Absolute Determination of Gravity from Free Fall Measurements using Spectral Analysis and Least Squares Fit

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Abstract

A practical and accurate (within 10^{-4} nm) free fall formula is derived for the purpose of determining absolute gravity from free fall measurements using least squares fit. Small correcting terms for air resistance and Coriolis force (Eötvös effect) are taken into account. A procedure using spectral analysis, least squares fit and variance-ratio significance F test eliminates non-random noise and computes gravity. The present study shows from real data that perturbing frequencies may create systematic errors up to 50 μGal. A method of computing an average set of data is suggested and the problem of computer expenses is considered.

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Introduction

For several decades absolute gravity measurements have been envisaged and realized using freely falling objects (Volet, 1947), (Faller et al, 1965), (Sakuma, 1971), (Hammond, 1978). Our purpose is to analyze a procedure for computing the absolute gravity from free fall measurements $(\tilde{t}_i, \tilde{z}_i)$, i = 1, ..., m where $\tilde{t}_i$ are the measured times corresponding to the vertical coordinate values $\tilde{z}_i$ of the falling object and m is the number of points measured along the free fall path. In practice, in the case of single free fall, these measurements are realized by dropping with zero initial velocity a corner cube reflector in the light beam of a laser interferometer; the case of symmetric free fall used by A. Sakuma where the freely falling object is first thrown upwards will not be taken up in the present report. A fringe counter with a resolution of one thousandth of a fringe triggers every ℓ fringes (for example ℓ=4000), a time measurement individually accurate to within $\pm$ 0.5 ns ($\pm 5 \times 10^{-10}$ s). In the numerical computations made later, we will have m=439 over a 55.5 cm drop lasting about 0.23 s , these figures correspond to Hammond's free fall gravimeter as used at the Bureau International des Poids et Mesures (BIPM) in Sèvres, France, at the international campaign of absolute gravimeters comparison in Oct. Nov. 1981.

In the first part, applying the principles of Classical Mechanics (Goldstein, 1980), we derive the equations for a point mass falling freely in a rotating reference frame and a linearly space dependent gravity field (constant gravity gradients). After discussion of the magnitude of each term, a precise and practical formula for the vertical component z(t) is found to be a Taylor series of degree four.

In the second part, a practical procedure is proposed for analyzing the data in terms of perturbing frequencies of various origin by least squares spectral analysis (Vaníček, 1971). Least squares fit of a second degree polynomial and of the perturbing frequencies determines gravity. The gravity value is derived from the second order coefficient of the second degree polynomial. Only the perturbing frequencies proving to be significant via an F test as explained later are included in our least squares fit. In practice we choose a very high confidence level i.e. 99.956%.

In the third part, results of real and synthetized data analysis by our method are presented and discussed.

PART 1

THEORY

Let us apply the principles of Classical Mechanics (Goldstein, 1980) to a freely moving point mass (m) in a reference frame Axyz (see Fig. 1), fixed with respect to the Earth's crust, under the influence of some gravitational field. Denoting by $\breve{\Omega}$ the Earth's instantaneous angular velocity vector and T its center of mass, the absolute acceleration $\vec{\gamma}_M$ of a point M having mass m is given by (Goldstein, 1980; p. 177):

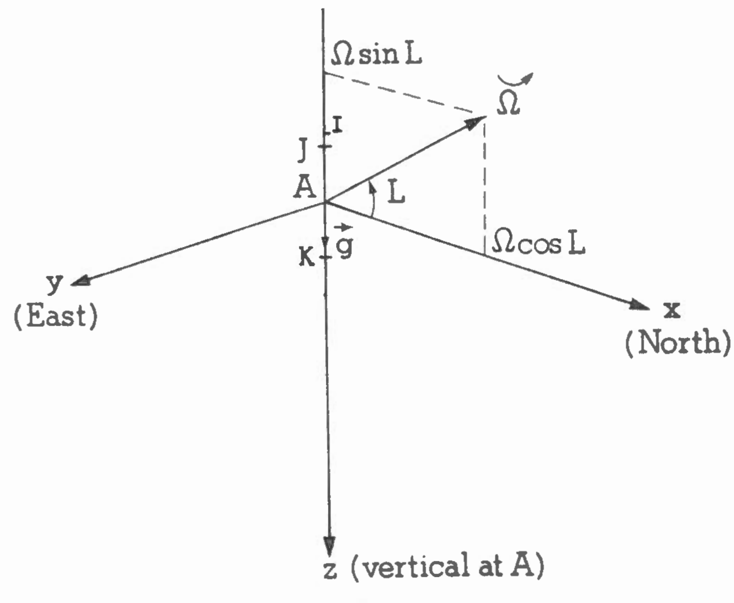
$$\vec{\gamma}_M = \vec{\gamma}_T + \frac{d\breve{\Omega}}{dt} \times \vec{TM} + \breve{\Omega} \times (\breve{\Omega} \times \vec{TM}) + \vec{\gamma}_{M_r} + 2\breve{\Omega} \times \vec{V}_{M_r} , \qquad (1)$$

where x is the vector cross-product, $\vec{\gamma}_T$ the acceleration of the Earth's center of mass T, $\vec{\gamma}_{M_r}$ the relative acceleration of M and $\vec{V}_{M_r}$ the relative velocity of M, both in the reference frame Axyz. Assuming for my theoretical derivation $d\breve{\Omega}/dt = \breve{0}$ and denoting by $\vec{\Phi}_M$ and $\vec{\Psi}_M$ the gravitational fields of the Earth and of other celestial bodies respectively, we apply Newton's Second Law of Motion and using (1) where $\vec{\gamma}_M=\vec{\Phi}_M+\vec{\Psi}_M$ for a body in free fall, we have:

$$\vec{\Phi}_M + \vec{\Psi}_M - \vec{\gamma}_T - \breve{\Omega}\times(\breve{\Omega}\times\overrightarrow{TM}) = \vec{\gamma}_{M_r} + 2\breve{\Omega}\times\vec{V}_{M_r} \quad . \tag{2}$$

One recognizes the tidal acceleration $\vec{\Psi}_M-\vec{\gamma}_T$, the centrifugal acceleration $-\breve{\Omega}\times(\breve{\Omega}\times\overrightarrow{TM})$ and the Coriolis acceleration $-2\breve{\Omega}\times\vec{V}_{M_r}$. The left-hand side of (2) is the complete theoretical expression of the physical gravity vector $\vec{g}(x,y,z)$ equal in magnitude to the force per unit mass on a body at rest with respect to an Earth fixed reference frame. In practice the air resistance may not always be neglibible (Faller & Hammond, 1970; p. 124).

<u>Figure 1</u>: <u>Axyz reference frame fixed with respect to the Earth's crust</u>.



<u>Legend</u>: L: latitude at point A, I: zero velocity position, J first point measured on the drop, K last point measured on the drop. A has been chosen at the middle of the free fall path (JA=AK). $\vec{g}$ is the gravity vector at point A.

Accounting for that effect, equation (2) becomes:

$$\vec{g}\,(x,\ y,\ z)\ -\lambda\vec{V}_{M_r} = \vec{\gamma}_{M_r} + 2\breve{\Omega} \times \vec{V}_{M_r}, \tag{3}$$

where λ is the positive air drag coefficient.

Since we want to discuss the effect of horizontal gravity gradients on the free fall equation of motion, we expand the gravity field in Taylor series up to the first degree in the neighbourhood of point A. Using the second partial derivatives of the gravity potential denoted by $e_{11}, e_{12}, e_{13}, e_{22}, e_{23}$ and $n=e_{33}$ where indices 1, 2, 3 denote partial derivatives with respect to x, y and z respectively, n being dg/dz where g(x,y,z) is the component of $\vec{g}(x,y,z)$ along the z axis (positive downwards, cf fig. 1), we have:

$$\vec{g}(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} + \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & n \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad , \tag{4}$$

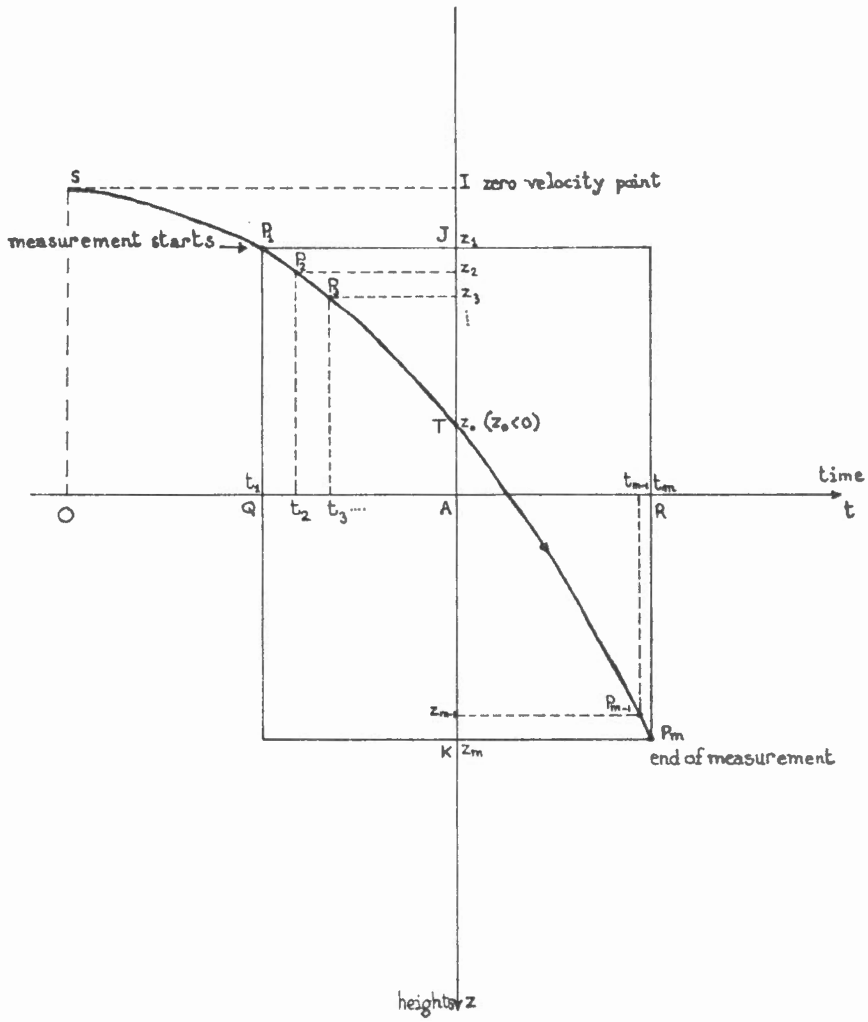
where g = g (0,0,0) (the origin (0,0,0) being the mid-point between the first and last measured point along the free fall path, i.e. $z_1+z_m=0$). From (4), one notices that Az is parallel to $\vec{g}$ at the origin A (Fig. 1). Denoting the latitude by L (angle between Ax and $\breve{\Omega}$), equation (3) expressed in terms of (4) becomes:

$$\ddot{x} + 2\dot{y}\Omega \sin L = e_{11}x + e_{12}y + e_{13}z - \lambda\dot{x}, \tag{5}$$

$$\ddot{y} - 2\dot{x}\Omega \sin L - 2\dot{z}\Omega \cos L = e_{12}x + e_{22}y + e_{23}z - \lambda\dot{y} \tag{6}$$

$$\ddot{z} + 2\dot{y}\Omega \cos L = g + e_{13}x + e_{23}y + nz - \lambda\dot{z} \tag{7}$$

Figure 2: Single free fall law of motion with measurement points P_1, P_2, P_3, ... and point A as time and vertical height origin.



Legend: Points A, J and K have the same location as in fig. 1, $z_1+z_m=0$, $t_1+t_m=0$. At point T, $z=z_o$ and $t=0$. $OQ \simeq QA \simeq 0.12$ s . $JA=AK \simeq 28$ cm.

where dots and double dots denote first and second derivatives with respect to time and $\Omega=|\vec{\Omega}|$.

Equations (5), (6) and (7) form a system of three linear differential equations of the second order with constant coefficients and could be solved by classical methods. Equation (7) is the most relevant to us since we are interested in the solution z(t) for the vertical motion of the freely falling body. $e_{13}x$ and $e_{23}y$ are always negligible. A reasonable order of magnitude for e_{13} and e_{23} derived from actual Eötvös torsion balance measurements can be 2×10^{-8} s^{-2} (Miller, 1934; p. 19). From experimental evidence (Sakuma, 1971; p. 455) (Zumberge, 1981; p. 90) and from the fact that x and y vary approximately linearly with time, x and y are always less than 2×10^5 nm. Thus, in all cases, $e_{13}x$ and $e_{23}y$ are less than 0.01 nm/s^2 (0.001 μ Gal) and will be neglected in equation (7).

The Coriolis term $2\dot{y}\Omega\cos L$ can be averaged by $2\dot{y}_o\Omega\cos L$ where $\dot{y}_o$ is the East-West deflection velocity at time t=0 (chosen at the middle of the drop cf fig. 2). A rigorous derivation of z (t) using the actual y in (7) shows no significant change.

Finally our fundamental equation of motion for a freely falling body accounting for Coriolis forces, gravity gradient (n) and air drag (λ) is the following:

$$\ddot{z} + 2\dot{y}_o\Omega\cos L = g + nz - \lambda\dot{z}. \tag{8}$$

A rigorous solution of equation (8) can be written in terms of exponential functions in time (Appendix A) but it would not be practical and accurate for numerical computation since it involves differences of large numbers. In fact a precision of 1 nm for z (t) would require computing the exponential functions with sixteen significant digits. To avoid that loss of precision, one can expand the rigorous solution in a Taylor series up to degree four. A lengthy but straightforward derivation yields from equation (8):

$$\boxed{\begin{aligned} z\,(t) = {} & z_o + \dot{z}_o t + 1/2\,(g + nz_o - \lambda\dot{z}_o - 2\dot{y}_o\Omega\cos L)t^2 \\ & + 1/6\,(n\dot{z}_o - \lambda g)t^3 + (ng/24)t^4 \end{aligned}} \quad , \tag{9}$$

where second and third order terms in n and λ have been neglected in the t^3 and t^4 coefficients introducing an error less than one part per million for these coefficients. An upper boundary of 10^{-5} nm for the truncation error, thus negligible against the length precision measurement ($\simeq$0.6 nm) can be estimated from the classical expression of the Taylor series remainder (Appendix A). Term $g+nz_o$ in the second degree coefficient can be reinterpreted as the value of gravity at point T (cf figure 2) corresponding to the time origin. Neglecting air drag and the Coriolis term one finds that gravity at T equals twice the coefficient of t^2 in formula (9). Point T is determined by height z_o above mid free fall track. Thus it can be precisely related to the floor of the laboratory room.

Before elaborating the practical procedure in Part II, let us analyze the structure of the coefficients in formula (9) and compute their numerical order of magnitude. In the coefficient of t^2 we recognize a horizontal velocity effect also called Eötvös effect (Sakuma, 1971; p. 454) which is the part of the Coriolis effect due to a non zero East-West initial velocity ($\dot{y}_o \neq 0$) and which amounts to 10 nm/s^2 for $\dot{y}_o$ = 0.07 mm/s (Zumberge, 1981; p. 90). Approximate magnitudes of the third and fourth degree coefficients can be estimated using n = 2.541 x 10^{-6} s^{-2} (254.1 µGal/m, actual value at Hammond's station in BIPM laboratory), L=45°,Ω = 2π/86400 rad/s, g = 9.8 x 10^9 nm/s^2. If the time origin is chosen at the middle of the drop, we have in the case of our example $\dot{z}_o$ = 2.367 m/s (velocity after a 0.24 s free fall, OQ$\simeq$QA approximately in fig. 2). λ can be estimated to 1.7 x 10^{-8} s^{-1} for a current operating pressure of 1.33 x 10^{-4} Pa (10^{-6} Torr) (Faller and Hammond, 1970; p. 122). The third and fourth order coefficients amount to 975 nm/s^3 and 1038 nm/s^4 respectively. Their maximum contributions (for t = 0.12 s) are about$\pm$1.7 nm and 0.2 nm respectively and they are not negligible because the standard error of a length measurement is 0.3 nm.

Having discussed the free fall equation of motion in detail and derived with all the required precision the solution given in formula (9), we consider that a polynomial of degree four in time is a suitable mathematical model for the actual times and heights measurements (t_i, z_i) i = 1, ..., m where m is the number of measured points along the free fall path. Let us now examine a practical procedure to take full advantage of formula (9).

PART II

PRACTICAL PROCEDURE

Actual free fallmeasurements are greatly perturbed by vibrations from various origin (microseismic, environmental, instrumental ...). Higher degree than two coefficients in formula (9) are badly affected and one has to forget about the possibility of determining both vertical gravity gradient and gravity at the same time. The procedure universally adopted is to measure vertical gravity gradient by relative measurements and use it to correct the data. A procedure suggested by N. Courtier and approved by J.E. Faller will be adopted here. It consists of correcting the height measurements $\tilde{z}_i$ for gravity gradient (and a small contribution due to air resistance in some cases) right from the beginning. Third and fourth order terms in formula (9) are subtracted from the observed time series $\tilde{z}_i$, i=1,...,m yielding the corrected height measurement time series z_i according to the formula:

$$z_i = \tilde{z}_i - \left[(1/6)\,(n\dot{z}_0 - \lambda g)t_i^3 + (ng/24)\,t_i^4\right] \quad i=1,\ \ldots,\ m. \tag{10}$$

We also apply the correction for finite light velocity to the observed times $\tilde{t}_i$ before processing the data any further. That correction is needed because recorded times $\tilde{t}_i$ correspond to positions of the corner cube at times t_i shifted by variable amounts Δt_i depending on $\tilde{z}_i$. The corrected times t_i are computed according to the formula (Zumberge, 1981; p. 219):

$$t_i = \tilde{t}_i + \tilde{z}_i/c, \tag{11}$$

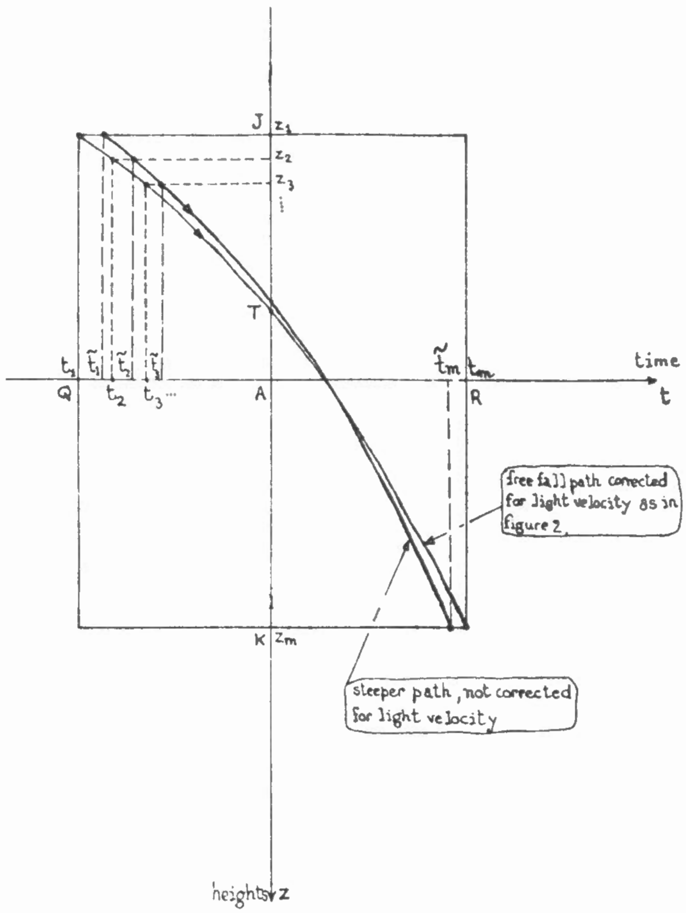
where c is the velocity of light. The effect of light velocity is to make the free fall path steeper as can be seen in fig. 3. It produces an observed gravity value too large. The correction is only defined up to an arbitrary constant that simply moves the time origin. Using centered $\tilde{z}_i$ in formula (11), there is in fact no change in the time origin. That problem of time origin does not affect the second order coefficient determining gravity but one has to solve it in order to know at what point corresponds the gravity value.

Having accounted for both effects (finite light velocity and vertical gravity gradient), z_i equals simply a polynomial of degree two in time t_i as given by the first three terms of the right hand side of formula (9). It is the ideal case. In practice, perturbations are present in the records. Our fundamental assumption is to model the perturbations using sine and cosine functions in time in addition to a Gaussian random noise r(t), we have:

$$z_i=z(t_i) = P(t_i) + \sum_{k=1}^{q} (a_k \cos f_k t_i + b_k \sin f_k t_i) + r(t_i) \quad , \tag{12}$$

where P(t) is a polynomial of degree two in time . The q frequencies f_k of amplitudes a_k and b_k represent non-random noise. The idea of the procedure is to get approximate values of the f_k by spectral analysis and to remove those waves by least squares fit of the mathematical model (12) to the original data where not only the amplitudes a_k and b_k are estimated but also the frequencies f_k. Since the time series (z_i, t_i), i = 1,...,m is unequally spaced in time, least squares spectral analysis (Vaníček, 1971) is

<u>Figure 3</u>: Effect of finite light velocity on absolute gravity determined by free fall measurements using an optical interferometer.



<u>Legend</u>: The free fall path not corrected for light velocity looks steeper than the corrected one. It would yield a gravity value too large (see explanation in the text).

the method to be used in that case.

The principle of the method is similar to a periodogram computation (Bloomfield, 1976; p.19) and can be recalled in a few lines. Various types of spectral functions can be defined. The simplest one is computed as a normalized function of the sum of squares of the weighted residuals obtained in a sine and cosine functions least squares fit to the data. More precisely, given a time series (l_k, t_k), k=1, ..., m, the least squares spectral function S (ω) is defined by:

$$S(\omega) = 1-(\sigma^2(\omega)/\sigma^2), \qquad (13)$$

where $\sigma^2 = \sum_{k=1}^{m} w_k \, l_k^2$; w_k, k=1,...,m being a set of weighting factors (taken equals unity all throughout the present study) and:

$$\sigma^2(\omega)=\sum_{k=1}^{m} w_k(l_k-\mu-a\cos\omega t_k-b\sin\omega t_k)^2. \quad (14)$$

$\hat{\mu}$, $\hat{a}$ and $\hat{b}$ are determined by the least squares condition $\sigma^2(\omega)$ minimum. Computing $S(\omega)$ at a given ω requires calculating $\hat{\mu}$, $\hat{a}$ and $\hat{b}$, then $\sigma^2(\omega)$ by formula (14) and $S(\omega)$ by formula (13). In practice, a more concise expression for $S(\omega)$ can be worked out avoiding the explicit computation of $\hat{\mu}$, $\hat{a}$ and $\hat{b}$. Since $0 \leq \sigma^2(\omega) \leq \sigma^2$ for all ω, we see that $0 \leq S(\omega) \leq 1$.

The method of spectral analysis by least squares does not differ fundamentally from the ordinary Fourier analysis. In fact when the number of data points is large and when they are equally spaced, our spectral function (13) tends to the ordinary Fourier spectral function as clearly shown by Bloomfield in his book referenced earlier.

It is interesting to note that weights can be introduced in a meaningfull least squares sense (inverse of variances). A confidence interval on $S(\omega)$, independent of ω, can also be computed, in the case of unit weight, using the Fisher distribution (Jeudy, 1982). Any peak lying outside that interval is evidence for non blank noise spectrum. This means that the time series contains deterministic information and the problem is to identify and fit the data to the significant frequencies. Our procedure is to make a fit for each of the frequencies above the upper limit of the confidence interval (picking up frequencies in the order of decreasing peak heights i.e. equivalent to the order of decreasing amplitude), and check for significance using a variance-ratio F test as explained in the appendix B. Once all the significant frequencies have been found in the spectrum, and fitted simultaneously to the data using formula (12), the residuals are again analyzed by spectral analysis for a blank noise test. The procedure is iterated until no more significant frequency is found. A final significance check is performed computing an F test for each of the frequency individually against the whole set of frequencies. At that stage the analyzed time series is represented by a systematic part i.e. formula (12) and a random part that is a Gaussian random variable having as variance the mean sum of squares of residuals of model (12). As mentioned earlier the frequencies themselves are readjusted (parameters $f_k, k=1,..,q$) using as approximate starting values, the values of the frequencies already adjusted and for the new frequency those provided by the spectral function computation ($\hat{a}$ and $\hat{b}$ in formula (14) are computed explicitly for the peaked frequencies). The procedure of readjusting the frequencies themselves makes the computation unbiased. Otherwise errors would accumulate as more and more frequencies are taken into account. In the free fall time series analysis, at the beginning $S(\omega)$ is not computed directly from the z_i, $i = 1,..., m$, but from the residuals of a second degree polynomial least squares fit.

PART III

RESULTS

A series of free fall measurements recorded from an absolute gravity determination at the International Gravimeters Calibration Campaign (Boulanger J. et al., 1982) was analyzed using the method described in Part II. Results are summarized in table 1.

Table 1: Significant frequencies in a set of real free fall data (99.956% confidence level). The data was provided by Dr. J.A. Hammond.

Frequency in Hz$\pm\sigma$ f_k	$a_k\pm\sigma$ in nm	$b_k\pm\sigma$ in nm	ρ value (F test)
6.04±0.42	0.92±0.45	3.66±0.32	60.64
29.30±0.32	1.88±0.25	-0.69±0.27	20.71
6115.80±0.40	1.58±0.25	-0.10±0.27	13.40
68.84±0.55	0.90±0.26	0.68±0.26	6.59

Legend: In accordance with formula (12) we have the following spectral decomposition for the real set of free fall data analyzed here: $z_i = (-67\ 283\ 356.52\pm0.37) + (2\ 367\ 125\ 912.9\pm7.1)\ t_i$
$-(4\ 904\ 631\ 953.9\pm88.8)t_i^2 + \sum_{k=1}^{4}(a_k \cos f_k\ t_i + b_k \sin f_k\ t_i) + r(t_i)$,

where the degree two polynomial coefficients respectively have units: nm, nm/s and nm/s^2. a_k, b_k and f_k as in table 1 above and r(t) a Gaussian random noise of standard error 3.7 nm. The 99.956% confidence level is passed when $\rho \geq 6.0$ where ρ is computed as explained in the appendix B.

We remark that the two first frequencies and possibly the last one can be attributed to residual effects due to the so called industrial microseisms (~ 5-50Hz) (Sakuma, 1971; p. 455).

The third frequency can be of instrumental origin as mentioned by R.L. Iliff, a collaborator of J. A. Hammond, in a private communication. The cause could be the servoed wavelength control whose multi-vibrator has a nominal frequency of 5 kHz.

Since the main point of the present work is to show how those perturbing frequencies create systematic errors up to several tens of μGal in a single set of measurements (i.e. one "drop" of the corner cube), it is interesting to see how the gravity value (up to an arbitrary constant) is modified as more and more frequencies are removed from the data set. Results are summarized in table 2.

Table 2: Systematic error due to vibrations.

Number of frequency eliminated	Gravity value in μGal $\pm\sigma$ up to an arbitrary constant
0	980 926 339.4±5.9
1	980 926 411.0±13.0
2	980 926 403.2±10.9
3	980 926 394.8±9.6
4	980 926 390.8±8.9

Legend: Frequencies are eliminated in the same order as they are shown in table 1.

Figure 4 represents values given in table 2. The shape of the curve suggests that some limiting value would be reached as more and more frequencies would be eliminated. But this would require to lower the confidence level we chose to the 99.956% value. It may not be justified when working with a large number of data set since one may hope cancellation of these systematic errors.

<u>Figue 4</u>: Effect of perturbing frequencies on the gravity value from one set of real data (i.e. one "drop").

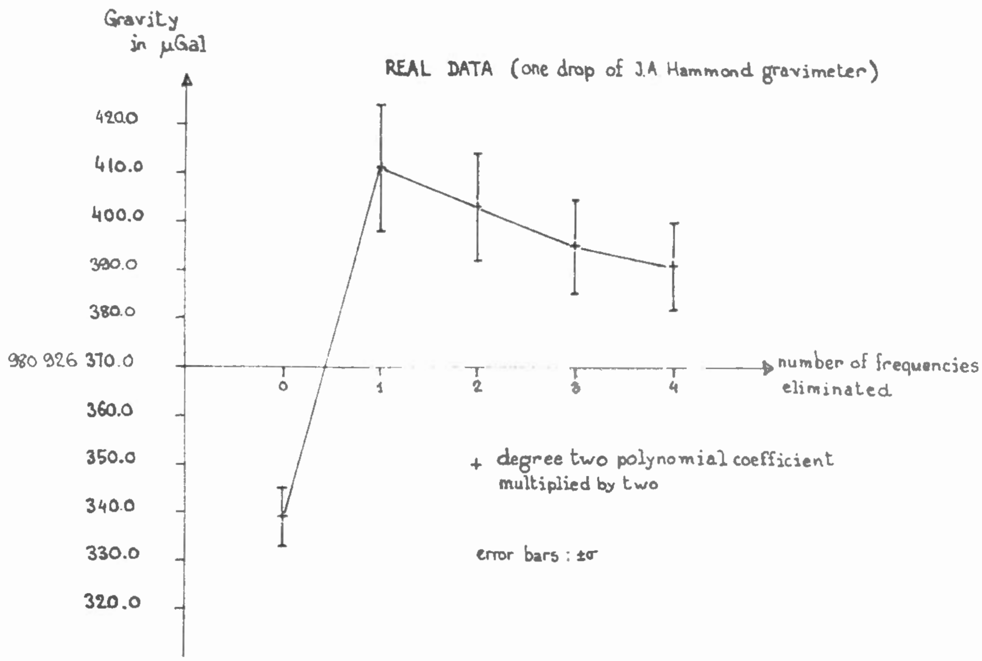
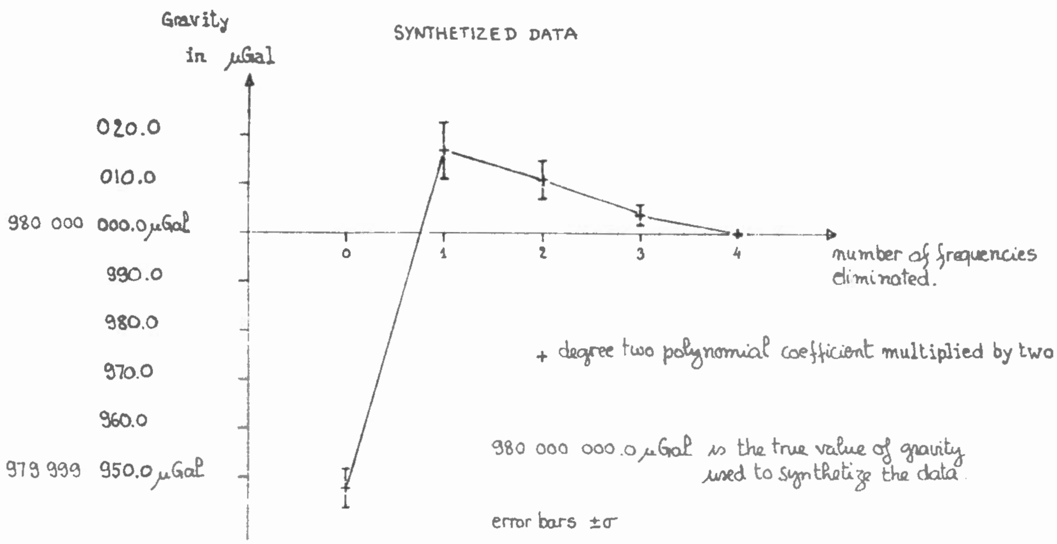


Figure 5 shows the results obtained using synthetized data computed with parameters similar to those found for the real data except that we have set random noise r(t) identically to zero (cf legend of table 1). As expected the method is unbiased and recovers the "true" gravity value within 0.003 µ Gal. (round off error).

<u>Figure 5</u>: <u>Synthetized data analysis</u>



Finally an example of gravity computation is given in table 3 below.

Table 3: Example of absolute gravity computation from one set of real free fall data*.

*(kindly provided by Dr. J.A. Hammond, October 28, 1981 at BIPM)

Second degree coefficient multiplied by two, as obtained after removal of four perturbing frequencies (6.0, 29.3, 6115.8 and 68.5 Hz) and correction for the light velocity effect and the gravity gradient effect:		
2 x 490 463 195.4	=	980 926 390.8 μGal
Laser wavelength correction:		
-(1-632 991.435 5/632 991.470 0) x g	=	53.5μGal
Reduction to the floor using the measured gravity gradient (2.541μGal/cm):		
111.4 x 2.541	=	+283.1μGal
Earth tide correction:		- 24.0μGal
Absolute gravity final value	=	980 926 596.4 μGal

Conclusion:

A practical and accurate (within 10^{-4} nm) free fall formula (cf equation (9) was derived for the purpose of determining absolute gravity from free fall measurements using least squares fit. Small correcting terms for air resistance and Coriolis force (Eötvös effect) are taken into account. In part II, a procedure was derived using spectral analysis, least squares fit and significance F test (cf Appendix B) to identify perturbing frequencies in the data. They may be caused by microseisms and also instrumental components. The main point of the present study is to show that these frequencies create systematic errors up to 50 μGal in one set of real data (i.e. one "drop"), (cf table 2). One may hope that these errors will cancel out over many data sets but this could not be checked due to lack of real data. Our procedure is considerably more sophisticated than a straightforward degree two polynomial least squares fit. Nevertheless after optimization of our software, it costs only about ten dollars to process one drop of 200 points. It would be much less (maybe not even a dollar) to process a 50 point drop as currently available on J.E. Faller absolute gravimeter. Anyhow it cannot be done in real time. Each drop will have to be recorded on a magnetic tape and processed on the Cyber.

One can think of an even faster procedure to be evaluated when more real data will be available. It would consist of computing an average drop $(\bar{t}_i, z_i)$, i=1,...,m out of several tens of individual drops, summing and averaging the times corresponding to the same heights z_i. Then only one set of data would have to be processed per station.

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APPENDIX A

RIGOROUS SOLUTION AND TAYLOR SERIES TRUNCATION ERROR

The rigorous solution of equation (8) equals

$$z(t) = -(g - 2\Omega\dot{y}_0 \cos L)/n + C\exp(r_1 t) + D\exp(r_2 t)$$

where $r_1 = -(\lambda+\Delta)/2$, $r_2 = (-\lambda+\Delta)/2$, $\Delta = (\lambda^2 + 4n)^{1/2}$ and C and D are two constants defined by

$$C + D = z_0 + (g - 2\dot{y}_0 \Omega\cos L) / n$$

$$C - D = \left(-\lambda(C+D) / \Delta\right) - \left(2\dot{z}_0/\Delta\right).$$

Solving numerically for C and D using the numerical values adopted in the main text, $\dot{y}_o = 0$ and $\lambda = 1.7 \times 10^{-8} s^{-1}$ we have in nanometers:

$$z(t) = (-3.86 \times 10^{15}) + (1.93 \times 10^{15}) \exp(r_1 t) + (1.93 \times 10^{15}) \exp(r_2 t)$$

where $r_1 = -0.001\,594\ s^{-1}$, $r_2 = 0.001\,594\ s^{-1}$.

Mathematically, the truncation error R (t) of Taylor series expansion (9) equals

$$R(t) = \frac{t^5}{5!} \times d^5 z(\theta)/dt^5$$

where $0 \leq \theta \leq t$. An upper boundary for R(t) for all t can be found by

$$|R(t)| \leqslant \left(\frac{\tau}{5!}\right)^5 \times (C|r_1|^5 + Dr_2^5) \times (1.0002) \leqslant 10^{-5} \text{ nm}$$

where τ is the maximum value of t and 1.0002 an upper boundary for both exponential functions (i.e. $\tau = 0.12$ s).

In conclusion of the present appendix, the Taylor series truncation error for the free fall law of motion is less than 10^{-5} nm.

Appendix B

Variance-ratio Significance F Test

When in a least squares fit new parameters are added to the mathematical model, one may wonder if these parameters are significant. In any case (provided the normal matrix is not ill conditioned) the sum of squares of the weighted residuals will diminish. The purpose of the test described now is to set a numerical limit to that decrease below which the parameters will not be considered to be significant.

Let us now enter into the mathematical details. Having a set of m observables being fitted to mathematical model one with u_1 parameters and yielding sum of weighted residuals r_1, u_2-u_1 parameters are added tranforming mathematical model one into mathematical model two. Denoting by r_2 the sum of weighted residuals (as mentioned above, in any case $r_2 \leq r_1$), the variance-ratio F test is based on the assumption that:

$$\rho = (r_1 - r_2)\ (m - u_2)\ /\ r_2\ (u_2 - u_1) \quad , \qquad \text{B1}$$

is a particular value of a ρ' random variable having an $F_{u_2-u_1, m-u_2}$ Fisher distribution (Hamilton, 1964; p. 139) in the case where mathematical model one (u_1 parameters) is true. This means that the following probability statement holds:

$$P\left[\rho' \geq F_{1-\alpha;\, u_2-u_1, m-u_2}\right] = \alpha \qquad \text{B2}$$

where α is a significance level, (in the present work, $\alpha = 0.00044$). When the two least squares fit have been computed, ρ is calculated by formula B1. If

$\rho \geqslant F_{1-\alpha;\, u_2-u_1, m-u_2}$ the ρ' random variable of which ρ is a particular value cannot be considered to have a Fisher distribution and the hypothesis that mathematical model one is true, is rejected. The $u_2 - u_1$ additional parameters are then considered to be significant.

Intuitively the test is based on the idea that if an additional parameter produces a sufficiently large relative decrease in the r values (i.e. the sum of squares of the weighted residuals when Σ is diagonal), it is significant. Formula B1 accounts for both: the number m of observations (as it enters into ρ and into $F_{1-\alpha;\, u_2-u_1, m-u_2}$)

and the Gaussian fluctuations of the random errors. ρ is not affected by the overall scale of random errors, it is a relative value. At a fixed variance-ratio (i.e. $(r_1 - r_2)/r_2$) and a fixed number (i.e. u_2-u_1) of additional parameters, the significance test is easier to pass with a large number of observations than with a small one, because ρ is proportional to $(m - u_2)$ and because $F_{1-\alpha;\, u_2-u_1,\, m-u_2}$ decreases and tends to a finite positive limit as m tends to infinity. In other words, for small sample of data, the relative devease in value has to be greater than for large samples in order to make a parameter significant.

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