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FORMULAE FOR PROBABILITIES ASSOCIATED WITH WIENER AND BROWNIAN BRIDGE PROCESSES.

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ABSTRACT. We generalize some of the available methods for computing probabilities associated with Wiener and Brownian bridge processes of the following forms:

$P\{ W(t) \le \tau+\delta t : a\le t\le b \}$, $a\ge 0$, $b<\infty$, $\tau>0$, $\tau+\delta b\ge 0$;

$P\{ \tau'+\delta' t\le W(t)\le\tau+\delta t : a\le t\le b \}$, $a\ge 0$, $b<\infty$, $\tau>0$, $\tau+\delta b\ge 0$, $\tau'<0$, $\tau'+\delta' a\le 0$;

$P\{ B(t) \le \tau : a\le t\le b \}$, $a\ge 0$, $b\le 1$, $\tau>0$;

$P\{ \tau' \le B(t) \le \tau : a\le t\le b \}$, $a\ge 0$, $b\le 1$, $\tau>0$, $\tau'<0$,

where $W(.)$ and $B(.)$ are Wiener and Brownian bridge processes, respectively.

Key Words: Wiener process, Brownian bridge process, weak convergence, strong approximation.

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1. Introduction.

Weak and strong convergence of empirical, empirical quantile, product limit and quantile-quantile processes to Brownian bridge and Wiener process are quite well known in these days (cf., for example, Breslow and Crowley (1974), M. Csörgö and Révész (1978), S. Csörgö and Horváth (1985), Aly, M. Csörgö and Horváth (1985), Aly and Bleuer (1983), and Doksum and Yandell (1984) in particular for further applications).

When applying these methodologies to statistical problems, the computation of probabilities of functionals of interest of the approximating Gaussian processes plays a crucial role. These required computations are frequently complicated. Without these computations, however, the said recent statistical advancements using weak and strong convergence methodologies can not be utilized in practice.

In this exposition we summarize and generalize some of the available methods for computing probabilities associated with Wiener and Brownian bridge processes of the following forms:

(1.1) $P\{ W(t) \le \tau : a \le t \le b \}$, $a \ge 0$, $b < \infty$, $\tau > 0$;

(1.2) $P\{ |W(t)| \le \tau : a \le t \le b \}$, $a \ge 0$, $b < \infty$, $\tau > 0$;

(1.3) $P\{ W(t) \le \tau + \delta t : a \le t \le b \}$, $a \ge 0$, $b < \infty$, $\tau > 0$, $\tau + \delta b \ge 0$;

(1.4) $P\{ |W(t)| \le \tau + \delta t : a \le t \le b \}$, $a \ge 0$, $b < \infty$, $\tau > 0$, $\tau + \delta b \ge 0$;

(1.5) $P\{ B(t) \le \tau : a \le t \le b \}$, $a \ge 0$, $b \le 1$, $\tau > 0$;

(1.6) $P\{ |B(t)| \le \tau : a \le t \le b \}$, $a \ge 0$, $b \le 1$, $\tau > 0$,

where $W(.)$ and $B(.)$ are Wiener and Brownian bridge processes, respectively.

Some special cases of the probabilities of (1.5) and (1.6) have been tabulated by S. Csörgö and Horváth (1981), Barr and Davidson (1973), Koziol and Byar (1975), and Schumacher (1984) among many others. We have developed a computer program package called WIENER PACK (cf. Chung (1987)), to compute the probabilities of the above forms for any given values of the parameters a, b, τ and δ. Inversely, the package also enables one to compute any one of the parameters - a , b and τ in (1.1), (1.2), (1.5) and (1.6) for a given probability level and given values for the other two of the parameters a, b and τ. In the last section, twelve tables for the critical value τ of (1.6) are computed using WIENER PACK and are provided in Tables 1 – 12.

Obviously, (1.1) and (1.2) are special cases of (1.3) and (1.4), respectively. It can also be shown that (1.5) and (1.6) are again special cases of (1.3) and (1.4), respectively. Hence, we will concentrate our discussions on the formulae for (1.3) and (1.4) in detail, and those for the other four (1.1), (1.2), (1.5) and (1.6) will be obtained as corollaries.

2. Preliminaries.

A stochastic process $\{ W(t) : 0 \le t < \infty \}$ is called a Wiener process if :

(i) $W(t) - W(s)$ is $N(0, t-s)$ for all $0 \le s \le t < \infty$ and $W(0)=0$ where $N(\mu,\sigma^2)$ stands for normal random variable with mean μ and variance σ^2;

(ii) $W(t)$ is an independent increment process, that is $W(t_2)-W(t_1)$, $W(t_4)-W(t_3), \ldots, W(t_{2i})-W(t_{2i-1})$ are independent random variables for all $0 \le t_1 < t_2 < \ldots < t_{2i-1} < t_{2i} < \infty$ ($i = 2, 3, \ldots$);

(iii) the sample path function $W(t)$ is continuous in t with probability 1.

Remark 2.1. Note that (i) and (ii) imply that the covariance function of a Wiener process is

(2.1) $\quad E(W(s)W(t)) = s \wedge t,$

since $W(s \vee t) - W(s \wedge t)$ and $W(s \wedge t) - W(0) = W(s \wedge t)$ are independent random variables. Conversely, a continuous Gaussian process having the covariance function of (2.1) is a Wiener process, since the process with the covariance function in (2.1) must also satisfy properties (i) and (ii).

A stochastic process $\{ B(t) : 0 \le t \le 1 \}$ is called a Brownian bridge if:

(i) the joint distribution of $B(t_1), \ldots, B(t_n)$; $0 < t_1 \le \ldots \le t_n < 1$, is Gaussian with $E(B(t)) = 0$;

(ii) the covariance function is

(2.2) $EB(t)B(s) = s \wedge t - st$;

(iii) the sample path function $B(t)$ is continuous in t with probability 1.

Remark 2.2. Note that (ii) implies that $B(0) = B(1) = 0$ a.s. and also that, if $B(t) = W(t) - tW(1)$ $(0 \le t \le 1)$ where $W(t)$ is a Wiener process, then $B(t)$ is a Brownian bridge.

We first state a few well known results which are required in our sequel.

Theorem A. ((4.1) of Doob (1949) and (2.5) of Chapter VI of Feller (1966)). Let $\{ W(t) : t \ge 0 \}$ be a Wiener process. Then

(2.3) $P\{ W(t) \ge c : 0 \le t \le b \} = 2\, P\{ W(b) \ge c \}$.

The equality of (2.3) is often referred to as the reflection principle and (2.3) is also the one-sided stable distribution function of index $\frac{1}{2}$ ((4.7) of Chapter II of Feller (1966)).

The next theorem states relationships between Wiener and Brownian bridge processes.

Theorem B. (Doob (1949), and (1.4.4) and (1.4.5) of M. Csörgö and Révész (1981)). Let $\{ W(t) : t \geq 0 \}$ be a Wiener process and define

$$(2.4) \qquad B(s) := \begin{cases} (1-s)\, W(\, s/(1-s)\,) & \text{if } 0 \leq s < 1, \\ 0 & \text{if } s = 1. \end{cases}$$

Then $\{ B(s) : 0 \leq s \leq 1 \}$ is a Brownian bridge process.

Conversely, let $\{ B(s) : 0 \leq s \leq 1 \}$ be a Brownian bridge process and define

$$(2.5) \qquad W(t) := (1+t)\, B(\, t/(1+t)\,),\ t \geq 0.$$

Then $\{ W(t) : t \geq 0 \}$ is a Wiener process.

The transformations in (2.4) and (2.5) are referred to as Doob's transformation. Using these transformations, it is easy to observe that (1.5) and (1.6) are special cases of (1.3) and (1.4), respectively.

The next theorem states that the uniform empirical process $\{ \alpha_n(y) : 0 \leq y \leq 1 \}$, where $\alpha_n(y) = \sqrt{n}\, (\, y - U_n(y)\,)$ and $U_n(y)$ is the uniform empirical distribution function on [0,1], can be approximated by a sequence of Brownian bridge processes.

Theorem C. <u>(Komlós, Major and Tusnády (1975), Theorem 4.4.1 of M. Csörgö and Révész (1981)). Given $\{ \alpha_n(y) : 0 \le y \le 1 \}$, there exists a sequence of Brownian bridge processes $\{ B_n(y) : 0 \le y \le 1 \}$ such that</u>

$$(2.6) \qquad \sup_{0 \le y \le 1} | \alpha_n(y) - B_n(y) | \overset{a.s.}{=} O (n^{-\frac{1}{2}} \log n).$$

Remark 2.3. Using Theorem C, we can state that the distributional properties of $\{ \alpha_n(y) : 0 \le y \le 1 \}$ coincide with those of $\{ B(y) : 0 \le y \le 1 \}$ as $n \longrightarrow \infty$. Doob (1949) is the first one who suggested that in order to study asymptotic distributional properties of $\alpha_n(y)$ as $n \longrightarrow \infty$, we may simply replace $\alpha_n(.)$ by $B(.)$. Later Donsker (1952) justified Doob's approach. For a more detailed exposition of weak convergence we refer to Billingsley (1968).

Now, we quote a result on the limiting distribution of a functional of the empirical process by Csáki (1981), which will be used to study the distribution of the corresponding functionals of Brownian bridge process in the light of (2.6).

Theorem D. <u>(Theorem 2.1 of Csáki (1981)). Let $0 < a < 1$, $v > 0$ and $v - ua \ge 0$, and let $U_n(.)$ be the uniform empirical distribution function on $[0,1]$. Then</u>

$$(2.7)\qquad \lim_{n\to\infty} P\{\ U_n(x) \geq (\ 1 + u/\sqrt{n}\ )\ x - v/\sqrt{n} : 0 \leq x \leq a\ \}$$

$$= \Phi\left[\frac{v - ua}{\sqrt{a(1-a)}}\right] - e^{-2v(v-u)}\,\Phi\left[\frac{2av - v - ua}{\sqrt{a(1-a)}}\right],$$

<u>where</u>

$$(2.8)\qquad \Phi\ (y) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}h^2}\, dh,$$

<u>is the standard normal distribution function.</u>

The following theorem is due to Anderson (1960) and it is one of the most frequently quoted formulae when computing probabilities of functionals of Wiener process.

<u>**Theorem E.** (Theorem 4.3 of Anderson (1960)). Let $\tau_1 > 0$, $\tau_2 < 0$, $b > 0$, $\tau_2+\delta_2 b \leq \tau_1+\delta_1 b$. Then we have</u>

$$(2.9)\qquad P\{\ \tau_2+\delta_2 t \leq W(t) \leq \tau_1+\delta_1 t : 0\leq t\leq b\ \}$$

$$= 1 - \Phi\ (\ w/\sqrt{b}\ ) - \Phi\ (\ -v/\sqrt{b}\ )$$

$$+ \sum_{k=1}^{\infty}\Big[\ e^{s_1}[\ \Phi(q_1)-\Phi(p_1)\ ] + e^{s_2}[\ \Phi(p_2)-\Phi(q_2)\ ]$$

$$- e^{s_3}[\ \Phi(q_3)-\Phi(p_3)\ ] - e^{s_4}[\ \Phi(p_4)-\Phi(q_4)\ ]\ \Big],$$

<u>where</u>

$$\tau' = \tau_1-\tau_2, \quad \delta' = \delta_1-\delta_2, \quad v = \tau_1+\delta_1 b, \quad w = \tau_2+\delta_2 b,$$

$$s_1 = -2 [k^2\tau'\delta' + k(\tau_1\delta_2-\tau_2\delta_1)],$$

$$s_2 = -2 [k^2\tau'\delta' - k(\tau_1\delta_2-\tau_2\delta_1)],$$

$$s_3 = -2 [k^2\tau'\delta' + k(\tau'\delta_2+\delta'\tau_2) + \tau_2\delta_2],$$

$$s_4 = -2 [k^2\tau'\delta' - k(\tau'\delta_1+\delta'\tau_1) + \tau_1\delta_1],$$

$$p_1 = (-2k\tau' + w) / \sqrt{b}, \quad q_1 = (-2k\tau' + v) / \sqrt{b},$$

$$p_2 = (-2k\tau' - w) / \sqrt{b}, \quad q_2 = (-2k\tau' - v) / \sqrt{b},$$

$$p_3 = (-2k\tau'+w-2\tau_2) / \sqrt{b}, \quad q_3 = (-2k\tau'+v-2\tau_2) / \sqrt{b},$$

$$p_4 = (-2k\tau'-w+2\tau_1) / \sqrt{b}, \quad q_4 = (-2k\tau'-v+2\tau_1) / \sqrt{b},$$

and $\Phi(.)$ is defined in (2.8).

3. Formulae concerning $P\{ W(t) \le \tau+\delta t : a\le t\le b \}$.

We begin with the special case of $a = 0$.

Theorem 3.1. Let $\{ W(t) : t \ge 0 \}$ be a Wiener process. Let $\tau > 0$, $b > 0$ and $\tau+\delta b \ge 0$. Then

(3.1) $$P\{ W(t) \le \tau+\delta t : 0\le t\le b \}$$

$$= \Phi [(\tau+\delta b)/\sqrt{b}] - e^{-2\tau\delta} \Phi [(-\tau+\delta b)/\sqrt{b}].$$

Proof. Applying Doob's transformation in (2.5), we obtain

$$P\{ W(t) \le \tau+\delta t : 0\le t\le b \}$$

$$= P\{ (1+t) B(t/(1+t)) \le \tau+\delta t : 0\le t\le b \}$$

$$= P\{ B(s) \leq \tau(1-s) + \delta s : 0 \leq s \leq b' \},$$

where B(.) is Brownian bridge process and $b' = b/(1+b)$.

Now applying Theorem C, we get

$$P\{ W(t) \leq \tau+\delta t : 0 \leq t \leq b \}$$

$$= \lim_{n\to\infty} P\{ \alpha_n(s) \leq \tau(1-s) + \delta s : 0 \leq s \leq b' \}$$

$$= \lim_{n\to\infty} P\{ U_n(s) \geq [1+(\tau-\delta)/\sqrt{n}]s - \tau/\sqrt{n} : 0 \leq s \leq b'\} .$$

If we apply Theorem D with $u = \tau - \delta$ and $v = \tau$, then we obtain the assertion of the theorem.▒

Remark 3.1. For $\delta = 0$, we obtain (cf. (1.5.1) of M. Csörgö and Révész (1981))

$$(3.2) \qquad P\{ W(t) \leq \tau : 0 \leq t \leq b \} = 2\, \Phi (\tau/\sqrt{b}) - 1 ,$$

which is also an immediate result of the reflection principle in (2.3).

Having (3.1) and Doob's transformation in (2.4), it leads to the following corollary.

Corollary 3.1. Let $\{ B(s) : 0 \leq s \leq 1 \}$ be a Brownian bridge process. Let $0 < b < 1$ and $c \geq 0$. Then

(3.3) $$P\{\ B(s) \le c : 0 \le s \le b\ \}$$

$$= \Phi\left[\ c/\sqrt{b(1-b)}\ \right] - e^{-2c^2}\ \Phi\left[\ -c(1-2b)/\sqrt{b(1-b)}\ \right].$$

Proof. Apply Doob's transformation in (2.4), and then Theorem 3.1 with $\tau = \delta = c$. Then we obtain

$$P\{\ B(s) \le c : 0 \le s \le b\ \}$$

$$= P\{\ (1-s)W(\ s/(1-s)\ ) \le c : 0 \le s \le b\ \}$$

$$= P\{\ W(t) \le c(1+t) : 0 \le t \le b/(1-b)\ \}$$

$$= \Phi\left[\ c/\sqrt{b(1-b)}\ \right] - e^{-2c^2}\ \Phi\left[\ -c(1-2b)/\sqrt{b(1-b)}\ \right]. \blacksquare$$

Remark 3.2. For $b = 1$, formula (3.3) yields

(3.4) $$P\{\ B(s) \le c : 0 \le s \le 1\ \} = 1 - e^{-2c^2},$$

which is the limiting distribution function of the one-sided Kolmogorov-Smirnov statistic

$$\sqrt{n} \sup_{-\infty<x<\infty} (\ F_n(x) - F(x)\ ),$$

where $F_n(.)$ is the empirical distribution function of a continuous distribution function $F(.)$(cf. Doob, 1949).

Remark 3.3. Koziol and Byar (1975) derived (3.3) denoting it by $G_b(c)$ ((2.3) of Koziol and Byar (1975)). There is a misprint in their formula (2.3). Namely, instead of having $(T-T^2)^{-\frac{1}{2}}$ in the probability argument of (2.3), we should have $(T-T^2)^{-\frac{1}{2}}d$. Hall and Wellner (1980) have also derived a formula for the probability statement of (3.3) and denoted it by $G_b{}^+(c)$ (cf. (2.8) of Hall and Wellner (1980)). However, their (2.8) is not correct.

We will now extend Theorem 3.1 to the general case of $a > 0$ which, of course, covers Theorem 3.1.

Theorem 3.2. Let $\{ W(t) : t \geq 0 \}$ be a Wiener process. Let $\tau \geq 0$, $a > 0$, $b > a$ and $\tau + \delta b \geq 0$. Then

$$(3.5) \qquad P\{ W(t) \leq \tau+\delta t : a\leq t\leq b \}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\tau+\delta a)/\sqrt{a}} e^{-\frac{1}{2}x^2} \Big[\Phi\left[(\tau+\delta b-\sqrt{a}x)/\sqrt{b-a} \right]$$

$$- e^{-2(\tau+\delta a-\sqrt{a}x)\delta} \Phi\left[(-\tau-2a\delta+b\delta+\sqrt{a}x)/\sqrt{b-a} \right] \Big] dx.$$

Proof. Apply the facts that, for a fixed a, $W(s+a)-W(a)$ and $W(a)$ are two independent random variables and $W(s+a)-W(a) = W(s)$ in distribution. Then

$$(3.6) \qquad P\{ W(t) \leq \tau+\delta t : a\leq t\leq b \}$$

$$= P\{ W(t+a) \leq \tau+\delta(t+a) : 0\leq t\leq b-a \}$$

$$= P\{\ W(t+a) - W(a) \le \tau+\delta(t+a) - W(a)\ :\ 0\le t\le b-a\ \}$$

$$= \int_{-\infty}^{\infty} P\{\ W(t+a)-W(a) \le \tau+\delta(t+a)-h\ :\ 0\le t\le b-a\ \}\ dP\{\ W(a) \le h\ \}$$

$$= \int_{-\infty}^{\infty} P\{\ W(t) \le \tau+\delta a-h+\delta t\ :\ 0\le t\le b-a\ \}\ dP\{\ W(a) \le h\ \}.$$

However, using the normality of Wiener process, we have

$$dP\{\ W(a) \le h\ \} = d\Phi(\ h/\sqrt{a}\ ) = \frac{1}{\sqrt{2\pi a}}\, e^{-h^2/2a} dh. \tag{3.7}$$

In addition, it follows from (3.1) that

$$P\{\ W(t) \le (\tau+\delta a-h)+\delta t\ :\ 0\le t\le b-a\ \} \tag{3.8}$$

$$= \Phi\ [\ (\tau+\delta b-h)/\sqrt{b-a}\ ] - e^{-2(\tau+\delta a-h)\delta}\ \Phi\ [\ (-\tau-2\delta a+\delta b+h)/\sqrt{b-a}\ ],$$

if $\tau + \delta a \ge h$. By substituting (3.7) and (3.8) into (3.6), we obtain

$$P\{\ W(t) \le \tau+\delta t\ :\ a\le t\le b\ \}$$

$$= \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{(\tau+\delta a)} e^{-h^2/2a} \Big[\ \Phi\ [\ (\tau+\delta b-h)/\sqrt{b-a}\ ]$$

$$- e^{-2(\tau\delta+\delta^2 a-\delta h)}\ \Phi\ [\ (-\tau-2a\delta+b\delta+h)/\sqrt{b-a}\ ]\ \Big]\ dh.$$

If we set $x = h/\sqrt{a}$, then we obtain the assertion of the theorem.▒

Remark 3.4. As $a \to 0$ in (3.3), (3.3) becomes (3.1). That is, (3.1) is a special case of (3.3).

Remark 3.5. For $\delta = 0$, we obtain

$$P\{\, W(t) \le \tau : a \le t \le b \,\} \tag{3.9}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tau/\sqrt{a}} e^{-\frac{1}{2}x^2} \left[2\,\Phi\,[\,(\tau-\sqrt{a}x)/\sqrt{b-a}\,] - 1 \right] dx \,,$$

which is also discussed by Rényi(1953, (3.6)) and M. Csörgö(1967, (3.4)).

As an analogue of Corollary 3.1 of Theorem 3.1, we have the following corollary.

Corollary 3.2. Let $\{\, B(s) : 0 \le s \le 1 \,\}$ be a Brownian bridge process. Let $a \ge 0$, $a < b < 1$ and $c > 0$. Then

$$P\{\, B(s) \le c : a \le s \le b \} \tag{3.10}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c/(1-a)} e^{-\frac{1}{2}h^2} \left[\Phi\left[\frac{c\sqrt{1-a} - (1-b)\sqrt{a}h}{\sqrt{1-b}\,\sqrt{b-a}} \right] \right.$$

$$\left. - e^{-2c[c-\sqrt{a(1-a)}h]/(1-a)}\, \Phi\left[\frac{c(2b-a-1)+(1-b)\sqrt{a(1-a)}h}{\sqrt{1-a}\,\sqrt{1-b}\,\sqrt{b-a}} \right] \right] dh.$$

Proof. Similar to the proof of Corollary 3.1. Namely we apply Doob's transformation in (2.4) and Theorem 3.2 with $\tau = \delta = c$. Then we derive

$$P\{ B(s) \le c : a \le s \le b \}$$

$$= P\{ (1-s)W(s/(1-s)) \le c : a \le s \le b \}$$

$$= P\{ W(t) \le c(1+t) : a/(1-a) \le t \le b/(1-b) \}.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{q} e^{-\frac{1}{2}h^2} \Big[\Phi [(c+cb'-\sqrt{a'}h)/\sqrt{b'-a'}]$$

$$- e^{-2(c^2+c^2a'-\sqrt{a'}ch)} \Phi [(-c-2ca'+cb'+\sqrt{a'}h)/\sqrt{b'-a'}] \Big] dh,$$

where $q = c(1+a')$, $a' = a/(1-a)$ and $b' = b/(1-b)$. If we apply algebraic manipulations, then we obtain the assertion of the corollary.▒

Remark 3.6. Csáki((1981), (2.17)) obtained (3.10) as a limiting distribution of the uniform empirical process on a limited range. His formula is utilized by M. Csörgö(1983, (4.2.10)).

Remark 3.7. For $a = 0$, (3.3) of Corollary 3.1 is a special case of (3.10). On the other hand, when $b = 1$ and $a \neq 0$, by using the symmetricity property, we have

(3.11) $$P\{\ B(s) \le c : a \le s \le 1\ \}$$

$$= P\{\ B(s) \le c : 0 \le s \le 1-a\ \}$$

$$= \Phi\ [\ c/\sqrt{a(1-a)}\ ] - e^{-2c^2}\ \Phi\ [\ c(1-2a)/\sqrt{a(1-a)}\ ].$$

4. Formulae concerning $P\{\ |W(t)| \le \tau+\delta t : a \le t \le b\ \}$.

Anderson's formula in (2.9) of Theorem E immediately provides the following theorem which also appeared in Gillespie and Fisher (1979).

Theorem 4.1. Let $\{\ W(t) : t \ge 0\ \}$ be a Wiener process. Let $\tau > 0$ and $\tau+\delta b \ge 0$. Then

(4.1) $$P\{\ |W(t)| \le \tau+\delta t : 0 \le t \le b\ \}$$

$$= 1 - 2\,\Phi\,(\ -u/\sqrt{b}\ ) - 2 \sum_{k=1}^{\infty} (-1)^{k+1}$$

$$e^{-2k^2\tau\delta} \left[\ \Phi[\ (-2k\tau+u)/\sqrt{b}\ ] - \Phi[\ (-2k\tau-u)/\sqrt{b}\ ]\ \right],$$

where $u = \tau + \delta b$.

Proof. (4.1) is an immediate result of (2.9).▒

Remark 4.1. For $\delta_1 = \delta_2 = 0$ in (2.9), Anderson's formula in Theorem E also implies that

(4.2) $$P\{ \tau_2 \le W(t) \le \tau_1 : 0\le t\le b \}$$

$$= 1 - 2 \sum_{k=1}^{\infty} (-1)^{k+1} \left[\Phi[(-k\tau'+\tau_1)/\sqrt{b}]+\Phi[(-k\tau'-\tau_2)/\sqrt{b}]\right] ,$$

where $\tau' = \tau_1 - \tau_2$.

However, Feller (1966) showed not only the assertion of (4.2) (cf. Feller (1966), (5.8) of Chapter X) but also another formula (cf. Feller (1966), (5.9) of Chapter X) for the probability in (4.2) which is

(4.3) $$P\{ \tau_2 \le W(t) \le \tau_1 : 0\le t\le b \}$$

$$= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \exp\left[\frac{-(2k+1)^2\pi^2 b}{2(\tau_1-\tau_2)^2}\right] \sin\left[\frac{(2k+1)\pi\tau_1}{\tau_1-\tau_2}\right] .$$

Feller (1966) noted that while the series in (4.2) converges quickly when b is small, that in (4.3) converges rapidly when b is large. We note also that M. Csörgö(1967, (2.5)) also derived (4.3).

Remark 4.2. For $\tau_1 = -\tau_2 = \tau$ in (4.2) or $\delta = 0$ in (4.1), we have

(4.4) $$P\{ |W(t)| \le \tau : 0\le t\le b \}$$

$$= 1 - 4 \sum_{k=1}^{\infty} (-1)^{k+1} \Phi [-(2k-1)\tau/\sqrt{b}]$$

$$= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left[\frac{-(2k+1)^2\pi^2 b}{8\tau^2} \right].$$

From Theorem 4.1, we have the following corollary for a Brownian bridge process.

<u>**Corollary 4.1.** Let $\{ B(s) : 0 \le s \le 1 \}$ be a Brownian bridge process. Let $0 < b < 1$ and $c > 0$. Then</u>

(4.5) $$P\{ |B(s)| \le c : 0 \le s \le b \}$$

$$= 1 - 2\,\Phi\left(-c/\sqrt{b(1-b)} \right) - 2 \sum_{k=1}^{\infty} (-1)^{k+1} e^{-2k^2c^2} \left[\Phi\left(\frac{-c(2k-2kb-1)}{\sqrt{b(1-b)}} \right) - \Phi\left(\frac{-c(2k-2kb+1)}{\sqrt{b(1-b)}} \right) \right].$$

Proof. Similar to the proof of Corollary 3.1. Apply Doob's transformation in (2.4), and then apply Theorem 4.1. Thus we get

$$P\{ |B(s)| \le c : 0 \le s \le b \}$$

$$= P\{ |W(t)| \le c(1+t) : 0 \le t \le b/(1-b) \}.$$

$$= 1 - 2\,\Phi\left(-\mu/\sqrt{b'} \right) - 2 \sum_{k=1}^{\infty} (-1)^{k+1} e^{-2k^2c^2} \left[\Phi[(-2kc+\mu)/\sqrt{b'}] - \Phi[(-2kc-\mu)/\sqrt{b'}] \right],$$

where $\mu = c(1+b')$ and $b' = b/(1-b)$, i.e. we obtain (4.5). ▒

Remark 4.3. For b = 1, as b $\to$ 1 in (4.5), we have

(4.6) $$P\{\ |B(s)| \le c : 0 \le s \le 1\ \}$$

$$= 1 - 2 \sum_{k=1}^{\infty} (-1)^{k+1} e^{-2k^2c^2},$$

which is the limiting distribution function of the two-sided Kolmogorov-Smirnov statistic

$$\sqrt{n} \sup_{-\infty<x<\infty} |\ F_n(x) - F(x)\ |,$$

where $F_n(.)$ is the empirical distribution function of a continuous distribution function $F(.)$(cf. Doob (1949)).

Remark 4.4. Hall and Wellner(1980, (2.9)) also derived (4.5) and denoted it by $G_b(c)$.

We will now extend Theorem E to the general case of a > 0.

Theorem 4.2. Let { W(t) : t ≥ 0 } be a Wiener process. Let $\tau_1 > 0$, $\tau_2 < 0$, $a > 0$, $\tau_2+\delta_2 b \le \tau_1+\delta_1 b$. Then we have

(4.7) $$P\{\ \tau_2+\delta_2 t \le W(t) \le \tau_1+\delta_1 t : a \le t \le b\ \}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{q''/\sqrt{a}}^{q'/\sqrt{a}} \Gamma(\ q''-\sqrt{a}h, \delta_2, q'-\sqrt{a}h, \delta_1, b-a\ )\ e^{-\frac{1}{2}h^2}\, dh,$$

where $q' = \tau_1 + \delta_1 a$, $q'' = \tau_2+\delta_2 a$ and

(4.8) $\Gamma(u',v', u, v, w) = P\{ u'+v't \le W(t) \le u + vt : 0\le t\le w \}$,

which has been discussed in (2.9) in Theorem E.

Proof. Similary to the proof of Theorem 3.2, applying that, for fixed a, W(s+a)-W(a) and W(a) are two independent random variables and W(s+a)-W(a) = W(s) in distribution, we get

$$P\{ \tau_2+\delta_2 t \le W(t) \le \tau_1+\delta_1 t : a\le t\le b \}$$

$$= P\{ \tau_2+\delta_2(t+a) \le W(t+a) \le \tau_1+\delta_1(t+a) : 0\le t\le b-a \}$$

$$= P\{ \tau_2+\delta_2(t+a)-W(a) \le W(t+a)-W(a) \le \tau_1+\delta_1(t+a)-W(a) : 0\le t\le b-a \}$$

$$= \int_{-\infty}^{\infty} P\{ \tau_2+\delta_2(t+a)-h \le W(t) \le \tau_1+\delta_1(t+a)-h : 0\le t\le b-a \}\, dP\{W(a)\le h\}.$$

Now we use (3.5), apply Theorem E and observe that Theorem E is applicable only if $h \ge q''$ and $h \le q'$, where $q' = \tau_1 + \delta_1 a$, $q'' = \tau_2+\delta_2 a$. Then

$$P\{ \tau_2+\delta_2 t \le W(t) \le \tau_1+\delta_1 t : a\le t\le b \}$$

$$= \frac{1}{\sqrt{2\pi a}} \int_{q''}^{q'} \Gamma(q''-h,\delta_2,q'-h,\delta_1,b-a)\, e^{-h^2/2a}\, dh,$$

where $\Gamma(u',v', u, v, w) = P\{ u'+v't \le W(t) \le u + v\,t : 0 \le t \le w \}$.

If we set $x = h/\sqrt{a}$, then we obtain the assertion of the theorem.▒

Remark 4.5. As $a \rightarrow 0$ in (4.7), (4.7) becomes (2.9).

Remark 4.6. For $\delta_1 = \delta_2 = 0$ in (4.7), we obtain

(4.9) $P\{\ \tau_2 \le W(t) \le \tau_1 : a\le t\le b\ \}$

$$= \frac{1}{\sqrt{2\pi}} \int_{\tau_2/\sqrt{a}}^{\tau_1/\sqrt{a}} \Gamma'(\ \tau_2-\sqrt{a}h, \tau_1-\sqrt{a}h, b-a\ )\ e^{-\frac{1}{2}h^2}\, dh,$$

where $\Gamma'(u',u,w\ ) = P\{\ u' \le W(t) \le u : 0\le t\le w\ \}$. $\Gamma'(.)$ can be obtained by either (4.2) or (4.3).

Remark 4.7. For $\tau_1 = -\tau_2 = \tau\ (\ > o\ )$, $\delta_1 = -\delta_2 = \delta$ and $\tau + \delta b \ge 0$, we have

(4.10) $P\{\ |W(t)| \le \tau+\delta t : a\le t\le b\ \}$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{q/\sqrt{a}} \Gamma(\ -q-\sqrt{a}h, \delta, q-\sqrt{a}h, \delta, b-a\ )\ e^{-\frac{1}{2}h^2}\, dh,$$

where $q = \tau + \delta a$. If $\delta = 0$ in (4.10), we obtain

(4.11) $P\{\ |W(t)| \le \tau : a\le t\le b\ \}$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\tau/\sqrt{a}} \Gamma'(\ -\tau-\sqrt{a}h, \tau-\sqrt{a}h, b-a\ )\ e^{-\frac{1}{2}h^2}\, dh.$$

From Remark 4.7, we have the following corollary for a Brownian bridge process.

Corollary 4.2. Let $\{ B(s) : 0 \le s \le 1 \}$ be a Brownian bridge process. Let $0 \le a < b < 1$ and $c > 0$. Then

$$(4.12) \qquad P\{ |B(t)| \le c : a \le t \le b \}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\mu/\sqrt{a'}} \Gamma(-\mu-\sqrt{a'}h, -c, \mu-\sqrt{a'}h, c, b'-a')\, e^{-\frac{1}{2}h^2}\, dh,$$

where $\mu = c(1+a')$, $a' = a/(1-a)$, $b' = b/(1-b)$ and $\Gamma(.)$ is defined in (4.8).

Proof. (4.12) is an immediate consequence of (4.10).▒

Remark 4.8. For $b = 1$, if we apply the symmetricity of Brownian bridge process, and then apply (4.5) in Corollary 4.1, we obtain a formula similar to that of (4.5).

Remark 4.9. Anderson and Darling (1952, (5.9)) discuss the probability in Corollary 4.2 and show that

$$(4.13) \quad P\{ |B(t)| \le c : a \le t \le b \}$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k e^{-2c^2k^2} M(2kc\sqrt{a'}, 2kc\sqrt{b'}, c\sqrt{a'}/a, c\sqrt{b'}/b, -\sqrt{a'b'}) ,$$

where $M(u,v,u',v',\rho)$ is the volume under the bivariate normal distribution surface with zero means, unit variances and correlation ρ which is above the rectangle with vertices $x = u \pm u'$ and $y = v \pm v'$.

5. Tables.

Using the computer programs WIENER PACK by Chung (1986) based upon (4.12), tables of the critical points τ of the distribution of $\sup_{(a\le s\le b)} |B(s)|$, for selected values of a, b and probability levels, are computed and presented in Tables 1 – 12.

The selected probability levels are:

$$P\{ \sup_{a\le s\le b} |B(s)| \le \tau \} = 0.99,\ 0.975,\ 0.95,\ 0.9,\ 0.8,\ 0.7,\ 0.6,\ 0.5,\ 0.4,\ 0.3,\ 0.2 \text{ and } 0.1.$$

Table 1. Critical points τ of the distribution of $\sup\limits_{a\le s\le b} |B(s)|$ for selected values of a and b, and

$$P\{ \sup_{a\le s\le b} |B(s)| \le \tau \} = 0.990.$$

a	b=0.1	b=0.2	b=0.3	b=0.4	b=0.5	b=0.6	b=0.7	b=0.8	b=0.9	b=1.0
0.00	0.85120	1.15054	1.34186	1.46960	1.55194	1.59958	1.62137	1.62723	1.62762	1.62762
0.05	0.85098	1.15054	1.34186	1.46960	1.55194	1.59958	1.62137	1.62723	1.62762	1.62762
0.10		1.15002	1.34183	1.46960	1.55194	1.59958	1.62137	1.62723	1.62762	1.62762
0.15		1.14061	1.34079	1.46939	1.55188	1.59955	1.62134	1.62720	1.62759	1.62759
0.20			1.33398	1.46744	1.55103	1.59903	1.62094	1.62683	1.62723	1.62723
0.25			1.31039	1.46054	1.54770	1.59686	1.61920	1.62518	1.62561	1.62561
0.30				1.44360	1.53949	1.59140	1.61469	1.62094	1.62137	1.62137
0.35				1.40686	1.52341	1.58072	1.60590	1.61267	1.61313	1.61313
0.40					1.49527	1.56293	1.59140	1.59903	1.59958	1.59958
0.45					1.44604	1.53568	1.56976	1.57886	1.57956	1.57956
0.50						1.49527	1.53949	1.55103	1.55194	1.55194
0.55						1.43314	1.49854	1.51434	1.51572	1.51572
0.60							1.44360	1.46744	1.46960	1.46960
0.65							1.36646	1.40842	1.41226	1.41226
0.70								1.33398	1.34183	1.34183
0.75								1.23639	1.25577	1.25577
0.80									1.15002	1.15002
0.85									1.01639	1.01639
0.90										1.01639
0.95										1.01639

Table 2. Critical points τ of the distribution of $\sup_{a \le s \le b} |B(s)|$ for selected values of a and b, and

$$P\{ \sup_{a \le s \le b} |B(s)| \le \tau \} = 0.975.$$

a	b=0.1	b=0.2	b=0.3	b=0.4	b=0.5	b=0.6	b=0.7	b=0.8	b=0.9	b=1.0
0.00	0.75893	1.02821	1.20243	1.32095	1.39966	1.44759	1.47165	1.47943	1.48021	1.48021
0.05	0.75840	1.02821	1.20243	1.32095	1.39966	1.44759	1.47165	1.47943	1.48021	1.48021
0.10		1.02707	1.20233	1.32092	1.39966	1.44759	1.47165	1.47943	1.48021	1.48021
0.15		1.01395	1.20029	1.32037	1.39943	1.44745	1.47156	1.47934	1.48012	1.48012
0.20			1.19067	1.31712	1.39786	1.44646	1.47079	1.47864	1.47943	1.47943
0.25			1.16263	1.30772	1.39297	1.44321	1.46817	1.47621	1.47702	1.47702
0.30				1.28744	1.38248	1.43605	1.46233	1.47079	1.47165	1.47165
0.35				1.24693	1.36376	1.42343	1.45201	1.46121	1.46217	1.46217
0.40					1.33290	1.40369	1.43605	1.44646	1.44759	1.44759
0.45					1.28114	1.37468	1.41333	1.42564	1.42702	1.42702
0.50						1.33290	1.38248	1.39786	1.39966	1.39966
0.55						1.26985	1.34163	1.36211	1.36462	1.36462
0.60							1.28744	1.31712	1.32092	1.32092
0.65							1.21161	1.26106	1.26736	1.26736
0.70								1.19067	1.20233	1.20233
0.75								1.09789	1.12349	1.12349
0.80									1.02707	1.02707
0.85									0.90476	0.90476
0.90										0.90476
0.95										0.90476

Table 3. Critical points τ of the distribution of $\sup_{a \le s \le b} |B(s)|$ for selected values of a and b, and

$$P\{ \sup_{a \le s \le b} |B(s)| \le \tau \} = 0.950.$$

a	b=0.1	b=0.2	b=0.3	b=0.4	b=0.5	b=0.6	b=0.7	b=0.8	b=0.9	b=1.0
0.00	0.68248	0.92685	1.08682	1.19755	1.27307	1.32108	1.34708	1.35678	1.35809	1.35809
0.05	0.68141	0.92684	1.08682	1.19755	1.27307	1.32108	1.34708	1.35678	1.35809	1.35809
0.10		0.92473	1.08655	1.19749	1.27306	1.32107	1.34707	1.35677	1.35809	1.35809
0.15		0.90770	1.08316	1.19638	1.27255	1.32078	1.34686	1.35659	1.35790	1.35790
0.20			1.07057	1.19154	1.27003	1.31914	1.34558	1.35544	1.35677	1.35677
0.25			1.03841	1.17961	1.26345	1.31464	1.34196	1.35214	1.35353	1.35353
0.30				1.15622	1.25076	1.30583	1.33480	1.34558	1.34707	1.34707
0.35				1.11239	1.22961	1.29139	1.32307	1.33483	1.33649	1.33649
0.40					1.19637	1.26987	1.30583	1.31914	1.32107	1.32107
0.45					1.14240	1.23934	1.28214	1.29778	1.30013	1.30013
0.50						1.19637	1.25076	1.27003	1.27306	1.27306
0.55						1.13249	1.20989	1.23502	1.23911	1.23911
0.60							1.15622	1.19154	1.19749	1.19749
0.65							1.08141	1.13783	1.14712	1.14712
0.70								1.07057	1.08655	1.08655
0.75								0.98154	1.01364	1.01364
0.80									0.92473	0.92473
0.85									0.81138	0.81138
0.90										0.81138
0.95										0.81138

Table 4. Critical points τ of the distribution of $\sup_{a \le s \le b} |B(s)|$ for selected values of a and b, and

$$P\{ \sup_{a \le s \le b} |B(s)| \le \tau \} = 0.9.$$

a	b=0.1	b=0.2	b=0.3	b=0.4	b=0.5	b=0.6	b=0.7	b=0.8	b=0.9	b=1.0
0.00	0.59852	0.81547	0.95967	1.06164	1.13340	1.18126	1.20937	1.22157	1.22383	1.22383
0.05	0.59631	0.81542	0.95967	1.06164	1.13340	1.18127	1.20937	1.22157	1.22383	1.22383
0.10		0.81144	0.95893	1.06144	1.13332	1.18122	1.20934	1.22155	1.22381	1.22381
0.15		0.78909	0.95327	1.05923	1.13222	1.18055	1.20885	1.22112	1.22339	1.22339
0.20			0.93664	1.05198	1.12813	1.17780	1.20670	1.21922	1.22155	1.22155
0.25			0.89946	1.03670	1.11918	1.17152	1.20167	1.21470	1.21714	1.21714
0.30				1.00952	1.10368	1.16057	1.19279	1.20670	1.20934	1.20934
0.35				0.96180	1.07963	1.14389	1.17933	1.19458	1.19753	1.19753
0.40					1.04358	1.12029	1.16057	1.17780	1.18122	1.18122
0.45					0.98707	1.08796	1.13572	1.15585	1.15998	1.15998
0.50						1.04358	1.10368	1.12813	1.13332	1.13332
0.55						0.97874	1.06272	1.09388	1.10070	1.10070
0.60							1.00952	1.05198	1.06144	1.06144
0.65							0.93570	1.00068	1.01461	1.01461
0.70								0.93664	0.95893	0.95893
0.75								0.85148	0.89241	0.89241
0.80									0.81144	0.81144
0.85									0.70742	0.70742
0.90										0.70742
0.95										0.70742

Table 5. Critical points τ of the distribution of $\sup_{a \le s \le b} |B(s)|$ for selected values of a and b, and

$$P\{ \sup_{a \le s \le b} |B(s)| \le \tau \} = 0.8.$$

a	b=0.1	b=0.2	b=0.3	b=0.4	b=0.5	b=0.6	b=0.7	b=0.8	b=0.9	b=1.0
0.00	0.50445	0.69053	0.81678	0.90855	0.97563	1.02293	1.05326	1.06868	1.07266	1.07266
0.05	0.49972	0.69027	0.81676	0.90855	0.97563	1.02293	1.05326	1.06868	1.07266	1.07266
0.10		0.68257	0.81475	0.90784	0.97532	1.02276	1.05314	1.06859	1.07258	1.07258
0.15		0.65279	0.80515	0.90340	0.97285	1.02117	1.05196	1.06759	1.07163	1.07163
0.20			0.78291	0.89241	0.96614	1.01648	1.04830	1.06441	1.06859	1.06859
0.25			0.73944	0.87260	0.95381	1.00760	1.04119	1.05815	1.06259	1.06259
0.30				0.84066	0.93465	0.99378	1.03004	1.04830	1.05314	1.05314
0.35				0.78826	0.90697	0.97429	1.01440	1.03451	1.03995	1.03995
0.40					0.86755	0.94812	0.99378	1.01648	1.02276	1.02276
0.45					0.80803	0.91363	0.96754	0.99385	1.00132	1.00132
0.50						0.86755	0.93465	0.96614	0.97532	0.97532
0.55						0.80151	0.89346	0.93267	0.94434	0.94434
0.60							0.84066	0.89241	0.90784	0.90784
0.65							0.76783	0.84361	0.86503	0.86503
0.70								0.78291	0.81475	0.81475
0.75								0.70181	0.75512	0.75512
0.80									0.68257	0.68257
0.85									0.58836	0.58836
0.90										0.58836
0.95										0.58836

Table 6. Critical points τ of the distribution of $\sup_{a \le s \le b} |B(s)|$ for selected values of a and b, and

$$P\{ \sup_{a \le s \le b} |B(s)| \le \tau \} = 0.7.$$

a	b=0.1	b=0.2	b=0.3	b=0.4	b=0.5	b=0.6	b=0.7	b=0.8	b=0.9	b=1.0
0.00	0.44305	0.60881	0.72309	0.80784	0.87147	0.91802	0.94958	0.96726	0.97286	0.97286
0.05	0.43549	0.60815	0.72299	0.80782	0.87146	0.91802	0.94958	0.96726	0.97286	0.97286
0.10		0.59669	0.71934	0.80633	0.87073	0.91760	0.94930	0.96704	0.97266	0.97266
0.15		0.56109	0.70618	0.79964	0.86676	0.91494	0.94731	0.96539	0.97112	0.97112
0.20			0.67958	0.78552	0.85775	0.90851	0.94227	0.96106	0.96704	0.96704
0.25			0.63158	0.76228	0.84278	0.89756	0.93350	0.95344	0.95985	0.95985
0.30				0.72689	0.82091	0.88161	0.92068	0.94227	0.94930	0.94930
0.35				0.67121	0.79064	0.86010	0.90351	0.92735	0.93525	0.93525
0.40					0.74886	0.83213	0.88161	0.90851	0.91760	0.91760
0.45					0.68726	0.79612	0.85439	0.88545	0.89617	0.89617
0.50						0.74886	0.82091	0.85775	0.87073	0.87073
0.55						0.68196	0.77950	0.82478	0.84095	0.84095
0.60							0.72689	0.78552	0.80633	0.80633
0.65							0.65463	0.73824	0.76616	0.76616
0.70								0.67958	0.71934	0.71934
0.75								0.60100	0.66403	0.66403
0.80									0.59669	0.59669
0.85									0.50848	0.50848
0.90										0.50848
0.95										0.50848

Table 7. Critical points τ of the distribution of $\sup_{a \le s \le b} |B(s)|$ for selected values of a and b, and

$$P\{ \sup_{a \le s \le b} |B(s)| \le \tau \} = 0.6.$$

a	b=0.1	b=0.2	b=0.3	b=0.4	b=0.5	b=0.6	b=0.7	b=0.8	b=0.9	b=1.0
0.00	0.39563	0.54554	0.65031	0.72933	0.78993	0.83556	0.86782	0.88724	0.89439	0.89439
0.05	0.38496	0.54424	0.65005	0.72926	0.78990	0.83555	0.86781	0.88723	0.89439	0.89439
0.10		0.52899	0.64448	0.72673	0.78858	0.83475	0.86727	0.88682	0.89401	0.89401
0.15		0.48835	0.62802	0.71780	0.78302	0.83094	0.86440	0.88444	0.89183	0.89183
0.20			0.59767	0.70091	0.77192	0.82288	0.85806	0.87905	0.88682	0.88682
0.25			0.54591	0.67477	0.75469	0.81015	0.84788	0.87029	0.87865	0.87865
0.30				0.63655	0.73061	0.79247	0.83370	0.85806	0.86727	0.86727
0.35				0.57821	0.69825	0.76936	0.81534	0.84230	0.85266	0.85266
0.40					0.65456	0.73998	0.79247	0.82288	0.83475	0.83475
0.45					0.59128	0.70278	0.76454	0.79955	0.81345	0.81345
0.50						0.65456	0.73061	0.77192	0.78858	0.78858
0.55						0.58697	0.68904	0.73938	0.75982	0.75982
0.60							0.63655	0.70091	0.72673	0.72673
0.65							0.56470	0.65480	0.68864	0.68864
0.70								0.59767	0.64448	0.64448
0.75								0.52096	0.59243	0.59243
0.80									0.52899	0.52899
0.85									0.44525	0.44525
0.90										0.44525
0.95										0.44525

Table 8. Critical points τ of the distribution of $\sup_{a \le s \le b} |B(s)|$ for selected values of a and b, and

$$P\{ \sup_{a \le s \le b} |B(s)| \le \tau \} = 0.5.$$

a	b=0.1	b=0.2	b=0.3	b=0.4	b=0.5	b=0.6	b=0.7	b=0.8	b=0.9	b=1.0
0.00	0.35589	0.49232	0.58886	0.66274	0.72044	0.76495	0.79753	0.81831	0.82695	0.82695
0.05	0.34190	0.49015	0.58833	0.66257	0.72038	0.76492	0.79751	0.81830	0.82695	0.82695
0.10		0.47117	0.58064	0.65879	0.71827	0.76361	0.79661	0.81760	0.82633	0.82633
0.15		0.42605	0.56111	0.64766	0.71110	0.75858	0.79279	0.81445	0.82347	0.82347
0.20			0.52749	0.62831	0.69808	0.74900	0.78522	0.80805	0.81760	0.81760
0.25			0.47245	0.59970	0.67893	0.73473	0.77381	0.79830	0.80863	0.80863
0.30				0.55908	0.65301	0.71562	0.75851	0.78522	0.79661	0.79661
0.35				0.49844	0.61894	0.69124	0.73921	0.76882	0.78161	0.78161
0.40					0.57366	0.66077	0.71562	0.74900	0.76361	0.76361
0.45					0.50895	0.62264	0.68720	0.72554	0.74255	0.74255
0.50						0.57366	0.65301	0.69808	0.71827	0.71827
0.55						0.50548	0.61140	0.66601	0.69049	0.69049
0.60							0.55908	0.62831	0.65879	0.65879
0.65							0.48757	0.58327	0.62252	0.62252
0.70								0.52749	0.58064	0.58064
0.75								0.45235	0.53136	0.53136
0.80									0.47117	0.47117
0.85									0.39115	0.39115
0.90										0.39115
0.95										0.39115

Table 9. Critical points τ of the distribution of $\sup_{a\le s\le b} |B(s)|$ for selected values of a and b, and

$$P\{ \sup_{a\le s\le b} |B(s)| \le \tau \} = 0.4.$$

a	b=0.1	b=0.2	b=0.3	b=0.4	b=0.5	b=0.6	b=0.7	b=0.8	b=0.9	b=1.0
0.00	0.32067	0.44495	0.53388	0.60286	0.65762	0.70077	0.73332	0.75515	0.76524	0.76524
0.05	0.30324	0.44165	0.53293	0.60252	0.65748	0.70070	0.73328	0.75512	0.76522	0.76522
0.10		0.41914	0.52300	0.59729	0.65440	0.69872	0.73189	0.75404	0.76428	0.76428
0.15		0.37011	0.50072	0.58406	0.64561	0.69241	0.72705	0.75006	0.76071	0.76071
0.20			0.46434	0.56259	0.63088	0.68143	0.71834	0.74272	0.75404	0.75404
0.25			0.40649	0.53196	0.61011	0.66585	0.70584	0.73210	0.74438	0.74438
0.30				0.48939	0.58275	0.64559	0.68964	0.71834	0.73189	0.73189
0.35				0.42681	0.54739	0.62026	0.66964	0.70147	0.71666	0.71666
0.40					0.50091	0.58904	0.64559	0.68143	0.69872	0.69872
0.45					0.43501	0.55034	0.61694	0.65802	0.67801	0.67801
0.50						0.50091	0.58275	0.63088	0.65440	0.65440
0.55						0.43231	0.54135	0.59940	0.62763	0.62763
0.60							0.48939	0.56259	0.59729	0.59729
0.65							0.41831	0.51871	0.56275	0.56275
0.70								0.46434	0.52300	0.52300
0.75								0.39075	0.47629	0.47629
0.80									0.41914	0.41914
0.85									0.34258	0.34258
0.90										0.34258
0.95										0.34258

Table 10. Critical points τ of the distribution of $\sup_{a \le s \le b} |B(s)|$ for selected values of a and b, and

$$P\{ \sup_{a \le s \le b} |B(s)| \le \tau \} = 0.3.$$

a	b=0.1	b=0.2	b=0.3	b=0.4	b=0.5	b=0.6	b=0.7	b=0.8	b=0.9	b=1.0
0.00	0.28779	0.40049	0.48201	0.54603	0.59763	0.63910	0.67127	0.69385	0.70536	0.70536
0.05	0.26693	0.39580	0.48044	0.54540	0.59735	0.63895	0.67118	0.69379	0.70531	0.70531
0.10		0.37009	0.46822	0.53853	0.59309	0.63611	0.66915	0.69221	0.70395	0.70395
0.15		0.31803	0.44359	0.52337	0.58271	0.62850	0.66324	0.68735	0.69963	0.69963
0.20			0.40513	0.50016	0.56648	0.61624	0.65344	0.67910	0.69221	0.69221
0.25			0.34523	0.46806	0.54446	0.59958	0.64003	0.66773	0.68198	0.68198
0.30				0.42422	0.51611	0.57848	0.62314	0.65344	0.66915	0.66915
0.35				0.36033	0.48001	0.55258	0.60271	0.63630	0.65384	0.65384
0.40					0.43295	0.52106	0.57848	0.61624	0.63611	0.63611
0.45					0.36642	0.48229	0.54993	0.59309	0.61591	0.61591
0.50						0.43295	0.51611	0.56648	0.59309	0.59309
0.55						0.36441	0.47534	0.53583	0.56743	0.56743
0.60							0.42422	0.50016	0.53853	0.53853
0.65							0.35403	0.45773	0.50578	0.50578
0.70								0.40513	0.46822	0.46822
0.75								0.33350	0.42413	0.42413
0.80									0.37009	0.37009
0.85									0.29722	0.29722
0.90										0.29722
0.95										0.29722

Table 11. Critical points τ of the distribution of $\sup_{a\le s\le b} |B(s)|$ for selected values of a and b, and

$$P\{ \sup_{a\le s\le b} |B(s)| \le \tau \} = 0.2.$$

a	b=0.1	b=0.2	b=0.3	b=0.4	b=0.5	b=0.6	b=0.7	b=0.8	b=0.9	b=1.0
0.00	0.25503	0.35592	0.42967	0.48831	0.53628	0.57559	0.60694	0.62995	0.64285	0.64285
0.05	0.23085	0.34953	0.42723	0.48721	0.53573	0.57529	0.60676	0.62983	0.64275	0.64275
0.10		0.32114	0.41271	0.47852	0.53008	0.57137	0.60387	0.62757	0.64082	0.64082
0.15		0.26742	0.38628	0.46164	0.51816	0.56243	0.59683	0.62174	0.63568	0.63568
0.20			0.34671	0.43719	0.50071	0.54904	0.58603	0.61264	0.62757	0.62757
0.25			0.28623	0.40438	0.47788	0.53157	0.57189	0.60066	0.61688	0.61688
0.30				0.36034	0.44915	0.51003	0.55457	0.58603	0.60387	0.60387
0.35				0.29654	0.41310	0.48405	0.53403	0.56884	0.58869	0.58869
0.40					0.36651	0.45282	0.51003	0.54904	0.57137	0.57137
0.45					0.30067	0.41475	0.48204	0.52644	0.55188	0.55188
0.50						0.36651	0.44915	0.50071	0.53008	0.53008
0.55						0.29931	0.40970	0.47128	0.50575	0.50575
0.60							0.36034	0.43719	0.47852	0.47852
0.65							0.29224	0.39679	0.44782	0.44782
0.70								0.34671	0.41271	0.41271
0.75								0.27816	0.37158	0.37158
0.80									0.32114	0.32114
0.85									0.25277	0.25277
0.90										0.25277
0.95										0.25277

Table 12. Critical points τ of the distribution of $\sup_{a\le s\le b} |B(s)|$ for selected values of a and b, and

$$P\{ \sup_{a\le s\le b} |B(s)| \le \tau \} = 0.1.$$

a	b=0.1	b=0.2	b=0.3	b=0.4	b=0.5	b=0.6	b=0.7	b=0.8	b=0.9	b=1.0
0.00	0.21817	0.30544	0.36996	0.42193	0.46517	0.50136	0.53110	0.55404	0.56831	0.56831
0.05	0.19096	0.29687	0.36616	0.42000	0.46409	0.50072	0.53069	0.55375	0.56809	0.56809
0.10		0.26667	0.34945	0.40929	0.45670	0.49534	0.52661	0.55050	0.56532	0.56532
0.15		0.21374	0.32209	0.39106	0.44336	0.48504	0.51832	0.54358	0.55925	0.55925
0.20			0.28305	0.36617	0.42513	0.47077	0.50665	0.53369	0.55050	0.55050
0.25			0.22484	0.33390	0.40221	0.45295	0.49207	0.52130	0.53953	0.53953
0.30				0.29151	0.37411	0.43160	0.47477	0.50665	0.52661	0.52661
0.35				0.23078	0.33948	0.40638	0.45469	0.48982	0.51187	0.51187
0.40					0.29528	0.37652	0.43160	0.47077	0.49534	0.49534
0.45					0.23314	0.34053	0.40502	0.44931	0.47700	0.47700
0.50						0.29528	0.37411	0.42513	0.45670	0.45670
0.55						0.23236	0.33731	0.39771	0.43425	0.43425
0.60							0.29151	0.36617	0.40929	0.40929
0.65							0.22832	0.32899	0.38131	0.38131
0.70								0.28305	0.34945	0.34945
0.75								0.22012	0.31223	0.31223
0.80									0.26667	0.26667
0.85									0.20483	0.20483
0.90										0.20483
0.95										0.20483

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