## PUBLICATIONS

of THE

# Dominion Observatory OTTAWA 

VOL. XVI, No. 14

# GHARTS FOR MEASURING AZIMUTH AND DISTANCE and for TRACING SEISMIC RAYS THROUGH THE EARTH 

P. L. Willmore and J. H. Hodgson by scanning the original publication.

Ce document est le produit d'une numérisation par balayage de la publication originale.

#  <br> TGUNARIMA  

## 2ทOITADLIRUY

Int 80

# V1otsvisadO noinimoC AVATTO 

H. ohn , IVX atol

SDVATEIC AVA MTOMISA OVIMYehTM SOT ZTHAFO
 HTSMA Giat HaUOSiHT



# Charts for Measuring Azimuth and Distance and for Tracing Seismic Rays Through the Earth 

BY<br>P. L. Willmore and J. H. Hodgson


#### Abstract

The paper presents two charts which are of help in seismology. With the aid of the first chart the distance between any two points on the earth and their relative azimuths can be determined with an accuracy sufficient for most seismological problems. The second chart enables one to trace rays through the earth's mantle, and is of help in studies which involve the relationship between depth of penetration and point of emergence. It is used to exhibit a complicated cusp arrangement associated with the phase pP.

Additional copies of both charts, mailed unfolded, may be obtained from the authors.


## INTRODUCTION

This paper will present two charts which have been found useful in the seismological work of the Dominion Observatory. One of them is a stereographic net for the determination of azimuths and distances. The other is a diagram, based on the structure of the earth as determined by Jeffreys and Bullen, tracing ray paths penetrating to various depths. The diagram can be used very efficiently in any problem involving ray tracing, and the method described can be applied to other postulated structures.

Neither of the matters treated involve very much original work. However the draughting of the charts is so complicated that it seems worthwile to make our charts available to all. These have been drawn by Mrs. I. H. Blüme, of the Observatory staff. Our very sincere thanks go to her for her painstaking and skillful work.

## THE USE OF THE STEREOGRAPHIC NET FOR DETERMINING AZIMUTH AND DISTANCE

## Statement of the Problem

Direct computations of the distance and azimuth between pairs of points on the earth's surface become tedious when large bodies of data have to be reduced. In approximate work, distances are often measured by stretching a tape over the surface of a globe, but the method is clumsy, and its accuracy is often limited by the absence of a sufficiently close network of parallels and meridians. Recently, Tsuboi (1951) published a simple and rapid method for finding the distances from a fixed station to any number of other points on the earth's surface, but his device contains a nomogram which has to be redrawn for each station. Moreover, azimuth cannot easily be measured either on the globe or by Tsuboi's method.

At the Dominion Observatory, interest in the direction of faulting in earthquakes has raised the problem in an acute form, for a single fault plane reduction may require a knowledge of the distance and azimuth from an earthquake epicentre to a hundred or more recording stations. Until recently, the computations were made on an electronic machine at considerable expense, but with the aid of Chart I, the results can be obtained
rapidly, to an accuracy which is fully comparable with that of the original data. The chart is similar to that which is used for the solution of spherical triangles in crystallography. It has been adapted for seismological use by numbering the seales appropriately.

## Description of the Chart

The chart consists of a stereographic projection of a hemisphere, on which the meridians and parallels have been drawn at intervals of two degrees. The meridians are numbered in both directions across the equator. In referring to points in the eastern hemisphere, the zero meridian is taken to be the left-hand half of the bounding circle of the projection, and the longitudes are the positive numbers reading from left to right. For the western hemisphere, the zero meridian is the right-hand boundary, and the longitudes are the negative numbers reading from right to left. The latitude corresponding to each parallel is entered on the bounding circle, and the angular distances of the parallels from the south pole are marked along the vertical axis. When in use, the chart is fixed to a smooth board, and covered by a sheet of tracing paper held in place by a single pin through the centre.

To find the distance and azimuth of a point $P$ (whose latitude and longitude are $\varphi^{\prime}, \lambda^{\prime}$ ) from $\mathrm{Q}(\varphi, \lambda)$ first subtract the angle $\lambda$ from the longitudes of both points. Plot $\mathbf{P}$ on the tracing paper at $\varphi^{\prime},\left(\lambda^{\prime}-\lambda\right)$ and $Q$ at $\varphi, O$, using the sign convention for longitude outlined in the preceding paragraph, see (Figure 1). Now rotate the tracing paper until


Figure 1
$Q$ is carried to the south pole of the projection, and let $P_{1}$ and $Q_{1}$ be the displaced positions of $P$ and $Q$. Then the distance $P Q$ being equal to $P_{1} Q_{1}$ can be read immediately on the vertical scale of the projection. The azimuth of $P$ is equal to the longitude of $P_{1}$, and is read on the equatorial scale of the projection.



In proof of this result, consider the operations to be the displacements of $P$ and $Q$ on the surface of a sphere, rather than on the projection. The subtraction of $\lambda$ from each longitude rotates $P$ and $Q$ about the polar axis until $Q$ falls on the zero meridian. The rotation of the tracing paper rotates $P$ and $Q$ about the axis of projection, so that $Q$ travels down the meridian QN. As neither rotation affects the distance between the two points $P_{1} Q_{1}$ is evidently equal to the original value of $\Delta$. Moreover, the azimuth of $P$ relative to $Q$ is by definition the angle between the great circle $P Q$ and the meridian $Q N$. This is unaltered by the first rotation, but the second rotation transforms the great circle QP into the meridian $Q_{1} P_{1}$. The transformation of the azimuth of $P$ into the longitude of $P_{1}$ follows immediately from this fact. The rotation of the tracing paper legitimately represents rotation on the sphere because of the symmetry about the projection axis. The choice of the stereographic projection in preference to others which have the necessary symmetry is made partly because it is the easiest to construct, and partly because it enables an entire hemisphere to be reproduced without excessive distortion.

## Discussion of Errors

Despite the care with which the chart has been drawn, some inaccuracies must certainly exist within it. In addition to these, errors can arise from any misplacement of the pin which locates the centre of rotation of the tracing paper, and from the inability of the observer to interpolate exactly between the $2^{\circ}$ lines. The effect of the eccentricity of the pin and of some of the other sources of error will be most marked when the tracing paper is turned through a large angle after the points are entered and before their positions are read. For this reason, more accurate readings may be expected when the epicentre is near the south pole than when it is farther north.

In order to determine the magnitudes of both the random and systematic terms, an observer was asked to determine the azimuth and distance from an epicentre in each of the northern and southern hemispheres to 20 recording stations. The stations were selected to be distributed as uniformly as possible over the earth. In order to reverse the effect of the observer's personal error the chart was used in its ncrinal position for half of the test, and was inverted for the remainder. The readings were compared with computed values given to $0^{\circ} .1$. The mean error in degrees, and the standard deviation $\sigma$ of a single observation under each of the four sets of conditions is given in Table I.

TABLE I

|  | Northern Epicentre | Southern Epicentre |
| :---: | :---: | :---: |
| Chart in normal position | $\begin{aligned} \text { Mean } \Delta_{\text {obs. }}-\Delta_{\text {cale. }} & =-0.03 \pm 0.05 \\ \sigma & =0.16 \\ M_{\text {ean }} \mathbb{Z}_{\text {obs. }}-Z_{\text {calc. }} & =+0.08 \pm 0.06 \\ \sigma & =0.20 \end{aligned}$ | $\begin{aligned} \text { Mean } \Delta_{\text {obs. }}-\Delta_{\text {calc. }} & =+0.02 \pm 0.04 \\ \sigma & =0.12 \\ \text { Mean } Z_{\text {obs. }}-Z_{\text {calc. }} & =-0.02 \pm 0.09 \\ \sigma & =0.28 \end{aligned}$ |
| Chart <br> Inverted | $\begin{aligned} \text { Mean } \Delta_{\text {obs. }}-\Delta_{\text {calc. }} & =-0.19 \pm .07 \\ \sigma & =0.22 \\ \text { Mean }_{\text {obs. }}-Z_{\text {cale. }} & =+0.06 \pm 0.07 \\ \sigma & =0.23 \end{aligned}$ | $\begin{aligned} & \text { Mean } \Delta_{\text {obs. }}-\Delta_{\text {calc. }}=-0.06 \pm 0.05 \\ & \sigma=0.14 \\ &{\text { Mean } Z_{\text {obs. }}-Z_{\text {cale. }}}=0.00 \pm 0.06 \\ & \sigma=0.20 \end{aligned}$ |

As expected, there is some tendency for the largest systematic errors to occur in the readings which refer to the northern epicentre, but the systematic terms are not large compared with their standard deviations. A much larger statistical test would therefore be necessary to separate the sources of error, but it is clear that they are of the same order as those which are inherent in good epicentral determinations.

The same operator made a further 36 observations on a northern epicentre with different stations and a different copy of the chart. In this case the mean error for one observation of distance was $0^{\circ} \cdot 16 \pm 0^{\circ} \cdot 13$ and the corresponding error in azimuth was $0^{\circ} \cdot 15 \pm 0^{\circ} \cdot 22$. Of all the 116 observations only 6 were in error $0^{\circ} \cdot 5$ or more, and the largest error was $0^{\circ} \cdot 6$. These figures are consistent with the standard deviations given and further illustrate that the chart is sufficiently accurate for most purposes.

## THE TRACING OF RAYS THROUGH THE EARTH

## Derivation of the Chart

It is frequently useful to trace rays through the earth and so to obtain some idea of their depth of penetration and of their point of emergence. As an example, one of us (Hodgson and Allen, 1954) recently had reason to suspect that a complicated cusp was associated with the phase pP . The method to be described permitted the very rapid verification of this fact. An analytical demonstration has since been given by Bullen (1955).

It is well known that the path of a ray in a spherically stratified earth is governed by the equation

$$
\begin{equation*}
p=\frac{r}{v} \sin i \tag{1}
\end{equation*}
$$

where $v$ is the velocity of propagation at a distance $r$ from the centre and $i$ the angle between the radius vector and the ray. The parameter $p$ is a constant for the ray, defined by the values of $r, v$ and $i$ at any point upon it. Given the relationship between $v$ and $r$, a ray of given $p$ can be traced a step at a time. The process is tedious in the general case. If, however, the relationship between $v$ and $r$ can be expressed in a suitable algebraic form, it may be possible to derive an equation for the ray which will eliminate the labour of the step-by-step computation.

A particular case is that in which $v$ and $r$ are connected by an equation of the form

$$
\begin{equation*}
v=a-b r^{2} \tag{2}
\end{equation*}
$$

Under these circumstances the ray paths are circles of radii

$$
\begin{equation*}
\rho=\frac{1}{2 p b} \tag{3}
\end{equation*}
$$

Equation (2) does not apply to the whole of the real earth. For any assumed velocity distribution however the earth may be divided into zones in each of which an equation of the given form will apply. The velocity values derived by Jeffreys (1939) from the Jeffreys-Bullen travel-time curves (1940), have been used in the present study. In Figure 2 those velocities are plotted as a function of $r^{2}$ and a series of straight lines have been fitted to the points. The intersections of these lines define the boundaries of zones


Figure 2
within which equations of the form (2) are assumed to hold. The equations appropriate to each zone are indicated on the diagram and are summarized in Table II. It should be stressed that the zone boundaries represent the intersections of lines chosen for convenience, and do not necessarily correspond to discontinuities in the earth. Figure 2 does not include any points relating to the crust. The Jeffreys-Bullen tables indicate two crustal layers,
but the total thickness of 33 km . is so thin in comparison with the other zones that a single layer with an average velocity of $6.2 \mathrm{~km} / \mathrm{sec}$. has been assumed. The crust has been designated in Table II as zone 0 .

TABLE II

| Zone <br> Number | Boundaries from |  | Equation |
| :---: | :---: | :---: | :---: |
|  | $\mathbf{r}=$ | $\mathrm{r}=$ |  |
| Zone 0 | 6371 | 6338 | $\mathrm{v}=6.20$ |
| 7one I | 6338 | 5966 | $v=18.02-2.561 \times 10^{-7} \mathrm{r}^{2}$ |
| Zone II | 5966 | 5873 | $v=33.68-6.961 \times 10^{-7} \mathrm{r}^{2}$ |
| Zone III | 5873 | 5735 | $v=26.21-4.795 \times 10^{-7} \mathrm{r}^{2}$ |
| Zone IV | 5735 | 5511 | $v=20.49-3.056 \times 10^{-7} \mathrm{r}^{2}$ |
| Zone V | 5511 |  | $\nabla=15 \cdot 40-1.380 \times 10^{-7} \mathrm{r}^{2}$ |

As an example we shall trace a ray having its vertex at $r=5100 \mathrm{~km}$., a point within zone V. Substituting into the appropriate equation of Table II one obtains $v=11.81$ $\mathrm{km} / \mathrm{sec}$. so that the ray parameter

$$
\begin{aligned}
p & =\frac{5100(\sin i=1)}{11.81} \\
& =431.84 \text { secs. }
\end{aligned}
$$

The ray parameter will define the ray throughout its entire course, so that the radius of the circle within each zone is obtained by inserting this value of $p$ and the appropriate value of $b$ in equation (3).

The following radii are obtained:
for zone V. . . . . . . . . . . . . . . . . . . . . . . . . . 8389 km.
for zone IV. . . . . . . . . . . . . . . . . . . . . . . . . 3789 km.
for zone III . . . . . . . . . . . . . . . . . . . . . . . . . . 2415 km.
for zone II. . . . . . . . . . . . . . . . . . . . . . . . . . . 1663 km.
for zone I. . . . . . . . . . . . . . . . . . . . . . . . . . . . $4521 ~ k m . ~$

The use of these radii will become clear upon examination of Figure 3, in which the surface of the earth and the boundaries of the several zones have been drawn. The point A represents the vertex of the ray at a point $r=5100 \mathrm{~km}$. Within zone V the ray is a circle of radius 8389 km . This circle may be drawn immediately with centre $\mathrm{C}_{1}$ to intersect the boundary between zones IV and $V$ at the points $\mathrm{B}, \mathrm{B}$.


Since the propagation velocity is the same on both sides of the zone boundary, there can be no discontinuity in the slope of the ray at the point B. The sections of the ray within zone IV and within zone $V$ must therefore have a common normal at B. Draw $\mathrm{BC}_{1}$ and cut off $\mathrm{BC}_{2}=3789$ to obtain the centre, $\mathrm{C}_{2}$, of the circular arc BD. Similarly $\mathrm{DC}_{3}=2415 \mathrm{~km}$. is measured off on $\mathrm{DC}_{2}, \mathrm{EC}_{4}=1663 \mathrm{~km}$. is measured off on $\mathrm{EC}_{3}$ and $\mathrm{FC}_{5}=4521 \mathrm{~km}$. is measured off on $\mathrm{FC}_{4}$ produced. Thus, combining a knowledge of the radii with the physical fact that adjacent sections of the ray must have a common normal at the boundary, the rays may be traced very rapidly.

When the points $G$, $G$, at the base of the crust are reached it is necessary to introduce a change of slope to allow for the change of velocity at the Mohorovicić discontinuity. By substituting $r=6338 \mathrm{~km}$., $v=6 \cdot 20 \mathrm{~km} / \mathrm{sec}$. and $p=431.84$ secs. into equation (1), we obtain for $\sin i_{0}$ the value 0.4224 , whence $i_{0}$ is approximately $25^{\circ}$. With this value the path of the ray through the crust may be drawn.

In Chart II, 50 rays have been drawn, with depths of penetration varying from 50 to 1275 km . at 25 km . intervals. For ease of reproduction the centre of the earth has not been shown. In use the chart should be mounted and the centre defined.

The use of the chart will be illustrated by an example. Suppose that one wished to draw the family of pP rays diverging from a focus at a depth of 450 km . On a sheet of transparent paper draw the zone boundaries to the same scale as in Chart II, so that the two can be superposed exactly. Hold them in this position by a drawing pin through the centres of the circles. In this way the tracing paper can revolve around the centre and the boundaries will still coincide.

Now indicate the focus by a dot at a depth of 450 km . This dot is at the same depth as the vertex of ray 17, so that when dot and vertex coincide curve 17 represents the path of the ray leaving the focus horizontally. It is the p section of the pP ray. Rotating the tracing paper so that the point of emergence coincides with the beginning point of ray 17 on the chart, the reflected section can be traced throughout its length. The first of the family of pP rays has thus been traced.

Next let the focus be placed on ray 18. It will be slightly above the vertex of this ray so that in one direction the ray is rising, in the other it is falling. Since we are tracing the paths of pP we are interested in the rising ray only. Tracing this to the surface again swing the tracing paper to give the reflected section of the ray. The process may be repeated for rays of all higher numbers until the characteristics of the family have been determined. The cusps and other interesting features are usually defined by tracing every third or fourth ray, but an initial trial is necessary to select the rays desired.

In producing a drawing it is expected that the rays will normally be copied directly from the chart by means of a french curve. For those who wish to work directly with a compass the radii of the arcs and the values of the angle $i_{0}$ are given in Table III. The terms used in Table III are defined in Figure 3.


TABLE III
RADII OF THE CHRCULAR ARCS
（See Figure $\$$ for definition of terms）

| Ray Number | $\begin{aligned} & \text { Deptin } \\ & \text { Vertex } \end{aligned}$ | i。 | $\underset{\mathrm{km}}{\mathrm{Rm}}$ ． | $\underset{\substack{\mathrm{k} \\ \mathrm{~km} \\ \hline \\ \hline \\ \hline}}{ }$ | $\mathrm{R}_{\mathbf{R}}^{\mathrm{km}}$ ． | $\underset{\mathrm{km}}{\mathrm{R}_{2}}$ | $\underset{\mathrm{km}}{\mathrm{R}}$ ． |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 50 | 52：32 | 2406 | － | － | － | － |
| 2 | 75 | $51: 29$ | 2440 | － |  |  |  |
| 3 | 100 | $50: 30$ | 2475 | － | － |  |  |
| 4 | 125 | 49：32 | 2510 | － | － |  |  |
| 5 | 150 | 48：37 | 2545 | － |  |  |  |
| 7 | 175 | 47： 44 | 2581 |  | 二 |  | － |
| 8 | 225 | $46: 53$ $46: 04$ | 2653 | － | 二 | 二 |  |
| 9 | 250 | 45：20 | 2686 | － | － | － |  |
| 10 | 275 | $44: 33$ | 2722 | 二 | － | 二 |  |
| 11 | 300 | 43：48 | 2759 | － | － |  |  |
| 12 | 325 | 43 ： 04 | 2796 | － | － |  |  |
| 13 | 350 | 42：22 | 2834 | － | － |  | － |
| 14 | 375 | 41 ： 44 | 2869 | － |  |  |  |
| 15 | 400 | 41 ： 04 | 2907 | － | － |  |  |
| 16 | 425 450 | $39: 53$ $38: 37$ | 2978 | 1096 | － |  |  |
| 18 | 475 | 37：28 | 3139 | 1155 | 二 | － | － |
| 19 | 500 | $36: 21$ | 3222 | 1186 | － |  |  |
| 20 | 525 | $35: 37$ | 3280 | 1207 | 1751 | － |  |
| 21 | 550 | 34：52 | 3341 | 1229 | 1784 | － |  |
| 22 | 575 600 | 34：09 | 3402 | 1252 | 1817 | － |  |
| 24 | 625 | $33: 27$ 32 | 3464 3526 | 1275 | 1850 | 二 |  |
| 25 | 650 | 32： 14 | 3580 | 1317 | 1912 | 3000 |  |
| 26 | 675 | 31：47 | 3626 | 1334 | 1937 | 3039 |  |
| 27 | 700 | 31 ： 21 | 3670 | 1350 | 1960 | 3079 |  |
| 28 | 725 | 30：55 | 3717 | 1368 | 1985 | 3115 | － |
| 29 | 750 | 30：31 | 3762 | 1384 | 2009 | 3152 | － |
| 30 | 775 | 30：05 | 3810 | 1402 | 2035 | 3193 | － |
| 31 | 800 | 29：42 | 3855 | 1418 | 2059 | 3231 |  |
| 32 33 | 825 | 29：17 | 3904 | 1436 | 2085 | 3272 | － |
| 33 34 | 885 | 28：53 | 3954 3989 | 1455 | 2112 | 3313 3343 |  |
| 35 | 900 | 28：21 | 4022 | 1480 | 2148 | 3343 3370 | 7463 |
| 36 | 925 | 28：06 | 4055 | 1492 | 2166 | 3398 | 7524 |
| 37 | 950 | 27：53 | 4084 | 1503 | 2181 | 3423 | 7579 |
| 38 | 975 | $27: 38$ | 4117 | 1515 | 2199 | 3451 | 7641 |
| 39 | 1000 | 27：24 | 4151 | 1527 | 2217 | 3479 | 7704 |
| 40 | 1025 | $27: 09$ | 4185 | 1540 | 2235 | 3507 | 7766 |
| 41 | 1050 | 26：56 | 4216 | 1551 | 2252 | 3533 | 7824 |
| 42 | 1075 | 26：42 | 4251 | 1564 | 2270 | 3562 | 7888 |
| 44 | 1125 | 26：15 | 4317 | 1588 | 2289 | ${ }_{3618}$ | 8952 |
| 45 | 1150 | 26：01 | 4353 | 1601 | 2325 | 3648 | 8078 |
| 46 | 1175 | $25: 49$ | 4385 | 1613 | 2342 | 3675 | 8137 |
| 47 | 1200 | $25: 35$ | 4421 | 1627 | 2361 | 3705 | 8204 |
| 48 | 1225 | 25：22 | 4460 | 1640 | 2381 | 3736 | 8272 |
| 49 50 | 1250 | 25： 20 | 4491 | 1652 | 2399 | 3764 | 8334 |
| 50 | 1275 | 24：57 | 4529 | 1666 | 2419 | 3795 | 8403 |

## Applications to some Problems

Figures 4 to 9 show some ray diagrams constructed with the aid of a chart similar to Chart II. They were actually drawn from an earlier model in which the rays were drawn with their vertex at even values of radius $r$ rather than of even values of depth. The rays shown in these figures are therefore not identical with any of those shown in Chart II.

In Figure 4 the rays giving rise to the " $20^{\circ}$ cusp" on the P curve are traced. It is shown that the points of reversal are at something less than $22^{\circ}$ and at about $19^{\circ}$, a little short of the $18^{\circ} \cdot 5$ given in the published tables. This is to be expected, since it would be a matter of accident if the ray corresponding to the exact end of the cusp were one of those available in Chart II. Figure 5 illustrates the cusp arrangement for pP rays originating at focal depth of 0.02 R . It will be noted that there are two cusps, an initial one between $20^{\circ}$ and $19^{\circ}$ due to ray geometry, and a second analagous to the " $20^{\circ}$ cusp". In Figures 6 to 9 the interaction of these two cusps with increasing focal depth has been illustrated. These figures will supplement the theoretical discussion given by Bullen (1955).

## CONCLUSION

To be of value the charts described in this paper should be mounted with a minimum of distortion. An extra supply of the charts has been printed, and copies will be mailed unfolded to anyone wishing them. Requests should be addressed to the authors.

## References

## Bullen, K. E.

1955 "Features of Seismic pP and PP Rays", M.N.R.A.S., Geoph. Suppl., in press.
Hodason, J. H. and Aluinn, J. F. J.
1954 "Tables of Extended Distances for PP and pP", Publications of the Dominion Observatory, 16, 351-362
Jefrymes, H.
1939 "Times of P, S and SKS, and Velocities of P and S", M.N.R.A.S., Geoph. Suppl., 4, 498-533.
Jefrpeys, H., and Bullen, K. E.
1940 "Seismological Tables", British Association for the Advancement of Science.
Tsuroi, C.
1951 "A Simple Instrument Useful for Finding the Angular Distance of a Point ( $\varphi, \lambda$ ) from a Fixed Point ( $\varphi_{0}, \lambda_{o}$ ) on the Earth's Surface", Geophysical Notes, Tokyo University, 4, No. 4.


