Thermal mismatch in Ångström's method of measuring thermal diffusivity

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Introduction

Consider the semi-infinite solid $x>0$, with surface temperature a harmonic function of time. If that surface temperature is given by $v = Ae^{i\omega t}$ and the initial temperature is zero, the temperature within the solid at time t is (Carslaw and Jaeger, 1959, p.65):

$$v(x,t) = Ae^{-kx}e^{i(\omega t-kx)} \tag{1}$$

where $k = \sqrt{\omega/2s}$, ω being the angular frequency of the temperature wave and s the thermal diffusivity of the medium through which it propagates. Equation 1 provides the basis for Ångström's method (Ångström, 1863) for the measurement of diffusivity. In this method a harmonic temperature variation is impressed on one end of a long, thin rod. If there are no heat losses from the surface of the rod the diffusivity of the material is calculated by measuring either the amplitudes or the relative phase of the temperature variations on two planes through the rod, and applying equation (1). If surface heat losses do occur equation (1) is modified to:

$$v(x,t) = Ae^{-k_1 x}e^{i(\omega t - k_2 x)}.$$

where

$$k_1 = \sqrt{\xi + (\xi^2+\omega^2)^{1/2}/2s}$$

$$k_2 = \sqrt{-\xi + (\xi^2+\omega^2)^{1/2}/2s}$$

.../2

in which ξ is a term that describes the surface heat loss (Carslaw and Jaeger, 1959, p.134). The effects of surface heat loss can be eliminated by measuring both the temperature wave amplitudes and the phase lag and putting

$$k = \sqrt{k_1 k_2}$$

in equation (1).

The rod is sufficiently long that mathematically it can be considered to be infinite, such that at its free end the temperature is the same as ambient temperature at all times. In this case, there is no energy reflected into the rod from the free end. Kanamori et al. (1968, 1969) used a modified version of Angstrom's method to measure diffusivity of samples of mathematically finite length, ℓ. By assuming that there is no heat flux across the free end at $x = \ell$, an approximate solution for the temperature in the rod can be derived. derived. The diffusivity can therefore be obtained as in Ångström's original method.

In the version of Ångström's method used by the geothermics group of the Earth Physics Branch it is desired to measure diffusivity and thermal conductivity of the same sample. Requirements of the divided bar method for conductivity measurements mean that a thin disc must be used. In order to obtain good thermal contact between the thermistors and the upper and lower faces of the sample it is necessary to hold the sample in a hydraulic press. This means that there is no free face of the sample across which no heat is transferred, so that the analysis technique of Kanamori et al. (1968) cannot be used. Nor is a disc that is typically 10-20mm thick sufficiently long that Ångström's method in its original form (equation 1) can be used. A long

.../3

($\simeq$28cm) matching rod of material that has thermal properties that are similar to those of the sample is therefore inserted between the upper face of the sample disc and the stainless steel hydraulic press. The far end of the matching rod remains at ambient temperature so that Ångström's assumptions are partly met. However, the effect of thermal impedance mismatch between sample and rod means that a correction is necessary to the apparent diffusivity obtained by measuring amplitude and phase of the temperature wave.

Transmission line analogy

The temperature, v, and rate of flow of heat, Q, in a rod of cross-sectional area a with no loss of heat from its surface satisfy:

$$a\rho C \frac{\partial v}{\partial t} = -\frac{\partial Q}{\partial x} \quad ; \qquad Q = -Ka\frac{\partial v}{\partial t} \tag{2}$$

(Carslaw and Jaeger, 1959, p.69) where ρC is the volumetric heat capacity of the material, K is its thermal conductivity, and x is the distance from the heat source in the direction of propagation. These are the equations for the potential and current in a transmission line of series resistance $1/Ka$ and shunt capacitance $a\rho C$ per unit length, and with zero inductance and leak conductance. Hence the theory of electromagnetic wave propagation in a transmission line can be applied directly to the problem of the propagation of a temperature wave in a solid rod.

The potential along a transmission line of characteristic impedance Z_c that is terminated with a load impedance Z_L is:

.../4

$$V = V_0 \, e^{i\omega t}\left[\cosh(-\gamma x) + \frac{Z_c}{Z_L}\sinh(-\gamma x)\right] \tag{3}$$

where x=0 at the source, V_0 is the signal amplitude at the source, and γ is the propagation constant.

Z_c is given by:

$$Z_c = \sqrt{(R + i\omega L)/(G + i\omega\theta)} \tag{4}$$

where R, L, G and θ are the resistance, inductance, leak conductance and shunt capacitance (all per unit length) and ω is the angular frequency of the voltage or current variation.

The propagation constant, γ, is given by:

$$\gamma = \sqrt{(R + i\omega L)(G + i\omega\theta)} = \alpha + i\beta \tag{5}$$

Substituting for the thermal properties of a rod, with L=0 and G=0, we obtain, for a temperature wave in the rod:

$$Z_c = 1 / a\sqrt{iK\omega\rho C} \tag{6}$$

$$\alpha = \beta = \sqrt{\frac{\omega}{2S}} = k \tag{7}$$

Substituting into equation (3) and re-arranging terms:

$$v(x,t) = \frac{v_0}{2}\left[e^{-kx}\left(\frac{Z_L+Z_c}{Z_L}\right)e^{i(\omega t-kx)} + e^{kx}\left(\frac{Z_L-Z_c}{Z_L}\right)e^{i(\omega t+kx)}\right] \tag{8}$$

The first term in equation (8) represents the incident wave, and the second term describes the reflected wave. When there is no thermal mismatch, i.e. when the characteristic impedance and terminating impedance are equal, the equation reduces to equation (1), with $A = v_0$.

Equation (8) can be used in Angstrom's method to allow for the effect of thermal mismatch between a sample disc of thermal impedance Z_c and a matching rod of impedance Z_L. When there is mismatch the amplitude ratio and the relative phase lag of a temperature wave at two points are affected, so that measurement of both parameters is required.

Some observations

Figures 1 - 3 show in various ways some examples of the effects of thermal mismatch between a sample of length 10mm and a long rod. Parameters have been assigned values that are similar to actual values for the materials that are of interest in the current work.

Fig. 1 shows the variation of amplitude of the temperature wave with distance from the heat source ($x = 0$) in two samples, one of diffusivity 1.40 mm^2s^{-1} and conductivity 3.40 $Wm^{-1}K^{-1}$ (a granite), and one of diffusivity 0.85 mm^2s^{-1} and conductivity 1.37 $Wm^{-1}K^{-1}$ (fused silica). In both cases the rod is a granite of diffusivity 1.40 mm^2s^{-1} and conductivity 3.40 $Wm^{-1}K^{-1}$. If the sample and rod are identical the amplitude decreases exponentially, with no sharp break at the sample-rod interface (assuming that there is a perfect thermal contact). However, when the sample and rod are of different material there is a sharp change at the interface in the way the amplitude decreases. Further, the amplitude within the sample is much higher than it is in the case of no thermal mismatch. Similarly the phase difference of the temperature wave between points in the interior of the sample and the interface is increased when there is mismatch for the given particular parameters. The apparent diffusivity of fused silica when a granite matching rod is used is thus less than the true diffusivity.

The variation of apparent diffusivity with actual diffusivity of samples matched to a granite rod of diffusivity 1.40 mm^2s^{-1} and conductivity 3.40 $Wm^{-1}K^{-1}$ is shown in Fig. 2. For each curve the volumetric heat capacity has been kept constant so that diffusivity is directly proportional to

.../7

conductivity; in curve 2 the volumetric heat capacity of the sample is the same as that of the rod. It is sometimes assumed that the volumetric heat capacity of rocks at a particular temperature and pressure is constant; Jessop _et al_. (1981), however, show that a wide variation between approximately 1.8 and 3.0 MJ $K^{-1}m^{-3}$ can be expected even within similar rock types. The curves in Fig. 2 are designed to show the extent of scatter that might be expected in diffusivity measurements of several samples of a particular rock type. The shaded area of Fig. 2 represent a range of +/- 2.5% about a line of slope unity, and is therefore the range within which the effect of thermal mismatch is secondary to the acceptable experimental error. It is clear that only for a narrow range of sample diffusivity (1.3 - 1.5 mm^2s^{-1}), when sample and rod have the same volumetric heat capacity, is no correction required for thermal mismatch.

Fig. 3 shows the variation of the ratio apparent/actual diffusivity as a function of period of the temperature wave. Curve 1 simulates a fused silica sample with a granite rod, curve 2 is for no mismatch, and curve 3 is for a sample of diffusivity 1.60 mm^2s^{-1} and rod of diffusivity 1.40 mm^2s^{-1}, sample and rod having the same volumetric heat capacity. The effect on the apparent diffusivity of varying the period is clearly greater when mismatch is greater.

The effects shown in Figs. 1-3 are for specific examples. Thermal mismatch between sample and rod occurs because of differences in diffusivity, conductivity, volumetric heat capacity and diameter of sample and rod. With so many variables it is not possible to construct standard tables or curves for correction of a measured apparent diffusivity. Each apparent diffusivity that is measured can be corrected by an iterative procedure. The conductivity

.../8

of the sample is known independently, and the diameter of both sample and rod are known. The conductivity assigned to the rod is assumed from several measurements on discs cut from similar material and the diffusivity assigned to the rod is the mean of several measurements on such samples under the assumption that there is, in those measurements, no thermal mismatch. With these known or assumed parameters, an initial estimate of the true diffusivity of the sample under consideration is made. The apparent diffusivity that this "true" value would produce is calculated and compared with the measured apparent diffusivity. The inital estimate is refined and the process is repeated until the calculated and measured apparent diffusivities agree with each other within prescribed limits. The current "true" diffusivity is then taken to be the actual diffusivity of the sample, with due regard to the experimental uncertainty.

.../9

References

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Figure captions

Figure 1 Variation of amplitude and phase of sinusoidal temperature wave along rock sample and into matching rod. Amplitude normalized to 1 and phase to 0° at sample-rod interface. Solid curve: sample and rod same material and diameter. Dashed curve: Sample has diffusivity 0.85 mm^2s^{-1} and conductivity 1.37 $Wm^{-1}K^{-1}$. Rod has diffusivity 1.40 mm^2s^{-1} and conductivity 3.40 $Wm^{-1}K^{-1}$ in both cases. Period of temperature wave is 240s.

Figure 2 Variation of apparent diffusivity with actual diffusivity for samples of varying thermal properties and rod of diffusivity 1.40 mm^2s^{-1} and conductivity 3.40 $Wm^{-1}K^{-1}$. Shaded area is range of experimental observations (+/- 2.5%) for no mismatch. Period is 240s. Volumetric heat capacity of sample is:

1. 1.80 MJ $K^{-1}m^{-3}$
2. 2.43 MJ $K^{-1}m^{-3}$ (same as rod)
3. 3.00 MJ $K^{-1}m^{-3}$

Figure 3 Variation of ratio apparent/actual diffusivity with period.

1. Sample 0.85 mm^2s^{-1}, 1.37 $Wm^{-1}K^{-1}$
2. Sample 1.40 mm^2s^{-1}, 3.40 $Wm^{-1}K^{-1}$
3. Sample 1.60 mm^2s^{-1}, 3.90 $Wm^{-1}K^{-1}$

Rod has diffusivity 1.40 mm^2s^{-1} and conductivity 3.40 $Wm^{-1}K^{-1}$ in each case.

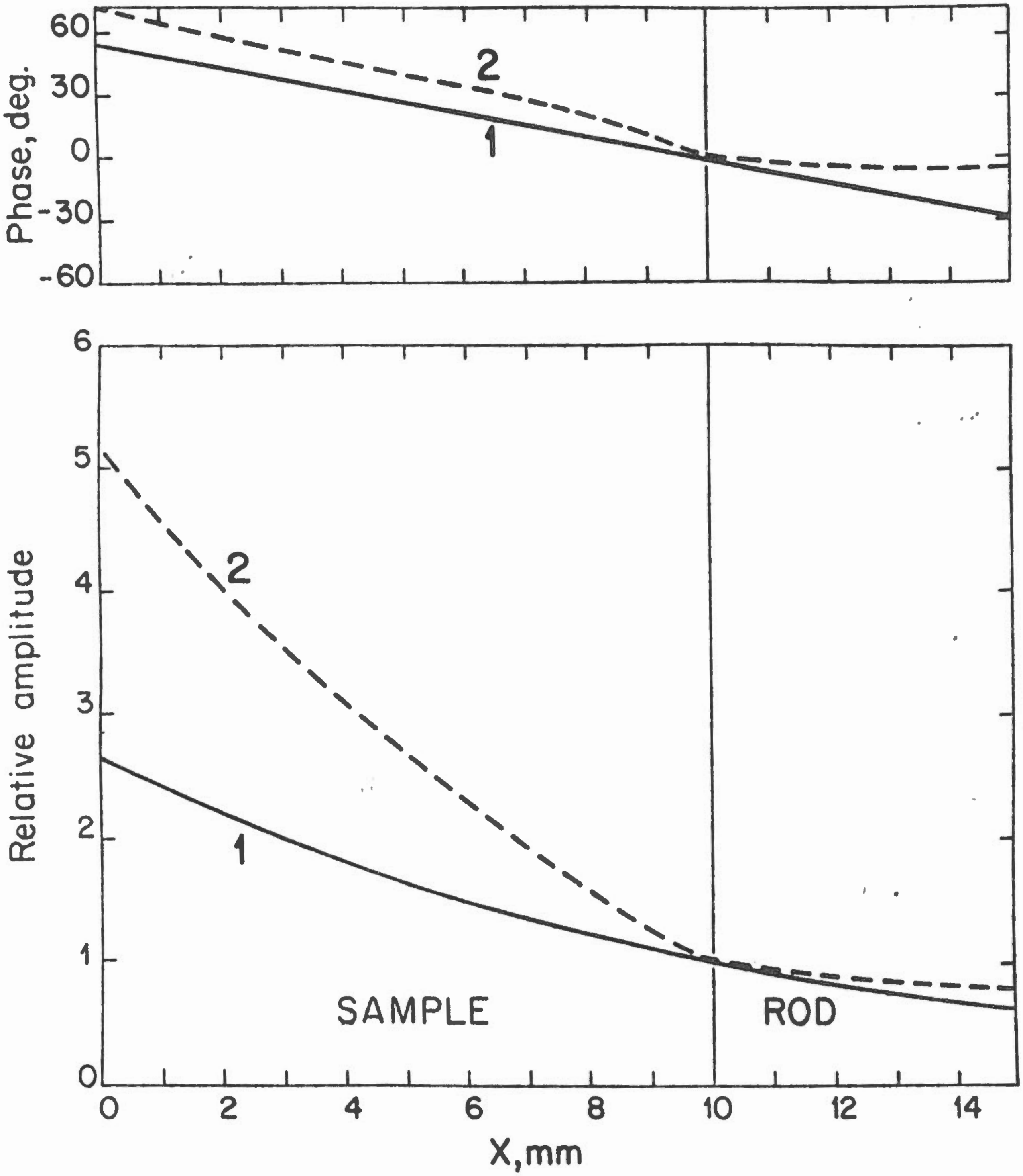
Phase, deg.
60
30
0
-30
-60
2
1
Relative amplitude
6
5
4
3
2
1
0
2
1
SAMPLE
ROD
0
2
4
6
8
10
12
14
X,mm

Figure 1

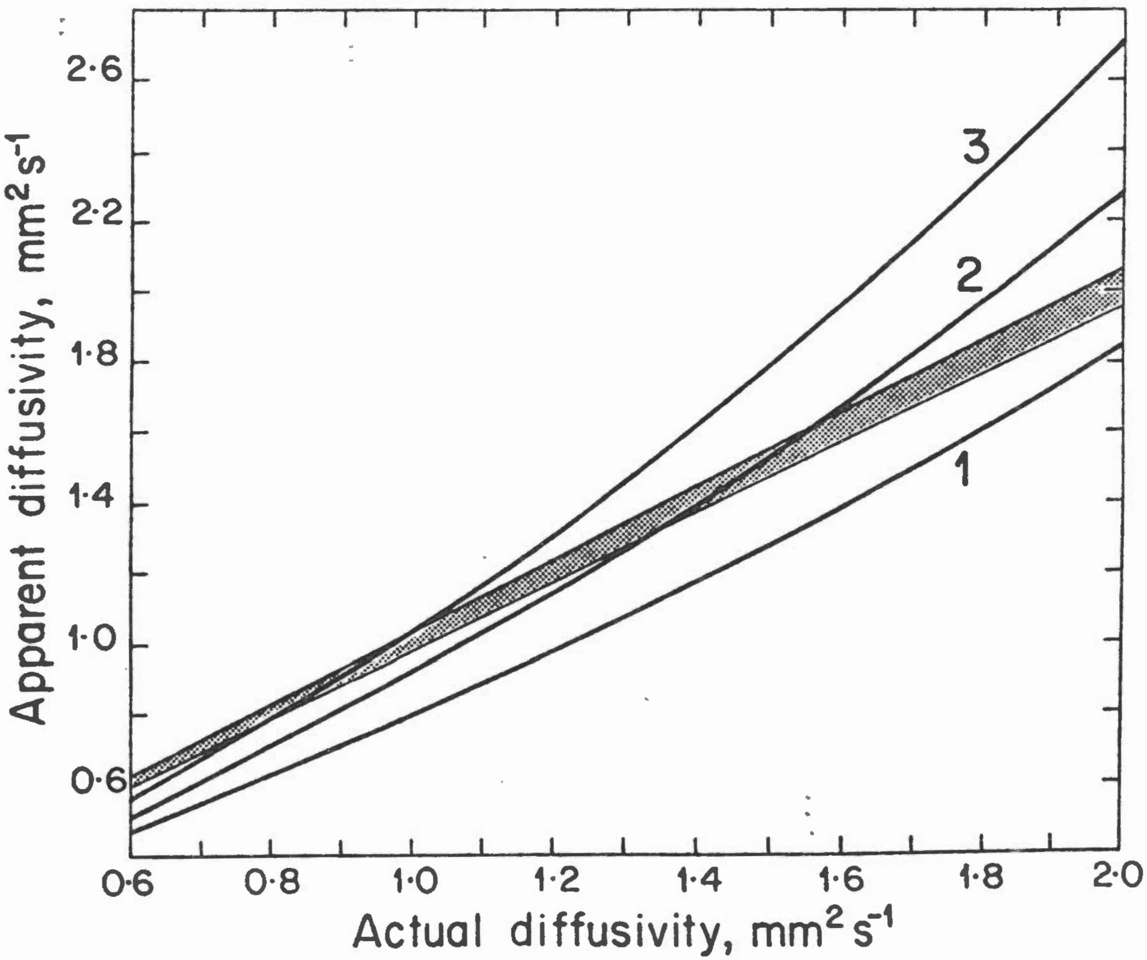
Apparent diffusivity, $mm^2 s^{-1}$
Actual diffusivity, $mm^2 s^{-1}$
0·6
1·0
1·4
1·8
2·2
2·6
0·6
0·8
1·0
1·2
1·4
1·6
1·8
2·0
3
2
1

Figure 2

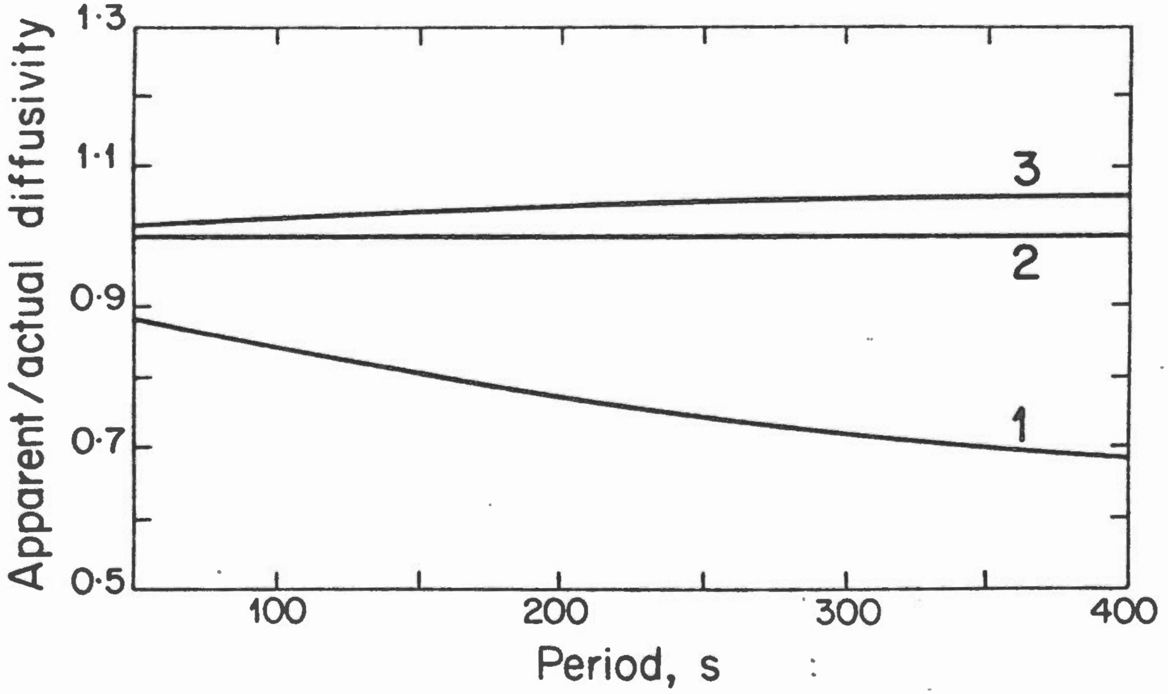
Apparent/actual diffusivity
1·3
1·1
0·9
0·7
0·5
3
2
1
100
200
300
400
Period, s

Figure 3