(\lambda+\mu)\zeta^3\{(\xi-x)(\xi-X)$$

$$+(\eta-y)(\eta-Y)+\zeta^2\}/(R_0^{\,5}r_0^{\,5}) - 2\mu\zeta/(R_0^{\,3}r_0^{\,3}) - 3(\lambda+\mu)\zeta^3/(R_0^{\,3}r_0^{\,5})]\ d\xi d\eta d\zeta$$

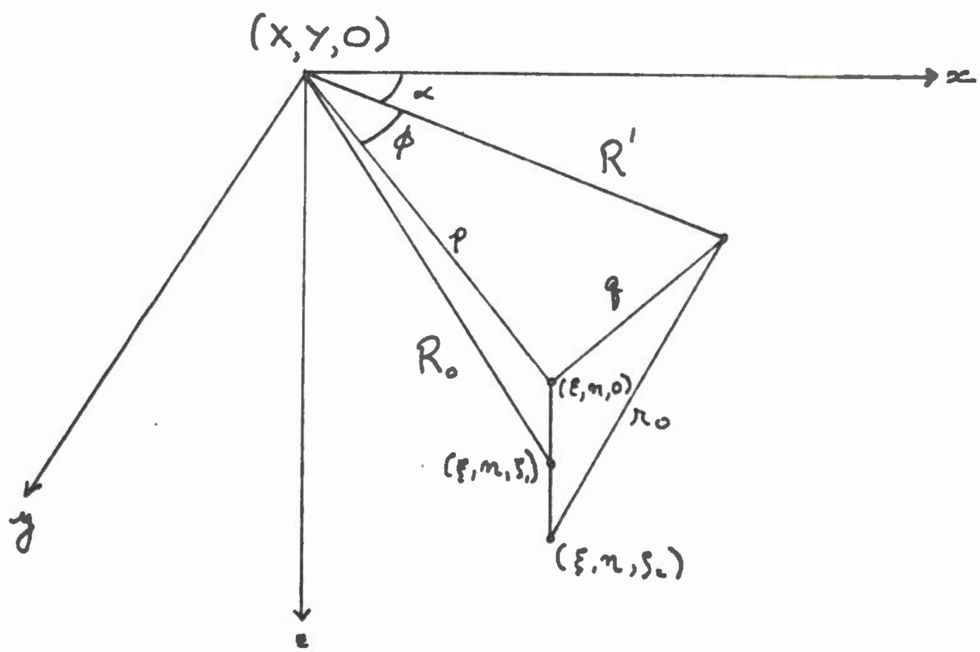
$$- 6\pi U(X,Y,0) \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{\zeta(\xi-X)/R_0^{\,5}\}\ d\xi d\eta d\zeta$$

$$- 6\pi V(X,Y,0) \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{\zeta(\eta-Y)/R_0^{\,5}\}\ d\xi d\eta d\zeta$$

$$+ 2\pi W(X,Y,0) \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty (1/R_0^{\,3}-3\zeta^2/R_0^{\,5})\ d\xi d\eta d\zeta \qquad (24)$$

In this equation, the triple integrals with respect to ξ, η and ζ over the half-space can be evaluated to obtain functions of x, y, X and Y. The effects of these functions are to introduce kernels in the surface integrals of the displacement components over the bounding plane of the half-space in (24) to represent the change in gravity at any point on the same bounding plane.

Evaluations of Triple Integrals in ξ, η and ζ



To facilitate the evaluation of the integrals in question, let us first introduce a few variables as indicated in the diagram above and explained by the relations below:

$$R' \cos\alpha = x-X, \quad R' \sin\alpha = y-Y, \quad p\cos(\phi+\alpha) = \xi-X, \quad p\sin(\phi+\alpha) = \eta-Y$$

$$q^2 = p^2+R'^2-2pR'\cos\phi\ , \qquad d\xi d\eta = p dp d\phi\ ,$$

$$1/R_0 = (p^2+\zeta_1^2)^{-1/2} = \int_0^\infty \exp(-k\zeta_1)\ J_0(kp)\ dk,$$

$$1/r_0 = (q^2+\zeta_2^2)^{-1/2} = \int_0^\infty \exp(-\ell\zeta_2)\ J_0(\ell q)\ d\ell.$$

Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta_1/R_0^3 r_0)\ d\xi d\eta = \int_0^\infty\int_0^\infty \exp(-k\zeta_1-\ell\zeta_2)\ k dk d\ell \int_0^\infty J_0(kp)\ p dp \int_0^{2\pi} J_0(\ell q)\ d\phi \qquad (25)$$

Now

$$J_0(\ell q) = \sum_{m=0}^{\infty} \varepsilon_m J_m(\ell p)\ J_m(\ell R')\cos m\phi, \qquad \varepsilon_0 = 1,\ \varepsilon_m = 2,\ m = 0 \qquad (26)$$

(Watson, 1922, p. 358).

and hence,

$$\int_0^{2\pi} J_0(\ell q)\ d\phi = 2\pi\ J_0(\ell p)\ J_0(\ell R') \qquad (27)$$

Substitution of (27) in (25) gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta_1/R_0^3 r_0)\ d\xi d\eta$$

$$= 2\pi \int_0^\infty\int_0^\infty \exp(-k\zeta_1-\ell\zeta_2)\ J_0(\ell R')\ k dk d\ell \int_0^\infty J_0(kp)\ J_0(\ell p)\ p dp \qquad (28)$$

Now let us assume that the Dirac delta function can be represented as

$$\delta(k-\ell) = \int_{\infty}^{\infty} D\,(k,p)\; J_0(\ell p)\; p dp.$$

Therefore, by inversion

$$D(k,p) = \int_{\infty}^{\infty} \delta(k-\rho)\; J_0(\ell p)\; \ell d\ell = k\; J_0(kp)$$

$$\text{or}\quad \delta(k-\ell) = k \int_{\infty}^{\infty} J_0(kp)\; J_0(\ell p)\; p dp \tag{29}$$

Using (29), (28) is reduced to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta_1/R_0{}^3 r_0)\; d\xi d\eta = 2\pi/H \tag{30}$$

$$\text{where}\qquad H^2 = R'^2+(\zeta_1+\zeta_2)^2 \tag{31}$$

Differentiating both sides of (30) with respect to ζ_2, we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta_1\zeta_2/R_0{}^3 r_0{}^3)\; d\xi d\eta = 2\pi(\zeta_1+\zeta_2)/H^3 \tag{32}$$

Similar differentiation of both sides of (30) and subsequent equations with respect to ζ_1, ζ_2, x and X and further algebraic manipulation of terms enables us to evaluate all our integrals with $\zeta_1=\zeta_2=\zeta$ as listed below:

(i) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta/R_0{}^3 r_0)\; d\xi d\eta = 2\pi/H,$ (ii) $\displaystyle\int_{\infty}^{\infty}\int_{\infty}^{\infty} (\zeta/R_0{}^3 r_0{}^3)\; d\xi d\eta = 4\pi/H^3,$

(iii) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\zeta(\xi-x)/R_0{}^3 r_0{}^5\}\; d\xi d\eta = 4\pi(X-x)/H^5,$

(iv) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\zeta^3(\xi-x)/R_0{}^5 r_0{}^5\}\; d\xi d\eta = (2\pi/3)(X-x)(1/H^5+20\zeta^2/H^7),$

(v) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta^3/R_0^5\; r_0{}^3)\; d\xi d\eta = (2\pi/3)(1/H^3+12\zeta^2/H^5),$

(vi) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta^3/R_0^5\, r_0^{\,5})\, d\xi d\eta = (4\pi/3)(1/H^5+20\zeta^2/H^7),$$

(vii) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\zeta(\xi-X)/R_0^{\,5} r_0^{\,3}\}\, d\xi d\eta = -\, 4\pi(X-x)/H^5,$$

(viii) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\zeta(\xi-x)(\xi-X)/R_0^{\,5} r_0^{\,5}\}\, d\xi d\eta = (4\pi/3)\{1/H^5-5(x-X)^2/H^7\},$$

(ix) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\zeta(\xi-x)(\eta-y)(\xi-X)/R_0^{\,5} r_0^{\,5}\}\, d\xi d\eta = (2\pi/3)(Y-y)\{1/H^5-5(X-x)^2/H^7\},$$

(x) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\zeta(\xi-x)^2(\xi-X)/R_0^{\,5} r_0^{\,5}\}\, d\xi d\eta = (2\pi/3)(X-x)\{1/H^5-5(X-x)^2/H^7\},$$

(xi) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta^3/R_0^3\, r_0^{\,5})\, d\xi d\eta = (2\pi/3)(1/H^3+12\zeta^2/H^5),$$

(xii) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\zeta(\xi-X)/R_0^{\,5}\}\, d\xi d\eta = 0,$$

(xiii) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta^2/R_0^{\,5})\, d\xi d\eta = (2\pi/3\zeta),$$

(xiv) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1/R_0^{\,3})\, d\xi d\eta = 2\pi/\zeta, \quad \text{with} \quad H^2 = R'^2+4\zeta^2$$

In order to introduce the total contribution of the half-space these integrals are then integrated with respect to ζ from 0 to ∞. The results are:

(i) $$\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta/R_0^{\,3} r_0)\, d\xi d\eta d\zeta = 2\pi \int_{0}^{\infty} \{1/(R'^2+4\zeta^2)^{1/2}\}\, d\zeta,$$

(ii) $$\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta/R_0^{\,3} r_0^{\,3})\, d\xi d\eta d\zeta = 2\pi/R'^2,$$

(iii) $$\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\zeta^2(\xi-x)/R_0^{\,3} r_0^{\,5}\}\, d\xi d\eta d\zeta = (\pi/3)\{(X-x)/R'^3\},$$

(iv) $$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{\zeta^4(\xi-x)/R_0^5 r_0^5\}\, d\xi d\eta d\zeta = (\pi/6)\{(X-x)/R'^3\},$$

(v) $$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{\zeta^3/R_0^5 r_0^3\}\, d\xi d\eta d\zeta = (2\pi/3)/R'^2,$$

(vi) $$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{\zeta^5/R_0^5 r_0^5\}\, d\xi d\eta d\zeta = (2\pi/9)/R'^2,$$

(vii) $$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{\zeta^2(\xi-X)/R_0^5 r_0^3\}\, d\xi d\eta d\zeta = -(\pi/3)\{(X-x)/R'^3\},$$

(viii) $$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{\zeta^3(\xi-x)(\xi-X)/R_0^5 r_0^5\}\, d\xi d\eta d\zeta = (\pi/18)\{(Y-y)^2-(X-x)^2\}/R'^4,$$

(ix) $$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{\zeta^2(\xi-x)(\eta-y)(\xi-X)/R_0^5 r_0^5\}\, d\xi d\eta d\zeta = (\pi/18)(Y-y)\{(Y-y)^2-2(X-x)^2\}/R'^5,$$

(x) $$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{\zeta^2(\xi-x)^2(\xi-X)/R_0^5 r_0^5\}\, d\xi d\eta d\zeta = -(\pi/18)(X-x)\{2(X-x)^2-(Y-y)^2\}/R'^5,$$

(xi) $$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{\zeta^3/R_0^3 r_0^5\}\, d\xi d\eta d\zeta = (2\pi/3)/R'^2,$$

(xii) $$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{\zeta(\xi-X)/R_0^5\}\, d\xi d\eta d\zeta = 0,$$

(xiii) $$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \{\zeta^2/R_0^5\}\, d\xi d\eta d\zeta = (2\pi/3)\int_0^\infty (1/\zeta)\, d\zeta,$$

(xiv) $$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty (1/R_0^3)\, d\xi d\eta d\zeta = 2\pi\int_0^\infty (1/\zeta)\, d\zeta,$$

When we make use of these results in (24), we obtain

$$\{2\pi(\lambda+3\mu)/\gamma\rho\}\Delta g(X,Y,0)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{(X-x)/R'^3\}u(x,y,0)\,dxdy\,[6\mu(-\pi/3) - 9(\lambda+\mu)(\pi/18)\{2(X-x)^2-(Y-y)^2\}/R'^2$$

$$- 9(\lambda+\mu)(\pi/18)\{2(Y-y)^2-(X-x)^2\}/R'^2 + 9(\lambda+\mu)(\pi/6) - 3(\lambda+\mu)(-\pi/3)]$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{(Y-y)/R'^3\}v(x,y,0)\,dxdy\,[6\mu(-\pi/3) - 9(\lambda+\mu)(\pi/18)\{2(X-x)^2-(Y-y)^2\}/R'^2$$

$$- 9(\lambda+\mu)(\pi/18)\{2(Y-y)^2-(X-x)^2\}/R'^2 + 9(\lambda+\mu)(\pi/6) - 3(\lambda+\mu)(-\pi/3)]$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1/R'^2)w(x,y,0)\,dxdy\,[6\mu(2\pi/3) + 9(\lambda+\mu)(\pi/18)\{(Y-y)^2-(X-x)^2\}/R'^2$$

$$+ 9(\lambda+\mu)(\pi/18)\{(X-x)^2-(Y-y)^2\}/R'^2 + 9(\lambda+\mu)(2\pi/9) - 2\mu.2\pi - 3(\lambda+\mu)(2\pi/3)]$$

$$- 6\pi U(X,Y,0).0 - 6\pi V(X,Y,0).0 + 2\pi W(X,Y,0)\{2\pi\int_0^{\infty}(1/\zeta)\,d\zeta - (2\pi/3).3\int_0^{\infty}(1/\zeta)\,d\zeta] \quad (33)$$

which, after collecting terms and noting that all terms involving the displacements w and W cancel out, reduces to:

$$\Delta g(X,Y,0)$$

$$= (\gamma\rho/2)\{(-2\mu)/(\lambda+3\mu)\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{(X-x)u(x,y,0)+(Y-y)v(x,y,0)\}/\{(X-x)^2+(Y-y)^2\}^{3/2}\,dxdy \quad (34)$$

Thus, under the assumptions of the homogeneity of density and the absence of body forces in the elastic half-space, we are able to express the change in gravity at any point on the bounding plane in terms of the displacement components on the surface alone. The reason that we only need to deal with a surface integral is that, in the absence of body forces, the displacement field inside the medium can be computed from a knowledge of the displacements at the surface. The advantage of only dealing with a surface integral is that in applying the method to the Earth, underground observations are not needed. We find it somewhat surprising that a knowledge of vertical displacements is not necessary for the problem as posed. This result calls into question the usual procedure of combining gravity and levelling results to study dynamic processes in the Earth's crust and ignoring horizontal strains.

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