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ANALYSIS OF ACCURACY IN THE DETERMINATION OF THE GROUND-STRESS TENSOR BY MEANS OF BOREHOLE DEVICES

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MINING RESEARCH CENTRE

Reprinted from Proceedings of the Ninth Symposium on Rock Mechanics held at the Colorado School of Mines, Golden, Colorado, 17-19 April, 1967. Chapter 3 pp 45-78

Reprint Series RS 109 Price 25 cents

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Ottawa, 1972

Chapter 3

ANALYSIS OF ACCURACY IN THE DETERMINATION OF THE GROUND-STRESS TENSOR BY MEANS OF BOREHOLE DEVICES

by W. M. Gray and N. A. Toews

The determination of the state of stress existing at a point in any solid body requires the determination of the six components of the stress tensor relative to a convenient set of axes. One general method of doing this in rock is to make strain or deformation measurements associated with local relief of stress in a number of boreholes penetrating a limited volume in the vicinity of the point at which the stress is to be determined. The stress is assumed to be constant throughout this volume.

Leeman [1] has described a number of the instruments that have been developed for making deformation or strain measurements in boreholes. Panek [2] has described the application of statistical methods to the determination of the average components of ground stress from sets of measurements made in boreholes by means of deformation meters of the type developed by the U.S. Bureau of Mines.

In this paper attention is given initially to the strain cell developed for use on the flattened ends of boreholes by the Council for Scientific and Industrial Research (CSIR), South Africa. The equations required for evaluating the average components of ground stress from measurements made by means of the CSIR strain cell are derived. These equations are similar in form to those applying to measurements made by the borehole deformation meter developed by the U.S. Bureau of Mines.

The results of an investigation of the properties of the equations are described. These results are important in the practical use of both the instruments mentioned above and any instrument based on similar principles.

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DERIVATION OF EQUATIONS FOR THE CSIR STRAIN CELL

For ease of reference, borehole directions and strain-gage orientation will be specified according to the convention adopted by Panek.[2] Thus we assume a basic rectangular system of coordinates g_1, g_2, g_3 equivalent to Panek's x, y, z, (Fig. 1). We also assume a second rectangular system h_1, h_2, h_3 where h_2 is in the direction of a particular borehole and h_1 lies in the g_1 g_2 plane. Quantities referred to the h-system do not appear directly in the final mathematical equations, but this system is important for the practical definition of borehole direction and gage orientations.

The derivation of the relationship between the strain measured by a particular strain gage and the stress components referred to the basic g-system of coordinates is simplified if a third coordinate system, m_1, m_2, m_3, is introduced. m_1 is in the direction of the strain gage and thus makes the angle θ with h_1. m_2 is identical with h_2.

Let M_{ij} be the components of the ground-stress tensor referred to the m-system, i.e.,

$$M_{11}=\sigma_{m_1} \qquad M_{12}=\tau_{m_1m_2}$$
$$M_{22}=\sigma_{m_2} \qquad M_{23}=\tau_{m_2m_3}$$
$$M_{33}=\sigma_{m_3} \qquad M_{31}=\tau_{m_3m_1}$$

The stress components M'_{ij} at the center point on the end surface of the borehole must be linear combinations of the M_{ij}. M'_{22}, the normal

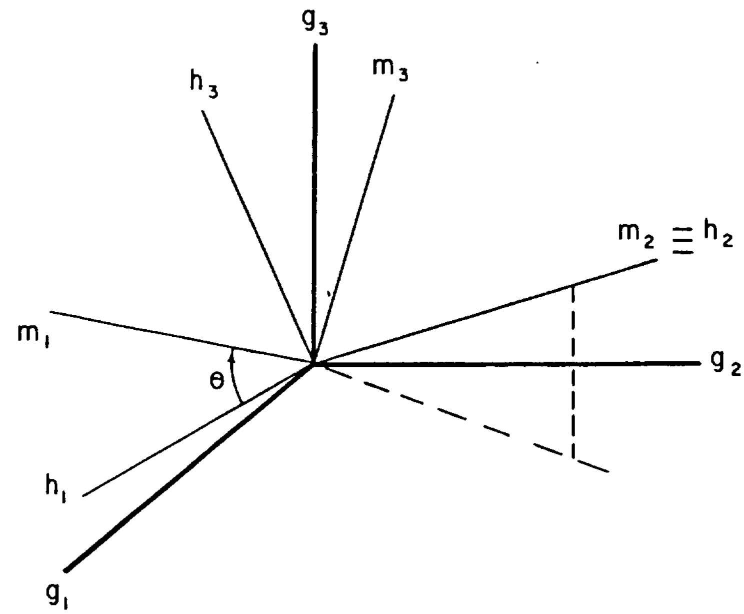

Fig. 1—Coordinate systems.

component at the surface, and M'_{12} and M'_{23} the shear components acting tangentially to the surface must all be zero. It can be shown (Appendix A), that M_{12}, M_{23}, M_{31} cannot contribute to M'_{11} and M'_{33}. Thus

and

$$\left.\begin{aligned} M'_{11} &= aM_{11} + bM_{33} + cM_{22} \\ M'_{33} &= bM_{11} + aM_{33} + cM_{22} \end{aligned}\right\} \quad [1]$$

Leeman[1] reports laboratory work showing that approximately

$$a = 1.5$$
$$b = 0.0$$

No laboratory determination of c was known to the authors when this work was done.

The point at which the strain gage is applied to the end surface of the borehole is assumed to be so close to the center that the stress conditions are the same as at the center.* Thus the strain at the point of measurement is

$$e_{11} = \frac{1}{E}[M'_{11} - \nu M'_{33}] \quad [2]$$

where E and ν are Young's modulus and Poisson's ratio for the rock, respectively. Since the m_1-axis is the direction of the strain gage, e_{11} is the strain that is measured when the stress is relieved by overcoring.

Substituting from Eqs. 1 we get

$$\left.\begin{aligned} e_{11} &= \frac{1}{E}[(aM_{11} + bM_{33} + cM_{22}) - \nu(bM_{11} + aM_{33} + cM_{22})] \\ &= \frac{1}{E}[(a - b\nu)M_{11} + (b - a\nu)M_{33} + c(1-\nu)M_{22}], \end{aligned}\right\} \quad [3]$$

or

$$e_{11} = AM_{11} + BM_{33} + CM_{22} \quad [4]$$

where

$$A = (a - b\nu)/E$$
$$B = (b - a\nu)/E$$

and

$$C = c(1-\nu)/E.$$

In the basic g-system of coordinates we shall let G_{ij} represent the components of the ground-stress tensor. These are the components that are to be determined. If the direction cosines of m_1, m_2 and m_3 in the

*It is important that the requirements of this assumption should be fulfilled; otherwise an accurate treatment would require the introduction of additional factors.

g-system are l_{1i}, l_{2i} and l_{3i}, respectively, the M_{ij} are related to the G_{ij} by the equations

$$M_{ij} = \sum_{r=1}^{3} \sum_{t=1}^{3} l_{ir}\, l_{jt}\, G_{rt}, \quad i, j = 1, 2, 3 \qquad [5]$$

For example,

$$M_{11} = l_{11}^2 G_{11} + l_{12}^2 G_{22} + l_{13}^2 G_{33} + 2l_{11}\, l_{12}\, G_{12} + 2l_{12}\, l_{13}\, G_{23} + 2l_{13}\, l_{11}\, G_{31}.$$

Substituting from Eq 5 in Eq 4 and dropping the subscripts from e_{11}, we obtain,

$$e = \sum_{r=1}^{3} \sum_{t=1}^{3} [A l_{1r}\, l_{1t}\, G_{rt} + B l_{3r}\, l_{3t}\, G_{rt} + C l_{2r}\, l_{2t}\, G_{rt}]$$

or

$$e = \sum_{r=1}^{3} \sum_{t=1}^{3} [A l_{1r}\, l_{1t} + B l_{3r}\, l_{3t} + C l_{2r}\, l_{2t}] G_{rt} \qquad [6]$$

An equation of this type is associated with each measurement that is made. The equations associated with a set of measurements may thus be written,

$$e_i = \sum_{r=1}^{3} \sum_{t=1}^{3} [A l_{1r}\, l_{1t} + B l_{3r}\, l_{3t} + C l_{2r}\, l_{2t}]_i\, G_{rt}, \; (i = 1, \ldots M) \qquad [7]$$

where M is the number of measurements.

PROPERTIES OF BASIC EQUATIONS

Eqs. 7 have been derived with reference to the CSIR Strain Cell. The equations are similar in form, however, to those that are applicable when the USBM Deformation Meter is used. The following discussion therefore applies in general to the use of either type of instrument and the measured values e_i may represent either strains or deformations.

Number of Independent Equations

Eqs. 7 express the measured values e_i as linear combinations of the stress components G_{ij} in the basic coordinate system. The coefficients of the G_{ij} depend only on instrumental constants, on the elastic constants of the medium, and on the direction cosines of the measurement directions (m_1) and of the borehole directions (m_2). (The direction cosines of the m_3 directions can be calculated from those of the m_1 and m_2 directions.)

A solution of Eqs. 7 for the six stress components G_{ij} is only possible if there are a minimum of six independent equations. Equations resulting from a number of measurements using the same borehole direction and gage direction are not independent. They merely improve the results through the averaging of observations. If different measurement directions are used in a single borehole only three independent equations can result. If more than three measurement directions are used, the equations are mutually dependent, but again they may serve to improve the results through averaging.

Since each borehole can furnish three independent equations it would seem to be possible to determine the six stress components by the use of two boreholes. It turns out, however, that the structure of the equations is such that there is one relation between the six equations for any pair of boreholes. Consequently only five independent equations can be derived by the use of two boreholes. The foregoing assertion can be proved in general terms, as shown in Appendix B, but it is enlightening to consider a particular example, from which a clear appreciation of the existing relationships can be gained.

Let us choose boreholes in the directions of the g_1 and g_2 axes and choose three measurement directions in each, as indicated below. The corresponding sets of direction cosines and the equations derived by using them in Eqs. 7 are as follows:

Borehole (1): m_2 is parallel to g_1

$$l_{21} = \quad 1, \quad 0, \quad 0$$

Measurement (1): m_1 is parallel to g_2

$$l_{11} = \quad 0, \quad 1, \quad 0$$
$$l_{21} = \quad 1, \quad 0, \quad 0$$
$$l_{31} = \quad 0, \quad 0, \quad -1$$

$$e_1 = CG_{11} + AG_{22} + BG_{33}$$

Measurement (2): m_1 is inclined 45° to g_2 direction

$$l_{11} = \quad 0, \quad 1/\sqrt{2}, \quad 1/\sqrt{2}$$
$$l_{21} = \quad 1, \quad 0, \quad 0$$
$$l_{31} = \quad 0, \quad 1/\sqrt{2}, \quad -1/\sqrt{2}$$

$$e_2 = CG_{11} + \tfrac{1}{2}(A+B)G_{22} + \tfrac{1}{2}(A+B)G_{33} + (A-B)G_{23}$$

Measurement (3): m_1 is parallel to g_3

$$l_{11} = \quad 0, \quad 0, \quad 1$$
$$l_{21} = \quad 1, \quad 0, \quad 0$$
$$l_{31} = \quad 0, \quad 1, \quad 0$$

$$e_3 = CG_{11} + BG_{22} + AG_{33}$$

Borehole (2): m_2 is parallel to g_2

$$l_{21} = 0, \quad 1, \quad 0$$

Measurement (4): m_1 is parallel to g_1

$$l_{11} = 1, \quad 0, \quad 0$$
$$l_{21} = 0, \quad 1, \quad 0$$
$$l_{31} = 0, \quad 0, \quad 1$$

$$e_4 = AG_{11} + CG_{22} + BG_{33}$$

Measurement (5): m_1 is inclined 45° to g_1 direction

$$l_{11} = 1/\sqrt{2}, \quad 0, \quad 1/\sqrt{2}$$
$$l_{21} = 0, \quad 1, \quad 0$$
$$l_{31} = -1/\sqrt{2}, \quad 0, \quad 1/\sqrt{2}$$

$$e_5 = \tfrac{1}{2}(A+B)G_{11} + CG_{22} + \tfrac{1}{2}(A+B)G_{33} + (A-B)G_{31}$$

Measurement (6): m_1 is parallel to g_3

$$l_{11} = 0, \quad 0, \quad 1$$
$$l_{21} = 0, \quad 1, \quad 0$$
$$l_{31} = -1, \quad 0, \quad 0$$

$$e_6 = BG_{11} + CG_{22} + AG_{33}$$

Collecting equations, we have,

$$\begin{aligned}
C\,G_{11} + A\,G_{22} + B\,G_{33} &= e_1 \\
C\,G_{11} + \tfrac{1}{2}(A+B)G_{22} + \tfrac{1}{2}(A+B)G_{33} + (A-B)G_{23} &= e_2 \\
C\,G_{11} + B\,G_{22} + A\,G_{33} &= e_3 \\
A\,G_{11} + C\,G_{22} + B\,G_{33} &= e_4 \\
\tfrac{1}{2}(A+B)G_{11} + C\,G_{22} + \tfrac{1}{2}(A+B)G_{33} + (A-B)G_{31} &= e_5 \\
B\,G_{11} + C\,G_{22} + A\,G_{33} &= e_6
\end{aligned}$$

These are six equations in five of the stress components. Considered as six equations in all six components they have a zero determinant since the coefficient of G_{12} is zero throughout. A zero determinant indicates that a solution cannot be obtained for all six components, a fact that is obvious in this particular case since G_{12} does not appear in the equations.

The determinant is zero, however, even when the angle between the boreholes is not a right angle, as shown in Appendix B, and again a solution cannot be found. The fundamental reason for this in both cases is that all the measurement directions project into only two directions in the plane of the boreholes (the g_1g_2 plane). The complete state of stress in a plane cannot be determined by measurements in two directions. At least one more measurement direction is needed and this can be obtained only by the use of a third borehole when instruments of the types under discussion are used.

Relationship between Parameters A, B, C

Returning to the equations for e_1 to e_6, let us consider solving for the five stress components. The shear components G_{23} and G_{31} appear only in the second and the fifth equations, respectively, and can therefore be determined if the remaining equations can be solved. The remaining four equations contain only the three normal components G_{11}, G_{22} and G_{33}. We may thus solve any three of these equations for G_{11}, G_{22} and G_{33} (since the respective determinants differ from zero). If we substitute the resulting expressions in the fourth equation we eliminate the stress components from it and obtain a compatibility relation connecting the parameters A, B, C and the measured values. The result of this elimination is

$$\frac{A-C}{B-C}=\frac{e_1-e_4}{e_3-e_6}, \qquad [8]$$

as can easily be verified.

The evaluation of the parameters A, B, C rests on the results of laboratory experiments (including the determination of the elastic constants of the rock) and of theory combined with a knowledge of the direction cosines of the boreholes and the measurement directions. Thus a value for the left side of Eq. 8 is known independently of measurement of the e's and the equation provides a means of checking that A, B, and C are consistent with the measured values.

Eq. 8 has been established with reference to boreholes in the directions g_1 and g_2. Similar equations can be derived using pairs of boreholes in the directions g_2 and g_3, or g_3 and g_1. In each of these pairs of boreholes only five independent measurements can be made.

In the three boreholes in the g_1, g_2 and g_3 directions, taken together, only six independent measurements can be made. If seven, eight or nine measurements are made, involving a maximum of three different independent directions in each borehole, a relation equivalent to Eq. 8 can be determined in addition to the stress components. When nine measurements are made there are two additional compatibility relations connecting the measurements, but these are independent of the parameters. Since they are also independent of the stress components they afford a means of checking the measurements for consistency.

In practice it may not be convenient to use an equation like Eq. 8 directly to establish the relationship between A, B and C. When the number of measurements is large and the stress components are determined by the method of least squares the relation expressed by Eq. 8 can be found in an indirect way that will be described below.

SOLUTIONS OF THE BASIC EQUATIONS BY THE METHOD OF LEAST SQUARES

Panek [2] has described the application of the method of least squares to the solution of a set of equations such as Eq. 7 for the components of the stress tensor. The basic relations of the method will be summarized here for completeness, with slight changes in notation. A new treatment of the solution will be given and its results will be discussed.

Eq. 7 is first rewritten in the following form which includes a residual error term,

$$e_i = \sum_{j=1}^{6} K_{ij}\, b_j + r_i \quad (i=1, \ldots M) \qquad [9]$$

where

$$\begin{aligned}
K_{i1} &= [Al_{11}^2 + Bl_{31}^2 + Cl_{21}^2]_i, & b_1 &= G_{11}\\
K_{i2} &= [Al_{12}^2 + Bl_{32}^2 + Cl_{22}^2]_i, & b_2 &= G_{22}\\
K_{i3} &= [Al_{13}^2 + Bl_{33}^2 + Cl_{23}^2]_i, & b_3 &= G_{33}\\
K_{i4} &= 2[Al_{11}l_{12} + Bl_{31}l_{32} + Cl_{21}l_{22}]_i, & b_4 &= G_{12}\\
K_{i5} &= 2[Al_{12}l_{13} + Bl_{32}l_{33} + Cl_{22}l_{23}]_i, & b_5 &= G_{23}\\
K_{i6} &= 2[Al_{13}l_{11} + Bl_{33}l_{31} + Cl_{23}l_{21}]_i, & b_6 &= G_{31}
\end{aligned}$$

The least squares solution of this equation consists in the set of b_i that minimizes the sum of the squares of the residual errors,

$$Q = \sum_{i=1}^{M} (r_i)^2 \qquad [10]$$

The normal equations for the least squares solution may be written

$$\sum_{j=1}^{6} a_{tj}\, b_j = g_t \quad (t=1, \ldots M) \qquad [11]$$

where

$$a_{tj} = \sum_{i=1}^{M} K_{it}\, K_{ij}$$

$$g_t = \sum_{i=1}^{M} K_{it}\, e_i$$

The solution for the b_j is

$$b_j = \sum_{k=1}^{6} c_{jk} g_k \quad (j=1, \ldots 6) \qquad [12]$$

where c_{jk} are the components of the matrix [C],* the inverse of the matrix [A] * whose components are a_{tj}.

Q′, the minimum value of Q that is associated with the solution, is called the sum of squares about regression. Division by (M−6), the "degrees of freedom" of Q′, gives the variance about regression

$$V = Q'/(M-6). \qquad [13]$$

If the coefficients K_{ij} in Eq. 9 are accurately known, the value of V is determined solely by the errors involved in making the measurements e_i and the standard error of measurement is estimated by the standard deviation about regression, s, given by

$$s = V^{\frac{1}{2}}. \qquad [14]$$

In practice the K_{ij}, as applying to the actual conditions under which a given set of measurements is made, may not be accurately known. One reason for such inaccuracy is that the values of these coefficients depend on the elastic constants of the rock in which the measurements are made. Unless accurate values of the average elastic constants applying to the rock *in situ* can be established, the functional relationship that is assumed to relate the strain or deformation measurements to the stress components will be incorrect.

A more complex reason for inaccuracy is that the rock may not be isotropic as assumed in developing the basic equations. The consequences of this possibility will not be discussed in detail and in what follows it will be assumed that the form of the basic equations is correct but that there may be inaccuracies in the coefficients.

Effect of Inaccuracies in the K_{ij}

If the functional relationship that is assumed to relate the measured values to the stress components is incorrect the minimum sum of squares, Q′, will tend to be larger than it would be if due to error in the e_i only. The variance about regression V, given by Eq. 13, is then an incorrect estimate of the basic error variance and an independent estimate is required. It is possible to make such an estimate if replicate measurements having the same borehole directions and the same measurement directions are made. The variability among such measurements is due

* Brackets are used to distinguish these matrices from the previously defined C and A.

solely to experimental error (including any error due to the variability of the rock properties from point to point).

Suppose that one such set of measurements is represented by $e_{i1}\ e_{i2}$ - - - e_{in}, where n is the number in the set. (n need not be the same for all sets and there may be a few unreplicated measurements.) From Eq. 9 the corresponding residual errors are given by

$$r_{it}=e_{it}-E_i \quad (t=1, \ldots n), \tag{15}$$

where $E_i=\sum_{j=1}^{6} K_{ij}\, b_j$ is constant because all the measurements are associated with the same borehole and measurement directions. Introducing $\bar{e}_i$, the mean value of e_{i1} - - - e_{in} defined by

$$\sum_{t=1}^{n}(e_{it}-\bar{e}_i)=0, \tag{16}$$

and summing the squares of the residual errors we have,

$$\begin{aligned} q_i &= \sum_{t=1}^{n}(r_{it})^2 \\ &= \sum_{t=1}^{n}(e_{it}-E_i)^2 \\ &= \sum_{t=1}^{n}[(e_{it}-\bar{e}_i)+(\bar{e}_i-E_i)]^2 \\ &= \sum_{t=1}^{n}(e_{it}-\bar{e}_i)^2+2\sum_{t=1}^{n}(e_{it}-\bar{e}_i)(\bar{e}_i-E_i)+\sum_{t=1}^{n}(\bar{e}_i-E_i)^2 \\ &= \sum_{t=1}^{n}(e_{it}-\bar{e}_i)^2+n(\bar{e}_i-E_i)^2. \end{aligned} \tag{17}$$

Summing q_i over all measurements (grouped in m sets) we obtain

$$\begin{aligned} Q' &= \sum_{i=1}^{m} q_i \\ &= \sum_{i=1}^{m}\ \sum_{t=1}^{n}(e_{it}-\bar{e}_i)^2+\sum_{i=1}^{m} n(\bar{e}_i-E_i)^2 \end{aligned} \tag{18}$$

The first component of Q′ is the sum of the squares of the deviations of the individual measurements from the means of the replicate sets to which they belong and may be called "the sum of squares within sets." The second component of Q′ is the sum of the squares of the deviations of the set means from the regression value, each square being weighted according to the number of measurements in the set. The second component may be called "the sum of the squares of means about regression."

Eq. 18 is a type well known in the analysis of variance.[3] The first component of Q′ has (n−1) degrees of freedom for each set of n replicated measurements. The total of freedom for all sets is (M−m).

The degrees of freedom of Q′ are (M−6). Thus the degrees of freedom of the second component of Q′ are (M−6)−(M−m)=(m−6).

The variance within sets is then given by

$$V_w = \sum_{i=1}^{m} \sum_{t=1}^{n} (e_{it} - \bar{e}_i)^2/(M-m) \qquad [19]$$

and the variance of means about regression is given by

$$V_m = \sum_{i=1}^{m} n(\bar{e}_i - E_i)^2/(m-6). \qquad [20]$$

Unreplicated measurements (n=1) contribute only to V_m since the corresponding terms in V_w are zero.

V_w is the basic estimate of the variance due to experimental error. If V_m can be shown by a variance ratio test[3] not to differ significantly from V_w it can be concluded that V_m is also caused by experimental error and V, from Eq. 13, can be used as the estimate of the variance due to experimental error. Then s, from Eq. 14, is the estimate of the standard error of measurement.

If on the other hand V_m is significantly greater than V_w the standard error of measurement must be estimated by

$$s = V_w^{\frac{1}{2}}. \qquad [21]$$

It must follow also that the mathematical formulation is incorrect in some respect. For this reason the residual sum of squares Q′ is greater than it would be if the correct formulation were known and used.

Q′ is associated with the given values of A, B and C. If the least squares solution is carried out again with a different value of one of the parameters, say C, keeping A and B unaltered, a different value of Q′ will in general be found. By solving repeatedly with different values of C a minimum value of Q″ may be found. Let the corresponding value of C be C″.

The quantity $\frac{A-C''}{B-C''}$ corresponds to the right side of Eq. 8, i.e., we would find

$$\frac{A-C''}{B-C''} = \frac{e_1 - e_4}{e_3 - e_6}$$

if the measurements e_1, e_3, e_4 and e_6 were made (neglecting experimental error and provided always that the rock is isotropic and ideally elastic).

Q' can be minimized by varying C under the same conditions under which Eq. 8 can be established. This means that, in general, to minimize Q' it is necessary to use at least seven different measurement directions, properly distributed in at least three boreholes.

Q' can also be minimized by varying A or B. But no absolute minimum occurs for a unique set of A, B and C values. Suppose that X is the value of $\frac{A-C}{B-C}$ for some set that minimizes Q'. For example, let $\frac{A-C''}{B-C''}=X$. It was discovered empirically that any set of values, A', B' and C' that satisfies the equation

$$\frac{A'-C'}{B'-C'}=X \qquad [22]$$

gives approximately the same residual sum of squares Q''.

Eq. 22 can be written

$$(X)B'/A'+(1-X)C'/A'=1. \qquad [23]$$

This is a linear relation between B'/A' and C'/A'. It is provided by the field data and may be used in conjunction with other data to establish the values of A, B and C that are most consistent with all the available information.

It should be noted that if two orthogonal boreholes are used, six different measurement directions can in general be used to determine five stress components and also minimize Q'. If, however, the boreholes are in the g_1 and g_2 directions (for example) and if G_{11} happens to equal G_{22}, it will be impossible to minimize Q' by varying C. It is shown in Appendix B that $\frac{A-C}{A-B}$ is indeterminate if G_{11} equals G_{22}. If three orthogonal boreholes (in the g_1, g_2, g_3 directions) are used, a minimum can be found for Q' even if G_{11} equals G_{22}, so long as G_{33} does not equal G_{22} also.

It can be inferred from the results of Appendix B that Q' can be minimized and $\frac{A-C''}{B-C''}$ can be found even if the boreholes are not orthogonal so long as certain special relationships (of relatively rare occurrence) do not exist between G_{11}, G_{22} and G_{33}. Computations carried out on field data have shown that the inference is justified.

Orientation of Boreholes and Measurement Directions

When the standard error of measurement, s, is known, the standard errors of the stress components are given [2] by,

$$\left.\begin{array}{l} s_{b_1} = s(c_{11})^{\frac{1}{2}} \\ s_{b_2} = s(c_{22})^{\frac{1}{2}} \\ \vdots \\ s_{b_6} = s(c_{66})^{\frac{1}{2}} \end{array}\right\} \quad [24]$$

where the c_{jk} are the components of the matrix [C] defined on page 53. The b_i represent the stress components as indicated on page 52.

The components c_{jk} depend only on the parameters A, B and C, on the number of measurement directions and on the geometrical relationships between the borehole and measurement directions and the coordinate axes. It is therefore possible to compute the factors multiplying s in Eq. 24 for any proposed measurement system, quite independently of the measurements. This has been done by means of an electronic computer for a number of hypothetical systems and the results are presented here as a general guide to the choice of a system to suit particular circumstances.

The various systems are illustrated in Fig. 2 and the corresponding values of the standard-error factors are tabulated in Table I. The particular values of A, B and C used in the calculations are shown also in Table I. They are typical values for the CSIR Strain Cell (when the measured strains are expressed in units of 10^{-6} and E is of order 10×10^6 psi).

System (a) was designed to be highly symmetrical. It uses three orthogonal boreholes oriented in the direction of the coordinate axes (g_1, g_2, g_3) which are disposed as in Fig. 1. There are four measurement directions in each borehole, two parallel to the other coordinate axes and two bisecting the angles between the axes (oblique directions). It is seen that G_{11}, G_{22} and G_{33} are determined with equal precision since the corresponding factors are equal. The same applies to G_{12}, G_{23}, and G_{31}. The difference between the factors relating to the normal and shear stresses reflects the fact that the two types of stresses enter differently into the equations. The actual difference is dependent on A, B and C (as shown later in Table II).

System (b) differs from (a) in that one oblique measurement direction is dropped from each borehole. The resulting factors indicate that the shear components are now determined less precisely than the normal components. In general, the precision with which a component, G_{ij}, is determined is closely linked to the number of measurements made in the direction bisecting the angle (g_i, g_j) and in the orthogonal direction.

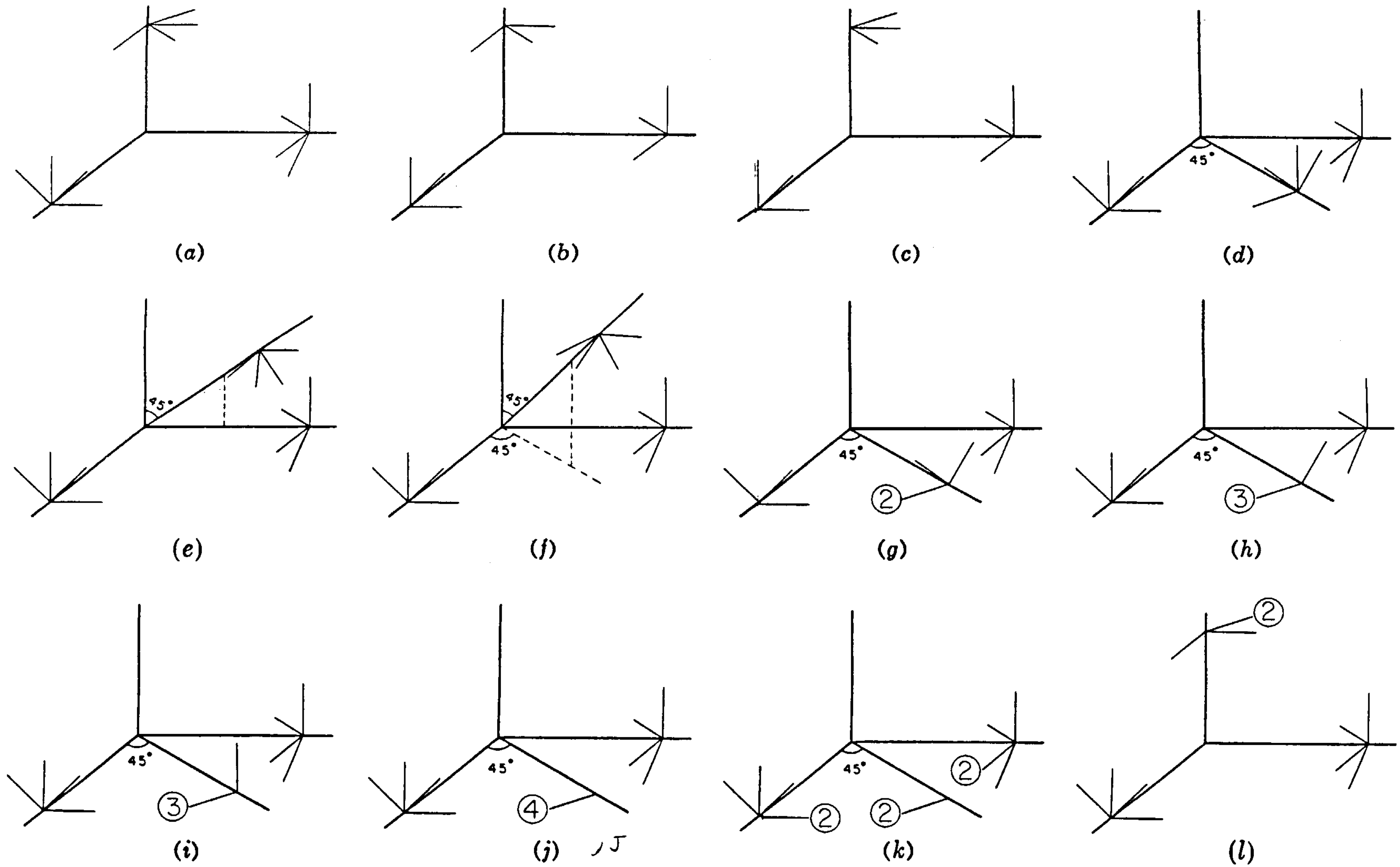

Fig. 2—Systems of boreholes and measurement directions.

Table I. Standard-Error Factors for Systems of Fig. 2

Stress Component	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)	(l)
G_{11} (b_1)	4.61	5.77	6.64	6.60	4.69	5.90	6.61	6.61	6.60	6.61	5.07	5.10
G_{22} (b_2)	4.61	5.77	5.12	6.60	6.16	5.90	6.61	6.61	6.60	6.61	5.07	5.10
G_{33} (b_3)	4.61	5.77	5.74	3.85	4.44	4.45	4.33	4.50	4.01	4.61	4.61	4.61
G_{12} (b_4)	4.49	7.07	4.49	6.77	7.78	6.43	5.80	5.51	5.59	5.27	5.64	5.31
G_{23} (b_5)	4.49	7.07	7.01	3.89	3.68	3.82	3.89	4.14	4.49	4.49	4.49	4.49
G_{31} (b^6)	4.49	7.07	7.21	3.89	4.49	3.82	3.89	4.14	4.49	4.49	4.49	4.49

Notes: (1) Calculated for: $A = 0.13346$, $B = -0.02389$, $C = -0.01647$.
(2) When A, B and C are all multiplied by the same constant, f, the standard-error factors are divided by f.

The results for System (c) show the effects of adding to System (b) one oblique measurement in the direction bisecting the angle $(g_2, -g_1)$, while dropping one measurement in the g_1 direction. The main effects are an increase in the factor for the normal stress component G_{11} (indicating decreased precision) and a decrease in the factor for the shear component G_{12}. The factors corresponding to other stress components are also affected, but to a lesser degree.

System (d) consists of three boreholes in the g_1 g_2 plane, each with four symmetrically distributed measurement directions. G_{33}, G_{23} and G_{31} are now determined more precisely than by System (a) but G_{11}, G_{22} and G_{12} are determined less precisely.

Systems (e) and (f) each include one oblique borehole. All boreholes include four symmetrically disposed measurement directions. The factors tend to lie between those for Systems (a) and (d) but G_{12} is rather poorly determined by System (e) because the direction of the oblique borehole approaches g_2. G_{12} is the component that cannot be determined at all when two boreholes in the g_1 and g_2 directions only are used.

Systems (g), (h), (i), (j) and (k) are modifications of (d) intended to have factors closer to those of System (a). The total number of measurement directions is twelve in all cases, but certain directions used in System (d) are omitted while others are replicated as indicated by the numbers on the diagrams. The best arrangement is that of System (k) but the factors are not as uniform as in System (a).

System (l) is similar to System (a) except that the measurement, in the g_3 borehole, that bisects the angle (g_1, g_2) is omitted and the measurement at right angles to it is duplicated. It might well be expected that the arrangement of System (l) should be equivalent to that of System (a) since only three measurement directions in each borehole can be independent. The factors given in Table I, however, reveal the curious result

Table II. Standard-Error Factors: Dependence on Parameters B and C with A = 0.1

(1) Borehole System (a) (Fig. 2)

B	0.00	—0.01	—0.02	—0.03	0.00	—0.01	—0.02	—0.03	0.00	—0.01	—0.02	—0.03
C	0.00	0.00	0.00	0.00	—0.01	—0.01	—0.01	—0.01	—0.02	—0.02	—0.02	—0.02
G_{11}, G_{22}, G_{33}	5.92	5.97	6.07	6.25	5.74	5.92	6.17	6.56	5.68	6.00	6.46	7.15
G_{12}, G_{23}, G_{31}	7.07	6.43	5.89	5.44	7.07	6.43	5.89	5.44	7.07	6.43	5.89	5.44

(2) Borehole System (k) (Fig. 2)

G_{11}, G_{22}	6.44	6.44	6.55	6.78	6.51	6.54	6.74	7.14	6.80	6.91	7.24	7.90
G_{33}	5.98	5.93	6.05	6.35	5.86	5.93	6.19	6.69	5.79	6.00	6.47	7.30
G_{12}	8.45	8.45	8.45	8.45	7.68	7.68	7.68	7.68	7.04	7.04	7.04	7.04
G_{23}, G_{31}	7.07	6.43	5.89	5.44	7.07	6.43	5.89	5.44	7.07	6.43	5.89	5.44

Note: When A, B and C are all multiplied by the same constant, f, the standard-error factors are divided by f.

that the symmetrical arrangement of System (a) is superior to that of System (l). The reason for this will be discussed below.

In System (k) the measurement directions are arranged similarly to those in System (l); it is impossible to arrange them similarly to those in System (a). Consequently the standard-error factors for Systems (k) and (l) are similar, differing from those of System (a) for similar reasons. The difference between the factors for Systems (k) and (l), caused by the different borehole arrangements, are dependent on A, B and C. If B and C were both zero the differences would be zero.

The dependence of the standard-error factors on A, B and C is illustrated in Table II where factors are given for borehole Systems (a) and (k). The computations were carried out with A equal to 0.1. B ranges from 0 to -0.03 and C, from 0 to -0.02. To use the table for another value of A, A′ say, the tabulated values of A, B and C must all be multiplied by the ratio A′/A and the standard-error factors must be divided by A′/A.

The results summarized in Table I have shown that three orthogonal boreholes provide the best configuration of three boreholes for measuring all six stress components with uniform precision. It requires four measurement directions in each borehole (System (a)) to achieve uniformity when the angle between these directions is 45 degrees. When the measurement directions are inclined to each other at 60 degrees and are disposed similarly in all boreholes three directions in each borehole give results equivalent to those for System (a). The standard-error factors tabulated for System (a) are simply multiplied by $(4/3)^{\frac{1}{2}}$. Thus the two arrangements are equally efficient since, in general, when the total number of measurements is multiplied by a number, f, (by making f measurements in each measurement direction of a given system) the standard-error factors are multiplied by $(1/f)^{\frac{1}{2}}$. By this criterion System (b) is less efficient than the above arrangements.

It remains to consider the result of using an orthogonal system of three boreholes all of which are inclined to the axes of coordinates. No simple relationships are to be expected but it can be shown by means of the theory for the combination of variances[3] that the standard-error factors should not differ greatly from those for System (a) if the measurement directions have the same relative orientations as in System (a).

To verify this point the standard-error factors were computed for an oblique system (0) of boreholes (b_1, b_2, b_3) derived from System (a) as follows: The system is first imagined to be situated so that b_1, b_2 and b_3 coincide with g_1, g_2 and g_3, respectively, as for System (a). Then the borehole axes are given a rigid rotation of 45 degrees about g_3, followed by a rotation of 30 degrees about g_1. The measurement directions of System (a) are also rotated rigidly with the borehole axes. The standard-error factors for Systems (0) and (a) are compared in Table III.

It will be seen that, for the same A, B and C, the average value of the factors is slightly less for System (a) than for System (0), and that the values are more uniform. Thus the stress components could be determined with a little more precision, and with more uniform precision, by means of System (a).

DISCUSSION AND CONCLUSIONS

The ideas put forward in this paper were developed in conjunction with computations on a limited amount of field data and on some artificial data. The least squares fits and the computation of standard-error factors were carried out on a CDC 3100 computer by means of programs based on the Gram-Schmidt ortho-normalization procedure.[4] A detailed account

Table III. Standard-Error Factors for Systems of Three Orthogonal Boreholes (Typical Values of A, B, C)

Stress Component	System (a) Parallel to Coordinate Axes	System (0) Oblique to Coordinate Axes
G_{11}	1.61	1.91
G_{22}	1.61	1.91
G_{33}	1.61	1.86
G_{12}	1.41	1.10
G_{23}	1.41	1.18
G_{31}	1.41	1.32
Average Factor	1.51	1.56

$A = 0.40$, $B = -0.10$, $C = -0.05$

Direction Cosines of Boreholes

System (a):

$$b_1 \equiv g_1: \quad 1, \quad 0, \quad 0$$

$$b_2 \equiv g_2: \quad 0, \quad 1, \quad 0$$

$$b_3 \equiv g_3: \quad 0, \quad 0, \quad 1$$

System (0):

$$b_1: \quad \cos 45°, \cos 45° . \cos 30°, \cos 45° . \cos 60°$$

$$b_2: \quad -\cos 45°, \cos 45° . \cos 30°, \cos 45° . \cos 60°$$

$$b_3: \quad 0, \quad -\cos 60°, \quad \cos 30°$$

of the work on particular sets of data is outside the scope of this paper and it will be reported elsewhere, but the general conclusions that follow from the theoretical and the numerical work will be discussed below.

Number of Boreholes Required

It has been shown theoretically that a solution for the complete stress tensor cannot be obtained from measurements in only two boreholes. Attempts to solve two sets of field data from pairs of boreholes inclined at less than 90 degrees to each other led to completely unrealistic results. The associated standard-error factors were in general so large that they indicated errors that would make any result meaningless.

In each of the foregoing cases, when the measurements from the two boreholes were combined with measurements from a third borehole an excellent solution was obtained. This proved that the data *per se* were not faulty.

It was also verified by using artificial and field data from pairs of orthogonal boreholes that it is possible to solve for five stress components and to find Q′, the minimum sum of squares of the residual errors, if the boreholes are parallel to two of the axes of coordinates. In one case Q changed very little as C was changed so that a value of C associated with minimum Q could not be determined. The full solution for the stress tensor showed that the normal stresses in the directions of the boreholes were almost equal. Thus the condition B2 of Appendix B was not properly satisfied and the indeterminacy of C was explained.

Choice of Borehole Directions

It has been shown that three orthogonal boreholes provide the best configuration of three boreholes for measuring all six stress components with uniform precision. (System (a) of Fig. 2).

It is possible, however, to obtain good results by the use of three boreholes in one plane. In System (k) (Fig. 2) pairs of nearest boreholes are inclined at 45 degrees to each other. A smaller angle should not be used if this can be avoided. Work on the combination of measurement directions, on the other hand, suggests that an angle of 60 degrees between nearest boreholes may be optimum (in the coplanar case only). This possibility has not yet been investigated.

In all cases the relative orientation of the boreholes is more important than their orientation with respect to the coordinate axes in determining the precision of the computed stress components.

Choice of Measurement Directions

When three orthogonal boreholes are used and when 45 degrees is chosen as the angle between nearest measurement directions, four measurement directions per borehole are required for the greatest and the most uniform precision in the determination of all the stress components. Equivalent results are given by three measurement directions per borehole if the angle between nearest directions is 60 degrees. The standard-error factors for the two cases differ only by a multiplier determined by the relative number of measurements. Ideally, then, the measurements made for each borehole direction should be based on sets of four different directions if the angle between nearest measurement directions is 45 degrees, or on sets of three different directions if the latter angle is 60 degrees.

When three coplanar boreholes directed as in System (k), Fig. 2, are used, the measurement directions shown for this system provide optimum uniformity of precision in the determination of the stress components. As mentioned in the previous section, it is possible that an equally good system can be based on coplanar boreholes inclined at 60 degrees to each other. The best measurement directions for such a system remain to be investigated.

In the above discussion we have taken account only of the variances of the stress components, i.e., the diagonal components, c_{11} --- c_{66}, of the variance-covariance matrix [C]. In general, the off-diagonal covariance terms, c_{jk}, are not zero. They must be taken into consideration when the variance of any function involving more than one stress component is calculated.[3]

System (a) of Fig. 2 (and the similar system employing three measurement directions inclined at 60 degrees, per borehole) is more efficient than the other systems because all of the corresponding covariances are zero except the three between the normal stress components. Reduction of these three covariances to zero requires that the four measuring directions in each borehole be reoriented to occupy directions at approximately ± 10 degrees to the oblique (45 degree) directions of System (a). The exact angle depends on the values of A, B and C and it is perhaps impractical to fulfill the requirements for all covariances to be zero.

Significance of Relationship Between Parameters A, B, C

The parameters A, B and C, introduced in Eq. 4, are characteristic of a given instrument and are dependent on the elastic properties of the rock in which the instrument is used. Some preliminary ideas on the significance of the ability to evaluate the function $X = \frac{A-C}{B-C}$ that occurs in Eq. 8, from field data, are discussed below.

USBM Deformation Meter

In the case of the USBM Deformation Meter the parameters given by theory are as follows:

$$\left.\begin{aligned} A &= f_1(\theta=0) = \frac{d(3-2\nu^2)}{E} \\ B &= f_3(\theta=0) = -\frac{d(1-2\nu^2)}{E} \\ C &= f_2 \qquad = -\frac{d\nu}{E}, \end{aligned}\right\} \qquad [25]$$

where f_1, f_2 and f_3 are functions given by Panek [2], d is the diameter of the borehole, E is Young's modulus and ν is Poisson's ratio.

We can therefore evaluate the function $\frac{A-C}{B-C}$, with the result,

$$X = \frac{A-C}{B-C} = -\frac{3-2\nu}{1-2\nu}. \qquad [26]$$

It is noteworthy that X is a function of Poisson's ratio only. Values of X calculated for a range of values of ν are shown in Table IV.

As an illustrative example Panek [2] has applied the method of least squares to a set of field measurements. The value of ν is given as $\frac{1}{3}$. It follows that the corresponding value of X is -7.0.

The parameters A, B and C calculated from Panek's data [2] are:

$$\begin{aligned} A &= 0.453 \times 10^{-6} \\ B &= -0.127 \times 10^{-6} \\ C &= -0.054 \times 10^{-6} \end{aligned}$$

The authors first verified Panek's least squares solution and then, by letting C vary, they found the solution giving the minimum sum of squares about regression, Q″. Q″ is 30 pct. lower than the original Q′ and

Table IV. Values of X calculated for range of values of ν.

ν	X
0	− 3.0
0.1	− 3.5
0.2	− 4.3
0.3	− 6.0
0.4	−11.0
0.5	− ∞

occurs for $C=C''=-0.172\times10^{-6}$. The corresponding value of X is

$$X=\frac{A-C}{B-C}=\frac{0.453+0.172}{-0.127+0.172}$$

$$=\frac{0.625}{0.045}=13.9 \qquad [27]$$

Substituted in Eq. 26 this leads to

$$\nu=0.567$$

This value of ν is outside the range allowed in the theory of elasticity. It is instructive, however, to calculate the set of parameters, A_o, B_o, C_o, that is obtained by substituting $\nu=0.567$ in Eqs. 25. The result is,

$$A_o= \ 0.390 \times10^{-6}$$

$$B_o=-0.0587\times10^{-6}$$

$$C_o=-0.0924\times10^{-6}$$

Of all the sets of parameters that satisfy Eq. 27, A_o, B_o, C_o are the only ones that also satisfy Eqs. 25. They therefore supply the only solution compatible with the measurement data that is also consistent with Eqs. 25.

The stress components associated with A_o, B_o, C_o are shown in Table V, where they are compared with the components associated with the original parameters, A, B, C.

It is important to notice that in the new solution seven quantities have been determined from the measurements, ν and the six stress components. Thus the statistical theory that applies when ν is accurately known in advance does not apply and it is not possible without further work to quote standard errors for the new solution.

Table V. Comparison of Solutions for Stress Components *

	New Solution	Original Solution [2]
ν	0.567	0.333 (assumed)
A	0.390×10^{-6}	0.4529×10^{-6}
B	-0.0587×10^{-6}	-0.1269×10^{-6}
C	-0.0924×10^{-6}	-0.0544×10^{-6}
G_{11}	9800 psi	8570 psi
G_{22}	5590 psi	4630 psi
G_{33}	5570 psi	4830 psi
G_{12}	840 psi	650 psi
G_{23}	−100 psi	−80 psi
G_{31}	1910 psi	1480 psi

* Compressive stresses are taken as positive.

The problem is a non-linear one. The correct solution for ν and the stress components has been found in a step-by-step manner but the standard errors can be found only by a unified solution. Work is continuing on a unified (non-linear) solution.

The above results are presented merely as an illustration of the type of information that can be sought, by means of an extended application of the method of least squares, on the basis of the special properties of Eq. 7. The full significance of the new results with respect to the particular case discussed cannot be judged without a careful study of other relevant data. At time of writing, however, these results are believed to be significant to the extent that a value of Poisson's ratio in the neighborhood of $\frac{1}{2}$, rather than $\frac{1}{3}$, is indicated by the measurements. Certainly this is the case if anisotropy of the rock in which the measurements were made can be ruled out and if the discrepancy cannot be assigned to some instrumental peculiarity not included in the theory.

Whatever the case may be, it has been demonstrated that a useful check can be placed on the field measurements. Since Poisson's ratio *in situ* is difficult to determine, Eq. 26 should prove to be of considerable value in the analysis of field measurements.

CSIR Strain Cell

In the case of the CSIR Strain Cell we must go back to the definition of A, B and C following Eq. 4 in order to evaluate X. We find

$$X=\frac{A-C}{B-C}=\frac{(a-b\nu)-c(1-\nu)}{(b-a\nu)-c(1-\nu)}$$

$$=\frac{(a-c)-(b-c)\nu}{(b-c)-(a-c)\nu}. \qquad [28]$$

A set of measurements made by means of the CSIR Strain Cell has been treated by the authors. When it is assumed that $a=1.53$ and $b=0$ in accordance with the values given by Leeman[1] it is found that $c=-0.105$ leads to a minimum sum of squares about regression. The corresponding value of X is -19.6 (since $\nu=0.115$).

The following relation between the parameters, applying to the actual conditions under which the measurements were made, is therefore available for comparison with laboratory determination of a, b and c:

$$\frac{(a-c)-(b-c)\nu}{(b-c)-(a-c)\nu}=-19.6 \qquad [29]$$

The relation is valid if the rock is isotropic and elastic and if the Strain Cell functioned as intended while the measurements were being made.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the benefit of discussions with Mr. W. L. van Heerden of the Council for Scientific and Industrial Research, South Africa, and Mr. K. Barron of the Mining Research Centre, Dept. of Energy, Mines and Resources, Canada.

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AUTHORS' NOTE

It is appropriate to include the following note to complete the discussion of the non-linear statistical problem introduced in this paper. It is of considerable interest that meaningful solutions to the non-linear problem can be found:

Unified solutions of the respective non-linear statistical problems have been carried out for the USBM Deformation Meter (to determine ν and the stress components) and for the CSIR Strain Cell (to determine C and the stress components, given A and B). In both cases the non-linear problem can be closely approximated by a linear problem. Q is accurately expressible as a quadratic, in the 95-pct confidence region surrounding its minimum, and thus the computed statistics can be interpreted in the normal way.[5]

When standard errors are included (in brackets) the "new solution" to the example relating to the USBM Deformation Meter is

ν	0.567	(0.057)
G_{11}	9800	(500)
G_{22}	5590	(460)
G_{33}	5570	(320)
G_{12}	840	(480)
G_{23}	−100	(240)
G_{31}	1910	(280)

APPENDIX A

Surface Stress at Center of End Surface of Borehole

The components of the stress acting in the rock in which the borehole is located are represented by M_{ij}, referred to axes m_1, m_2, m_3 where m_2 is the direction of the borehole axis. The end of the borehole is assumed to be plane and normal to the axis. The stress components at the center of the end surface are represented by M'_{ij}. M'_{22}, M'_{12} and M'_{23} must equal zero at the free surface.

It is required to show that M'_{11} and M'_{33} depend only on M_{11}, M_{22} and M_{33}. (A knowledge of M'_{13} is not required as it does not affect the reading of a strain gage oriented in the m_1 direction, when stress is relieved.)

As indicated in Fig. A-1, the shear stress M_{12} can be replaced by the normal stresses P and Q acting in directions inclined 45 degrees to the borehole axis, where $P=M_{12}$ and $K=-M_{12}$.

Suppose that P causes a stress x M_{12} in the m_1 direction at C, the center point on the end surface of the borehole. By symmetry Q must cause a stress $-x$ M_{12} in the m_1 direction at C. The effects of P and Q cancel each other and thus the shear component M_{12} contributes nothing to M'_{11} at C.

Similarly, M_{12} cannot contribute to M'_{33}. The same procedure can also be used to show that M_{23} and M_{31} cannot contribute to M'_{11} and M'_{33}.

It should be noted that the symmetry argument holds only for the center point C. Thus the strain gage must be applied within an area surrounding C in which the stress conditions are the same as at C, within experimental error.

APPENDIX B

General Proof that Only Five Independent Measurements Can be Made in Two Boreholes

Choose boreholes B_1 and B_2 in the plane g_1 g_2 so that B_1 makes the angle α and B_2 the angle $\pi/2-\alpha$ with g_1 (see Fig. B-1). Choose measurement direction 1 to 3 in B_1 and 4 to 6 in B_2 so that 1 and 4 lie in the g_1 g_2 plane, 3 and 6 are normal to the plane and 2 and 5 are inclined at 45 degrees to it. The special choice of borehole directions does not limit the validity of the proof as the final result is not affected by a transformation to a new system of reference. The choice of measurement directions is also sufficiently general since, in each borehole, the measurements chosen can be derived from measurements in any three independent directions.

Consider first borehole B_1. It has direction cosines $l_{21}=\cos\alpha, \sin\alpha, 0$. The sets of direction cosines associated with the chosen measurement

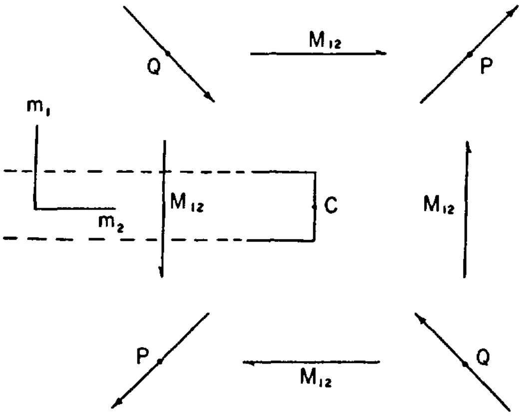

Fig. A-1—Boreholes in shear-stress field.

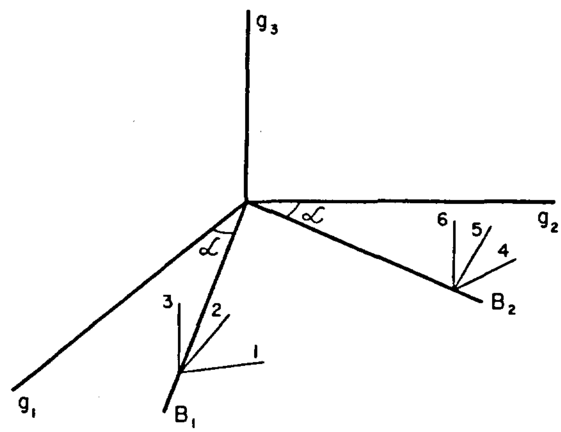

Fig. B-1—System of two non-orthogonal boreholes.

directions, and the equations derived from them by substitution in Eqs. 7 are as follows:

Measurement (1):

$$l_{11} = -\sin a, \quad \cos a, \quad 0$$
$$l_{21} = \cos a, \quad \sin a, \quad 0$$
$$l_{31} = 0, \quad 0, \quad -1$$

$$e_1 = (A \sin^2 a + C \cos^2 a)\, G_{11} + (A \cos^2 a + C \sin^2 a)\, G_{22} + BG_{33} - 2(A-C) \sin a \cos a\, G_{12}.$$

Measurement (2):

$$l_{11} = -\sin a/\sqrt{2}, \cos a/\sqrt{2}, \quad 1/\sqrt{2}$$
$$l_{21} = \cos a, \quad \sin a, \quad 0$$
$$l_{31} = -\sin a/\sqrt{2}, \cos a/\sqrt{2}, \quad -1/\sqrt{2}$$

$$e_2 = [\tfrac{1}{2}(A+B) \sin^2 a + C \cos^2 a] G_{11} + [\tfrac{1}{2}(A+B \cos^2 a + C \sin^2 a]\, G_{22} + \tfrac{1}{2}(A+B)\, G_{33} - (A+B-2C) \sin a \cos a\, G_{12} + (A-B) \cos a\, G_{23} - (A-B) \sin a\, G_{31}.$$

Measurement (3):

$$l_{11} = 0, \quad 0, \quad 1$$
$$l_{21} = \cos a, \sin a, 0$$
$$l_{31} = -\sin a, \cos a, 0$$

$$e_3 = (B \sin^2 a + C \cos^2 a)\, G_{11} + (B \cos^2 a + C \sin^2 a)\, G_{22} + AG_{33} - 2(B-C) \sin a \cos a\, G_{12}.$$

The equations for borehole B_2 may be got similarly, or by substituting $(\pi/2-\alpha)$ for α in the equations for borehole B_1. The results are:

Measurement (4):

$$e_4=(A\cos^2\alpha+C\sin^2\alpha)\,G_{11}+(A\sin^2\alpha+C\cos^2\alpha)\,G_{22}$$
$$+BG_{33}-2(A-C)\sin\alpha\cos\alpha\,G_{12}.$$

Measurement (5):

$$e_5=[\tfrac{1}{2}(A+B)\cos^2\alpha+C\sin^2\alpha]\,G_{11}+[\tfrac{1}{2}(A+B)\sin^2\alpha+C\cos^2\alpha]\,G_{22}$$
$$+\tfrac{1}{2}(A+B)\,G_{33}-(A+B-2C)\sin\alpha\cos\alpha\,G_{12}$$
$$+(A-B)\sin\alpha\,G_{23}-(A-B)\cos\alpha\,G_{31}.$$

Measurement (6):

$$e_6=(B\cos^2\alpha+C\sin^2\alpha)\,G_{11}+(B\sin^2\alpha+C\cos^2\alpha)\,G_{22}+AG_{33}$$
$$-2(B-C)\sin\alpha\cos\alpha\,G_{12}.$$

It will be noted that the components G_{23} and G_{31} appear only in the second and fifth equations above. The remaining four equations contain four unknowns and it is sufficient to prove that these four equations are not independent, to prove that all six equations are not independent. The determinant of the four equations is

$$\begin{vmatrix} A\sin^2\alpha+C\cos^2\alpha, & A\cos^2\alpha+C\sin^2\alpha, & B, & -2(A-C)\sin\alpha\cos\alpha \\ B\sin^2\alpha+C\cos^2\alpha, & B\cos^2\alpha+C\sin^2\alpha, & A, & -2(B-C)\sin\alpha\cos\alpha \\ A\cos^2\alpha+C\sin^2\alpha, & A\sin^2\alpha+C\cos^2\alpha, & B, & -2(A-C)\sin\alpha\cos\alpha \\ B\cos^2\alpha+C\sin^2\alpha, & B\sin^2\alpha+C\cos^2\alpha, & A, & -2(B-C)\sin\alpha\cos\alpha \end{vmatrix}$$

$$=\begin{vmatrix} (A-C)(\sin^2\alpha-\cos^2\alpha), & (C-A)(\sin^2\alpha-\cos^2\alpha), & 0, & 0 \\ (B-C)(\sin^2\alpha-\cos^2\alpha), & (C-B)(\sin^2\alpha-\cos^2\alpha), & 0, & 0 \\ A\cos^2\alpha+C\sin^2\alpha, & A\sin^2\alpha+C\cos^2\alpha, & B, & -2(A-C)\sin\alpha\cos\alpha \\ B\cos^2\alpha+C\sin^2\alpha, & B\sin^2\alpha+C\cos^2\alpha, & A, & -2(B-C)\sin\alpha\cos\alpha \end{vmatrix}$$

$$=(A-C)(B-C)(\sin^2\alpha-\cos^2\alpha)^2\begin{vmatrix} 1, & -1, & 0, & 0 \\ 1, & -1, & 0, & 0 \\ - & - & - & - \\ - & - & - & - \end{vmatrix}$$

$= 0$, since the first two rows are identical.

Thus the equations are not independent. Subtraction of the fourth and sixth equations from the first and third gives the following equations:

$$(A-C)\,(\sin^2 \alpha - \cos^2 \alpha)\,(G_{11}-G_{22}) = (e_1 - e_4)$$

$$(B-C)\,(\sin^2 \alpha - \cos^2 \alpha)\,(G_{11}-G_{22}) = (e_3 - e_6)$$

Division of the above equations gives the relation

$$\frac{A-C}{B-C} = \frac{e_1-e_4}{e_3-e_6} \qquad [B1]$$

which is similar to that derived in the text for the special case of orthogonal boreholes. The division of the above equations can only be carried out if

$$(\sin^2 \alpha - \cos^2 \alpha)\,(G_{11}-G_{22}) \neq 0.$$

If this condition is not fulfilled the expression $\frac{e_1-e_4}{e_3-e_4}$ is indeterminate and the relation between the parameters cannot be established.

The above condition implies

$$G_{11}-G_{22} \neq 0 \qquad [B2]$$

except when $\sin^2 \alpha$ equals $\cos^2 \alpha$, i.e., α equals 45 degrees. The latter is not a case of any interest since it means that the two boreholes have the same direction.

Thus the relation given by Eq. B1 can be established for two boreholes in the $(g_1\, g_2)$ plane, inclined at any angle different from zero, provided that G_{11} is not equal to G_{22}.

DISCUSSION OF CHAPTER 3

G. B. Barla, Columbia University, New York, N.Y.—The stress relief method to determine the absolute value of principal stresses at a point of a solid rock has been extensively studied, as both the papers [1, 2] presented at this Symposium confirm.

Here the rock is regarded as an isotropic, homogeneous medium and classical linear theory of elasticity is applied. Then, an analysis of the method which removes some of these assumptions must be considered as a natural extension of present knowledge.

The nonlinear mechanical effects and the rheological properties, which most of the rocks exhibit, are not taken into account. These effects may be assessed separately by using the nonlinear theory of continuous media and the linear theory of creep.

Physical Nonlinearity

Within the theory of elasticity, one can speak of two types of nonlinearity, geometrical and physical, which may be regarded as independent of each other. There are four types of problems which can be considered [3]: 1) those having both physical and geometrical linearity, 2) those which are physically nonlinear but geometrically linear, 3) those physically linear but geometrically nonlinear, 4) those both physically and geometrically nonlinear.

If a rock is regarded as a material which exhibits physical nonlinearity even when sustaining small deformations, a theory of elasticity of Type 2, for which kinematic linearity is retained but where physical nonlinearity is permitted, can be very appropriate.

Mechanical constitutive equations for this special case have already been developed. The plane elastostatic problem has been formulated, and an approximate solution of the problem of uniform extension of an infinite plate containing a circular hole has also been worked out by using perturbation techniques.[4]

An examination of some simple states of stress and deformations can lead to define, with a carefully planned experimental program, material functions which take into account the features arising from physical nonlinearity.

Creep Effect

Many rocks deform with time, while the stress condition does not change. Conversely, their stress condition may change while further deformation is impossible.

In both cases the stresses at any time are determined not only by the deformation at the present time, but also by their total prior deformation. Therefore, a theory of creep can be developed on the assumptions that the rock is regarded as a homogeneous isotropic medium and the relationship between the creep deformation and the stresses is linear. In these instances, the creep characteristics of the rock may be defined by two constants and two creep functions. A general theory to calculate stresses and deformations in a body, with consideration given to creep when the solution of the corresponding-instantaneous linearly elastic problem is available, has been developed.[5] It is possible to adapt the general theory to the case in discussion.

Careful experimental programs are required to account for the effects of creep, as well as for the effects of physical nonlinearity.

Conclusion

In both preceding cases, a model is introduced for the specific purpose of taking into account two properties of rocks which have been neglected in the existing solution. It is necessary to show that these models are in good agreement with the experimental data. It should be pointed out that these models are of interest not only in the investigation of the particular problem mentioned, but that they may be very conveniently used to analyze more general problems occurring in Rock Mechanics.

Work along these lines is being carried on at the Henry Krumb School of Mines, Columbia University, New York. We hope to report upon it in the future.

W. L. van Heerden, Rock Mechanics Division, National Mechanical Engineering Research Institute, South African Council for Scientific and Industrial Research, Pretoria, South Africa—Dr. Gray and Mr. Toews provide in their paper an excellent analysis of the conditions under which the components of the ground stress tensor can be determined from a set of measurements made by means of borehole devices. Since it is also shown that the choice of borehole and measuring directions can affect the precision with which the various stress components can be determined, the paper should be extremely useful in planning future stress measurement investigations of this nature.

Since the paper deals to a large extent with a strain gage device developed by the South African Council for Scientific and Industrial Research (CSIR), the contributor would like to give some additional information regarding the use of this device, which was not available at the time the paper was presented.

As mentioned in the paper the CSIR device was developed to measure the strain on the flattened face of a borehole drilled into rock. During recent months extensive laboratory tests were conducted by the CSIR to determine, for the flat borehole end, the three stress concentration factors, a, b and c, as defined by Gray and Toews.

Testing Procedure

(a) *Photoelastic investigations.* A three-dimensional photoelastic investigation was conducted on a block of Bakelite to determine the stress concentration factors a and b. The block, which had a height to width ratio of 2, had a hole drilled into one face, the end of the hole being machined flat. Uniform uniaxial compression was applied to the block by means of compressed air acting through rubber membranes. After the block was stress frozen, a slice containing the center of the flat borehole end was analyzed in order to determine factor a. A subslice was subsequently cut from the original slice so that the factor b could also be determined.

(b) *Direct loading tests.* Direct loading tests were conducted on blocks of norite and sandstone and on blocks and cylinders of steel and aluminum. A small hole was drilled into each block and cylinder and the ends of the holes were machined flat. A small rosette strain gage was cemented to the center of the end face of each hole. The blocks and cylinders in each instance had a height to width or height to diameter ratio of at least 2, so that a uniform stress distribution could be obtained at the center where the hole was drilled. The blocks and cylinders were subsequently loaded in a uniaxial compression testing machine while the strain readings were recorded at suitable increments of load. In some of the specimens the hole was drilled normal to the direction of the applied load while in others the hole was drilled in a direction parallel to the applied load so that from the strain readings obtained the three stress concentration factors a, b and c could be calculated.

Results

The results obtained for the stress concentration factors a and b are summarized in Discussion Table I while the results for the stress concentration factor c are

Discussion Table I. Results of Tests Aimed at Determining Stress Concentration Factors a and b

Material	Shape of model	Dimensions	Hole diameter (inches)	Depth of hole (inches)	Modulus of elasticity (psi)	Poisson's ratio ν	a	b
Bakelite	Rectangular block	5 in. $\times$ 6 in. $\times$ 12 in.	0.75	2	5000	0.48	1.25	-0.071
Aluminum	Cylinder	6 in. diam. $\times$ 12 in.	0.75	2	10.65×10^6	0.35	1.28	-0.022
Steel	Block	6 in. $\times$ 6 in. $\times$ 12 in.	0.75	2	29.00×10^6	0.287	1.24	-0.065
Sandstone	Block	6 in. $\times$ 6 in. $\times$ 12 in.	0.90	2	4.74×10^6	0.25	1.31	-0.19
Norite	Block	8 in. $\times$ 8 in. $\times$ 24 in.	0.90	3	13.83×10^6	0.261	1.22	-0.098
Average values for stress concentration factors a and b (Sandstone block not included)							1.25	-0.064

Discussion Table II. Results of Tests Aimed at Determining Stress Concentration Factor c

Material	Shape of model	Dimensions (inches)	Hole diameter (inches)	Depth of hole (inches)	Modulus of elasticity (psi)	Poisson's ratio (ν)	c
Aluminum	Cylinder	6 in. diam. $\times$ 12	0.75	6	10.65×10^6	0.35	-0.740
Steel	Cylinder	6 in. diam. $\times$ 12	0.75	6	30.6×10^6	0.308	-0.720
Norite	Block	$6 \times 6 \times 12$	0.90	6	13.98×10^6	0.273	-0.688

given in Discussion Table II. It should be noted that the result obtained for sandstone must be treated with caution since the sandstone used did not have a linear stress-strain relationship. It does, however, show that even for a poor material such as sandstone the stress concentration factors a and b do not deviate much from the average values obtained for better quality materials.

It can be seen from Discussion Table II that stress concentration factor c seems to be dependent on Poisson's ratio, ν. From the results obtained it is found that c is approximately given by the following relationship:

$$c = -0.75\ (0.645 + \nu)$$

It is believed that these results would be valuable in conjunction with the paper by Gray and Toews because it was shown in their paper that a useful check can be made on a set of field measurements provided the three stress concentration factors a, b and c are accurately known.

Acknowledgments

This work was conducted by the South African Council for Scientific and Industrial Research on behalf of the Chamber of Mines of South Africa.

Authors' Closure—The authors have used stress concentration factors extrapolated from Mr. van Heerden's experimental results in the analysis of a set of data obtained[6] in rock for which $E = 11.55 \times 10^6$ lb in^{-2} and $\nu = 0.115$, and which appeared to be elastically isotropic. Using $a = 1.25$, $b = -0.064$ and $c = -0.57$ we find the values of A, B and C shown in Column (1) of Discussion Table III. The value of Q, the residual sum of squares, associated with the set A, B, C is also given.

Columns (2) and (3) show the sets that are obtained by minimizing Q through variation of C and B respectively. Column (4) shows values of A, B and C obtained by extrapolation from results given by Hiramatsu and Oka,[7] together with the Q obtained when these values are applied to the analysis of Barron's data. It is seen that the set in Column (4) comes remarkably close to minimizing the residual sum of squares.

It may be remarked that the stresses associated with Columns (1) to (4) differ widely.

It will be noticed that when the values in Columns (1), (3) and (4) are scaled to $A = 0.1$ they fall outside Table II. Moreover Hiramatsu and Oka[7] have found, by assuming equal measurement errors (expressed as strain), that stresses determined by means of the CSIR Strain Cell are subject to higher errors than those determined by means of the USBM Deformation Meter. Calculations have therefore been done to extend Table II and the results are given in Discussion Table IV.

It is evident that the normal stress components are indeterminate if A + B +

Discussion Table III

	(1)	(2)	(3)	(4)
A	0.1090	0.1090	0.1090	0.1212
B	−0.0180	−0.0180	−0.0514	−0.0563
C	−0.0436	−0.0119	−0.0436	−0.0476
Q	13.12×10^4	6.98×10^4	6.93×10^4	6.94×10^4

Note: Strains have been expressed in units of 10^{-6}.

Discussion Table IV. Standard-Error Factors: Dependence on Parameters B and C with A = 0.1

Borehole System (a) (Figure 2)

C \ B	0.00	−0.01	−0.02	−0.03	−0.04	−0.05	−0.06	−0.07
0.00	5.92 7.07	5.97 6.43	6.07 5.89	6.25 5.44	6.58 5.05	7.19 4.71	8.29 4.42	10.38 4.16
−0.01	5.74	5.92	6.17	6.56	7.21	8.32	10.41	14.93
−0.02	5.68	6.00	6.46	7.15	8.31	10.42	14.94	29.11
−0.03	1.84	6.29	7.04	8.24	10.38	14.93	29.11	∞
−0.04	6.10	6.89	8.14	10.32	14.90	29.10	∞	29.07
−0.05	6.74	8.02	10.24	14.85	29.08	∞	29.06	14.81
−0.06	7.91	10.16	14.80	29.06	∞	29.05	14.79	10.13
−0.07	10.09	14.76	29.03	∞	29.03	14.76	10.10	7.83

Note: The standard error factors for the shear stress components appear below the corresponding factors for the normal components, in the C = 0.00 row. They are omitted from the remainder of the table since they do not vary with C. The figure 1.84 in the first column is correct.

C = 0, and are subject to large errors when this condition is nearly fulfilled. The values in Column (4) of Discussion Table III come close to fulfilling the condition and this accounts for Hiramatsu and Oka's conclusion regarding the relative accuracy of stress determinations made by means of the CSIR Strain Cell.

The analysis employed here assumes that strains are measured symmetrically about the center of the end of the borehole. In some versions of the strain cell, including that used by Barron, the strain gages are offset from the center. By an extension of the reasoning used in Appendix A it can be shown that if a gage is offset in the direction of measurement it may be sensitive to the shear stress component M_{12}. Sensitivity to M_{23} and M_{31} should be zero if the gage is symmetrical about a radius. It is thus possible that, even if A, B and C are not affected, an accurate determination of the stresses would require the knowledge of an additional parameter, the sensitivity to M_{12}. A standardized gage arrangement, symmetrical about the center, would appear to be desirable for purposes of comparison. It appears that further experimental work is required to establish agreed parameters for the strain cell.

For the USBM Deformation Meter,

$$A + B + C = \frac{d}{E}(2 - \nu).$$

A + B + C, therefore, cannot be zero and the indeterminacy discussed above cannot arise.

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