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DEPARTMENT OF ENERGY, MINES AND RESOURCES

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COMPUTER PROGRAM FOR THE ANALYSIS OF MULTIVARIATE SERIES AND EIGENVALUE ROUTINE FOR ASYMMETRICAL MATRICES

F. P. Agterberg and G. D. Cameron

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ABSTRACT

A computer program is presented by which the transition matrix for a multivariate series can be calculated. The solution is based on the assumption that the series satisfies a Markov scheme of the first order. By means of the eigenvalue routine, the transition matrix is divided into separate components of two types: (1) components consisting of real numbers only, and (2) components consisting of complex numbers.

The trend factors extracted from components of the first type indicate linear combinations of the variates with a variation pattern that is close to a smooth curve when the eigenvalue for the component is close to unity.

Components of the second type occur in conjugate pairs and may describe cyclical variations of the variates.

COMPUTER PROGRAM FOR THE ANALYSIS OF MULTIVARIATE SERIES AND EIGENVALUE ROUTINE FOR ASYMMETRICAL MATRICES

INTRODUCTION

This computer manual is largely based on a method discussed by Quenouille (1957) for the analysis of multiple time series with econometric applications. The practical example used by Quenouille at the end of his monograph will also be used as an example in this manual.

Apart from a listing of the program and operational instructions, this paper contains a discussion of the theory that underlies the method and emphasis is put on those aspects which may be useful for the analysis of multivariate series in geology.

The number of computations that will be carried out by the program and the methods that will be used are controlled by a header card that contains ten different indexes. For instance, the program can be used to calculate means and standard deviations for data on a number of variates and to produce a graphical plot of the standard deviates. A job can be discontinued when these calculations have been made. One or more out of sixteen different methods can be chosen to compute the transition matrix for the series.

The program is an extension of techniques used by one of us to determine the dominant components of real eigenvalue from the transition matrix (Agterberg, 1966). When there are p variates, p separate components can now be extracted from the transition matrix by sending it to the eigenvalue routine.

Some of the components may form pairs of conjugate matrices consisting of complex numbers. A pair of conjugate complex components indicates that there is an oscillatory constituent in the multivariate system. Thus, cyclical variations for groups of variates can be isolated from the trends in the system. Average values for the periodicity and possible shifts in phase angle for individual variates may be computed. This method of analyzing the oscillatory constituent in a geological multivariate series has not been discussed before. The isolation of an oscillatory constituent from the system is done with the help of an application of Sylvester's theorem that is fully discussed by Frazer, Duncan, and Collar (1958).

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Although this results in numerically prècise estimates of an oscillatory constituent, little is known about the interpretation of the results. The application to an artificial example will be discussed in this paper. However, this manual, in the first place, shows how the necessary computations can be carried out using a digital computer. The method should be applied to more cases before definite conclusions on its points of advantage and limitation can be drawn.

The program was written for the CDC 3100 computer with 16K memory. The maximum number of variates is eight; the maximum number of simultaneous observations is 100. Problems with more variates and observations can be treated by changing the dimension statements but more memory core will be necessary in that case. A machine method for calculating the eigenvalues and eigenvectors of an asymmetrical matrix is discussed by Francis (1961), who uses the QR transformation. The latter method was programmed for the CDC 1604 and 3600 computers by the University of Wisconsin Computing Center.

The following theoretical parts will successively deal with the formation of the transition matrix which is computed in the main part of the program, and the extraction of the components from the transition matrix which is performed by the eigenvalue routine.

FORMATION OF TRANSITION MATRIX

A multivariate series can be represented by an array of data $X_{i,k}$ where the subscripts i and k denote variate and position of observation, respectively. When there are n observations and p variates, $X_{i,k}$ consists of n columns $\underline{x}_k$ and pxn elements $x_{i,k}$.

A series $X_{i,k}$ possesses the Markov property when each of its observations can be predicted from the observation which precedes it in the series. The transition matrix U by which the prediction is carried out is computed from the entire series. The Markov schemes used in this paper are linear. By including more than one previous observation in the prediction, higher order Markov schemes could be constructed. Such schemes will not be considered here.

The matrix $X_{i,k}$ consists of the observations for the variates. It may be advisable to apply some transformation to the raw data before the computations are carried out. The program of this paper has the logarithmic transformation as an option. When a transformation is applied, the transformed data will be called $X_{i,k}$ instead of the raw data.

The transition matrix can be computed either from the deviations from the mean or from the standard deviates for the variates. These deviations and standard deviates will be called $y_{i,k}$ and $z_{i,k}$, respectively. For all values of i and k:

$$y_{i,k} = x_{i,k} - \bar{x}_i ,$$

and $$z_{i,k} = \frac{y_{i,k}}{s(y_i)} = \frac{x_{i,k} - \bar{x}_i}{s(x_i)} ,$$

where the bar symbol represents the mean and s the standard deviation:

$$s(x_i) = s(y_i) = \sqrt{\frac{\Sigma y^2_{i,k}}{n-1}} .$$

Graphical plots of the $z_{i,k}$ values can be obtained by the program for successive values of i.

The $y_{i,k}$ values form a matrix $Y_{i,k}$, and the $z_{i,k}$ values form $Z_{i,k}$. For $Z_{i,k}$, the Markov scheme consists of the following p equations:

$$\left.\begin{array}{l} z_{1,k+1} = u_{11}z_{1,k} + u_{12}z_{2,k} + \cdots\cdots + u_{1p}z_{p,k} + e_{1,k+1} \\ z_{2,k+1} = u_{21}z_{1,k} + u_{22}z_{2,k} + \cdots\cdots + u_{2p}z_{p,k} + e_{2,k+1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ z_{p,k+1} = u_{p1}z_{1,k} + u_{p2}z_{2,k} + \cdots\cdots + u_{pp}z_{p,k} + e_{p,k+1} \end{array}\right\} \quad (1).$$

The values $e_{i,k+1}$ are random numbers.

The set of equations (1) may also be written as

$$\underline{z}_{k+1} = U\underline{z}_k + \underline{e}_{k+1} \qquad (2),$$

where $\underline{z}_{k+1}$, $\underline{z}_k$, and $\underline{e}_{k+1}$ are column vectors as in Eq. (1), and the transition matrix U consists of pxp elements u_{ij}.

The transpose of $\underline{z}_k$ is a row vector that will be denoted by $\underline{z}_k'$. When both sides of Eq. (2) are postmultiplied by $\underline{z}_k'$ and when the results are summed for the entire series, it follows that

$$\sum_{k=1}^{n} \underline{z}_{k+1}\underline{z}'_k = U \sum_{k=1}^{n} \underline{z}_k\underline{z}'_k .$$

The correlation matrix R_0 for $Z_{i,k}$ satisfies:

$$R_0 = {}^1/_{n-1} \sum_{k=1}^{n} \underline{z}_k \underline{z}_k' \qquad (3).$$

By correlating each variate to all p variates from the observations that are one place behind in the series, a correlation matrix R_1 for lag 1, is obtained with

$$R_1 = {}^1/_{n-1} \sum_{k=1}^{n} \underline{z}_{k+1} \underline{z}_k' \qquad (4).$$

It follows that Eq. (2) may be written as

$$R_1 = UR_0$$

from which U can be solved by

$$U = R_1 R_0^{-1} \qquad (5).$$

In the previous derivation, the matrix $Z_{i,k}$ can be replaced by $Y_{i,k}$ with slightly different results. When the variance-covariance matrices that correspond to R_0 and R_1, are called C_0 and C_1, respectively, the resulting estimate for U becomes $C_1 C_0^{-1}$.

Further estimates of U can be obtained by computation of the matrix R_2.

$$R_2 = {}^1/_{n-1} \sum_{k=1}^{n} \underline{z}_{k+2} \underline{z}_k' \qquad (6)$$

leading to the estimate $U = R_2 R_1^{-1}$.
This estimate can be obtained by the present computer program as well as the estimate $C_2 C_1^{-1}$. In general, $U = R_s R_{s-1}^{-1}$ or $U = C_s C_{s-1}^{-1}$, with $s = 1, 2, 3, \ldots s$. For possible advantages of using estimates of U with $s>1$, see Quenouille (1957).

End corrections

These are required before Eq. (4) can be applied. By defining the cyclical scheme with $\underline{x}_{n+1} = \underline{x}_1$ and $\underline{x}_{n+2} = \underline{x}_2$, Eq. (4) can be maintained in the form as it was reported before. A visual appreciation of the graphical plot of $Z_{i,k}$ may learn whether the application of a cyclical scheme is reasonable or not. In general, the following equations for the lagged correlation matrices should be preferred (Quenouille, 1957, p. 51):

$$r_{(ij)s} = \frac{\sum_{k=1}^{n-s} x_{i,k+s}x_{j,k} - \left(\sum_{k=1}^{n-s} x_{i,k+s}\right)\left(\sum_{k=1}^{n-s} x_{j,k}\right) \Big/ (n-s)}{\sqrt{\left[\left\{\sum_{k=1}^{n-s} x_{i,k+s}^2 - \left(\sum_{k=1}^{n-s} x_{i,k+s}\right)^2 \Big/ (n-s)\right\}\left\{\sum_{k=1}^{n-s} x_{j,k}^2 - \left(\sum_{k=1}^{n-s} x_{j,k}\right)^2 \Big/ (n-s)\right\}\right]}} \quad (7)$$

with $s = 1$ and $s = 2$.
Either this end correction or that for the cyclical scheme can be obtained by the program.

EXTRACTION OF COMPONENTS FROM TRANSITION MATRIX

In its canonical form, the matrix U becomes:

$$U = V\Lambda V^{-1} \quad (8).$$

The matrix Λ is a diagonal matrix with the eigenvalues of U along its diagonal. V consists of p columns $\underline{v}_i$ which are the eigenvectors of U. The subscript i indicates the eigenvalue λ_i to which $\underline{v}_i$ corresponds. The λ_i are defined so that their magnitude decreases when i increases. The inverse of V will be written as T with p rows $\underline{t}_i'$.

Eq. (8) may also be written as

$$U = \lambda_1\underline{v}_1\underline{t}_1' + \lambda_2\underline{v}_2\underline{t}_2' + \ldots\ldots + \lambda_i\underline{v}_i\underline{t}_i' + \ldots\ldots + \lambda_p\underline{v}_p\underline{t}_p' \quad (9),$$

or $U = \sum_{i=1}^{p} \lambda_i\underline{v}_i\underline{t}_i'$

The component U_i for root λ_i is defined by $U = \lambda_i\underline{v}_i\underline{t}_i'$.

Hence $U = \sum_{i=1}^{p} U_i$ which can be used to check the precision of the computations once all components U_i have been computed.

Raising U to the power s results in

$$U^s = \lambda_1^s\underline{v}_1\underline{t}_1' + \lambda_2^s\underline{v}_2\underline{t}_2' + \ldots\ldots + \lambda_p^s\underline{v}_p\underline{t}_p' \quad (10).$$

The previous equations apply to both real and imaginary values for λ_i. When the largest root λ_1 is real, U^s will converge to the form $\lambda_1^s \underline{v}_1 \underline{t}_1'$. It may also occur that U^s converges to $\lambda_1^s \underline{v}_1 \underline{t}_1' + \lambda_2^s \underline{v}_2 \underline{t}_2'$ when λ_1 and λ_2 form a conjugate pair of complex roots with

$$\lambda_1 = \mu + i\,\omega \text{ and } \lambda_2 = \mu - i\,\omega$$

Both possibilities will be considered.

If the dominant root is real, the elements of U^{s+1} can be divided by the elements of U^s and the elements of the resulting matrix will all be equal to λ_1, when convergence has been reached. In that case, U_1 is found from $U_1 = U^s/\lambda_1^{s-1}$, and $\underline{v}_1$ and $\underline{t}_1'$ can be readily extracted from the matrix U^s/λ_1^s.

When U_1 is subtracted from U, raising of the difference $U-U_1$ to a high power will yield λ_2 when this is a real number. All components can be successively estimated, provided that the method is extended to account for the case of complex roots.

In the latter case, the coefficients of the vectors $\underline{v}_2$ and $\underline{t}_2'$ are conjugate to those of $\underline{v}_1$ and $\underline{t}_1'$. The same applies to the elements of U_1 and U_2 in this case. The sum $U_1 + U_2$ consists of real elements and will be written as $U_{1,2}$.

In order to determine $U_{1,2}$, $\underline{v}_1$ and $\underline{t}_1'$, use is made of Sylvester's theorem. When all roots are distinct, Sylvester's theorem for any polynomial of U is:

$$P(U) = \sum_{i=1}^{p} P(\lambda_i) \;\; Z_0(\lambda_i) \qquad (11),$$

where $Z_0(\lambda_i)$ is the square matrix

$$Z_0(\lambda_i) = \prod_{j \neq i} (\lambda_j I - U) \Big/ \prod_{j \neq i} (\lambda_j - \lambda_i) \qquad (12).$$

I represents the pxp unit matrix.

The equations (9) and (10) are special cases of (11) with $Z_0(\lambda_i) = \underline{v}_i \underline{t}_i'$ and $P(U) = U$ and $P(U) = U^s$, respectively.

When the roots of greatest modulus are $\lambda_1 = \mu + i\,\omega$ and $\lambda_2 = \mu - i\,\omega$, P(U) can be chosen as

$$P(U) = U^s P_0(U) \qquad (13)$$

with $P_0(U) = (\lambda_1 I - U)(\lambda_2 I - U)$.

$P_0(U)$ is independent of s.

For large s:

$$U^s P_0(U) = \lambda_1^s P_0(\lambda_1) Z_0(\lambda_1) + \lambda_1^s P_0(\lambda_2) Z_0(\lambda_2) \quad \text{or}$$

$$U^s(\lambda_1 I - U)\ (\lambda_2 I - U) = N \qquad (14),$$

where N is the pxp null matrix.

As in the case for a real dominant root, this relationship applies to all elements of U^s. Hence, with Eq. (13), it is found for an element e_s of U^s that

$$(\mu^2 + \omega^2) e_s - 2\mu e_{s+1} + e_{s+2} = 0 \qquad (15a),$$

where e_{s+1} and e_{s+2} refer to elements of U^{s+1} and U^{s+2} that take the same position as e_s in U^s. For another element indicated by f:

$$(\mu^2 + \omega^2)\ f_s - 2\mu f_{s+1} + f_{s+2} = 0 \qquad (15b).$$

The modulus $r = \sqrt{\mu^2 + \omega^2}$ or

$$r = \sqrt{\frac{e_{s+1}\ f_{s+2} - f_{s+1}\ e_{s+2}}{e_s f_{s+1} - f_s e_{s+1}}} \qquad (16)$$

can be tested for convergence.

When convergence has been reached, μ and ω are solved by Eqs. (15a) and (15b).

The component $U_{1,2}$ is found as follows. When in Eq. (13), $P_0(U)$ is put equal to $P_0(U) = \lambda_2 I - U$, for large s:

$$Z_0(\lambda_1) = \frac{U^s(\lambda_2 I - U)}{\lambda_1^s(\lambda_2 - \lambda_1)} \qquad (17).$$

The elements of $Z_0(\lambda_2)$ are the conjugates of those of $Z_0(\lambda_1)$. From $Z_0(\lambda_1) = \underline{v}_1\underline{t}'_1$ and $Z_0(\lambda_2) = \underline{v}_2\underline{t}'_2$, it follows that the first rows of $Z_0(\lambda_1)$ and $Z_0(\lambda_2)$ can be used as estimates for $\underline{t}'_1$ and $\underline{t}'_2$, respectively. The first columns of these matrices can be used to obtain corresponding estimates for $\underline{v}_1$ and $\underline{v}_2$ after the elements of these columns are divided by the first element. This method was followed by Quenouille (1957) and these estimates of $\underline{t}'_1$, $\underline{t}'_2$, $\underline{v}_1$, and $\underline{v}_2$ will be referred to as "Quenouille's estimates".

In this computation, the value λ_1^s is readily computed by writing λ_1 in its polar form:

$$\lambda_1 = r(\cos\theta + i\sin\theta)$$

with $\theta = \arctan \omega/\mu$.

Hence: $$\lambda_1^s = r^s(\cos s\theta + i\sin s\theta).$$

The combined components $U_{1,2} = \lambda_1 Z_0(\lambda_1) + \lambda_2 Z_0(\lambda_2)$ are subtracted from U. When the difference is raised to a high power, it will converge either to a form determined by U_3 when the dominant root is real, or to a form determined by $U_{3,4}$ which designates the combined components for a pair of complex roots. It is defined that $U_{i,i+1}$ refers to the combined components for the pair of complex roots λ_i and λ_{i+1}.

Method of programming used in the eigenvalue routine

The process of convergence for higher powers can be very slow when successive eigenvalues are approximately equal to one another. The following scheme of powering results in precise solutions for the eigenvalues in most instances.

From U, the following matrices are successively computed: $U^2, U^4, U^8, U^{16}, U^{32}, U^{64}, U^{128}, U^{256}, U^{512}, U^{1024}, \ldots\ldots$ In consequence, very high powers of U can be reached within a relatively short time.

For each power U^s, the matrices U^{s+1}, U^{s+2}, and U^{s+3} are also computed. When e and f refer to the first two elements of the first rows of these matrices, the values $p_1 = e_{s+1}/e_s$ and $p_2 = e_{s+3}/e_{s+2}$ can be computed. When convergence is satisfactory:

$$\Delta p = \left| p_1 - p_2 \right| < c_R,$$

where c_R is a small positive constant that can be entered on the header card of the program (columns 31-40).

When $p > c_R$, values r_1 and r_2 can be computed as follows:

$$r_1 = \sqrt{\left| \frac{e_{s+1} f_{s+2} - f_{s+1} e_{s+2}}{e_s f_{s+1} - f_s e_{s+1}} \right|} \quad (16a),$$

and r_2 is the expression for r_1 with s replaced by s+1.

When $\Delta r = \left| r_1 - r_2 \right| < c_I$, convergence for the case of two complex roots is accepted. c_I can also be entered on the header card (columns 41-50). If no values are entered in columns 31-50 of the header card, c_R and c_I are put equal to .000005 and .0000005, respectively.

On the CDC 3100 computer, the exponent of decimal numbers should have a resultant between 10^{-308} and 10^{308}. In order to avoid overflow of the exponent, which may happen for very high powers of U, all elements of U^s are divided by the first element of U^s (e_s) when $e_s > 150$. Subsequent computations are corrected using the relationship

$$(cU)^s = c^s U^s,$$

where c is a constant.

Standardized estimates of trend vectors and eigenvectors

The rows of the T-matrix ($\underline{t}_i'$) are referred to as trend vectors, and the columns of the V-matrix ($\underline{v}_i$) are called the eigenvectors. In both cases of a real root and of a pair of complex roots, the trend vector and eigenvector are first estimated by putting the first coefficient of v_i equal to 1. In the case of a pair of complex roots, these estimates are printed and labelled "Quenouille's estimates".

When λ_i is real, the linear combination $\underline{t}_i' z$ is called the ith trend factor and the individual values of $\underline{t}_i' z_k$ are the trend factor scores for $\underline{t}_i' z$.

Before they are printed out, the trend vector and the eigenvector are standardized by multiplication by $1/_{STF}$ and STF, respectively. STF represents the standard deviation of the trend factor scores. The latter are also multiplied by $1/_{STF}$ before they are printed out and appear in the graphical plot of the program.

When λ_i and λ_{i+1} form a pair of complex roots, the Quenouille's estimates of the real and imaginary parts of the trend vector ($\underline{t}'_{iR}$ and $\underline{t}'_{iI}$) are divided by STFR and STFI, respectively. The latter two standard deviations apply to the values of the "real trend factor scores" ($\underline{t}'_R \underline{z}_k$) and the "imaginary trend factor scores" ($\underline{t}'_I \underline{z}_k$), respectively. Real and imaginary trend factor scores are listed and plotted using standard scale. The real and imaginary parts of the eigenvector are multiplied by 2xSTFR and -2xSTFI, respectively.

When there are p variates, there are p patterns of trend factor scores which are shown graphically. The coefficients by which the individual elements describe these patterns are given by the coefficients of the eigenvectors.

In the case of complex roots, the elements describe the pattern of the real trend factor scores by the coefficients of the real part of the (standardized) eigenvector, and the pattern of the imaginary trend factor scores by those of the imaginary part of the eigenvector. Summarizing, it may be said that the variation pattern for the multivariate series Z_k is analysed in terms of the trend factors by using the following equation.

When q of the p eigenvalues are real, each observation $\underline{z}_k$ is divided into

$$\underline{z}_k = \sum_{q} \underline{v}_i (\underline{t}'_i \underline{z}_k) + \sum_{\frac{p-q}{2}} \left[\underline{v}_{jR} (\underline{t}'_{jR} \underline{z}_k) + \underline{v}_{jI} (\underline{t}'_{jI} \underline{z}_k) \right] \qquad (18),$$

$\underline{z}_k$ = column vector for observation k

$\underline{v}_i$ = eigenvector for real root λ_i

$(\underline{t}'_i \underline{z}_k)$ = trend factor score for real root λ_i

$\underline{v}_{jR}$ = real part of eigenvector for complex roots λ_j and λ_{j+1} (j labels pairs of complex roots while i goes from 1 to p)

$(\underline{t}'_{jR} \underline{z}_k)$ = real trend factor score for complex roots λ_j and λ_{j+1}

$\underline{v}_{jI}$ = imaginary part of eigenvector (λ_j and λ_{j+1})

$(\underline{t}'_{jI} \underline{z}_k)$ = imaginary trend factor score (λ_j and λ_{j+1}).

INTERPRETATION

Eq. (18) is equivalent to

$$\underline{z}_k = V(T\underline{z}_k), \text{ because } T = V^{-1} \qquad (19).$$

From Eq. (2), it follows that

$$\left.\begin{aligned} \underline{z}_{k+1} &= V\Lambda(T\underline{z}_k) + \underline{e}_{k+1} \\ \underline{z}_{k+s} &= V\Lambda^s(T\underline{z}_k) + \underline{e}_{k+s} \end{aligned}\right\} \qquad (20).$$

When both sides of these equations are premultiplied by T, it follows that for each trend factor i

$$\left.\begin{aligned} \underline{t}_i'\underline{z}_{k+1} &= \lambda_i(\underline{t}_i'\underline{z}_k) + \underline{e}^*_{k+1} \\ \underline{t}_i'\underline{z}_{k+s} &= \lambda_i^s(\underline{t}_i'\underline{z}_k) + \underline{e}^*_{k+s} \end{aligned}\right\} \qquad (21),$$

where the values of $\underline{e}^*$ are random numbers. When the expression E(..) denotes 'expectation', it follows that

$$\left.\begin{aligned} E(\underline{t}_i'\underline{z}_{k+1}) &= \lambda_i(\underline{t}_i'\underline{z}_k) \\ E(\underline{t}_i'\underline{z}_{k+s}) &= \lambda_i^s(\underline{t}_i'\underline{z}_k) \end{aligned}\right\} \qquad (22).$$

These relationships hold also true when i refers to a complex root.

When λ_i is real, it represents the first serial correlation coefficient of a one-variate Markov scheme. As a check for the precision of λ_i, the first serial correlation coefficient for the series of $\underline{t}_i'\underline{z}_k$ can be computed.

The first trend factor $\underline{t}_1'\underline{z}$ represents the linear combination of the p variates that, out of all possible linear combinations, has the strongest serial correlation or 'most' trend in its pattern of $\underline{t}_1'\underline{z}_k$ - values. When λ_2 is also real, the second trend factor represents the linear combination with most trend for a reduced system from which the effect of the first trend factor has been eliminated.

Linear combinations of the variates that result in a relatively smooth pattern may have physical significance. In geological multivariate series, the measured variates may have been controlled by physical agencies that were subject to a gradual change. Such controlling agencies may be linearly related to the first one or more trend factors.

The individual variates follow a trend factor with amplitudes given by the coefficients of the eigenvector.

It may not be possible to assign direct geological significance to the coefficients for the individual variates in the trend vectors and eigenvectors when approximate linear relationships between the variates do exist. For example, when there is such a relationship $\underline{a}'\underline{z}_k \approx 0$ for all observations k, with $\underline{a}'$ being a row vector, then $\underline{t}_i'\underline{z}_k + c\underline{a}'\underline{z}_k$, where c is an arbitrary constant, may have nearly the same variation pattern as $\underline{t}_i'\underline{z}_k$, but the coefficients of the new linear relationship may be quite different. Approximate linear relationships are likely to show up in the estimated coefficients of trend vectors and eigenvectors. As yet, no solution has been found to eliminate their effect.

Examples of geological series with a first trend factor of real root that may have physical significance are discussed by one of us (Agterberg, 1966).

For a trend factor of real root, the expectation $E(\underline{t}_i'\underline{z}_k)$ decreases exponentially when s increases for all points k. Hence, the expectations $E(\underline{v}_{ij}\underline{t}_i'\underline{z}_k)$ of all variates j also show an exponential decrease for increasing s.

When the roots form a complex pair, the following interpretations can be made. If

$$\mu \pm i\omega = r(\cos\theta \pm i\sin\theta),$$

and

$$\underline{t}_R'\underline{z}_k \pm i\underline{t}_I'\underline{z}_k = p_k \left[\cos\psi_k \pm i\sin\psi_k\right]$$

it follows that

$$E\left(\underline{t}_R'\underline{z}_{k+s} \pm i\underline{t}_I'\underline{z}_{k+s}\right) = (\mu \pm i\omega)^s \left(\underline{t}_R'\underline{z}_k \pm i\underline{t}_I'\underline{z}_k\right)$$

results in:

$$\left.\begin{aligned} E\left(\underline{t}_R'\underline{z}_{k+s}\right) &= r^s\, p_k \cos(\theta s + \psi_k) \\ E\left(\underline{t}_I'\underline{z}_{k+s}\right) &= r^s\, p_k \sin(\theta s + \psi_k) \end{aligned}\right\} \qquad (23).$$

The expressions for the expected real and imaginary parts of the trend factor are the same, except for a shift in phase angle of 90°. The expression $r^s p_k \cos(\theta s + \psi_k)$ consists of the parts $r^s p_k$, which reflects an exponential decrease that is different for each point k, and a set of stationary oscillations

$\cos(\theta s + \psi_k)$, with a phase given by ψ_k that is different for each point k, but angular frequency θ that is independent of position along the series. The corresponding periodicity T is equal to $T = 2\pi/\theta$. It represents the number of sampling intervals over which an oscillation is completed.

When the eigenvectors $\underline{v}_R \pm i\underline{v}_I$ are written as $\underline{q}(\cos \underline{\varphi} \pm i \sin \underline{\varphi})$, where $\underline{q}$ and $\underline{\varphi}$ are column vectors consisting of the elements q_j and φ_j for the variates j, the pattern which the variate j describes for the components $U_{j,j+1}$ is

$$\begin{aligned} z_{j,k+s} &= r^s p_k q_j \cos(\theta s + \psi_k + \varphi_j) + i \sin(\theta s + \psi_k + \varphi_j) + \\ &+ r^s p_k q_j \cos(\theta s + \psi_k + \varphi_j) - i \sin(\theta s + \psi_k + \varphi_j) \\ &= 2 r^s p_k q_j \cos(\theta s + \psi_k + \varphi_j) \qquad (24). \end{aligned}$$

This is the pattern of the real part of the trend factor

$$r^s p_k \cos(\theta s + \psi_k)$$

with the following two additions:

(1) a shift in phase angle equal to φ_j

(2) a factor of $2q_j$ added to the amplitude.

Artificial example

The previous interpretation for an oscillatory constituent in the multivariate system is illustrated by the following example. In Table 1, a three-variate series is listed which was obtained as follows:

$$x_{1,k} = \sin(10k - 10)° + e_1(0, 0.5)$$

$$x_{2,k} = \sin(10k - 30)° + e_2(0, 0.5)$$

$$x_{3,k} = \sin(10k - 60)° + e_3(0, 0.5) \text{ with} = 1, 2, \ldots, 36.$$

The random numbers e (0, 0.5) come from a normal distribution with zero mean and standard deviation equal to 0.5.

$x_{i,k}$ describes a complete sine-curve that begins at k=1. A random normal residual was added to each of the 36 individual values which are 10° apart. $x_{2,k}$ and $x_{3,k}$ describe similar sine-curves but with shifts in phase angle of 20° and 50°, respectively.

TABLE 1

Input listing for three-variate series of artificial example (AREX)

AREX 300000101010001010101 .000005 .0000005

1	.23200	-.24502	-.84704
2	.24215	.42235	-.71079
3	1.56952	-.25050	.01650
4	.33850	.03715	-.24052
5	.60879	1.03902	.05035
6	.91404	.22250	.37400
7	.72203	.61979	-.03785
8	1.58869	.92654	.12602
9	1.10531	2.33853	.15500
10	.52150	1.92669	1.02079
11	1.01481	.85581	-.04296
12	-.32331	1.20600	.69353
13	.60053	1.20431	.68419
14	.66904	.92219	-.04069
15	.91429	1.31903	.77150
16	-.27900	.50954	.87581
17	.43552	.38029	1.36819
18	-.42135	.79750	.63353
19	.01100	.78252	1.45204
20	.08885	-.29335	.75529
21	.40098	.78950	.68900
22	-.67700	-.09315	.72252
23	-.95979	-1.28452	.26415
24	-.41754	-.31450	-.36800
25	-.40303	-.05329	.30635
26	-.25219	-1.29354	-1.10702
27	-.59231	-.86253	-.63000
28	-1.48150	-.55519	-.58279
29	-1.41131	-.49931	-1.00704
30	-1.87219	-.64400	-.02703
31	-.35503	-.43981	-.96819
32	-.99704	-1.25519	-1.59931
33	-.01229	-.99353	-1.24300
34	1.260N	-1.11704	-.55681
35	-.05652	-1.39329	-1.18519
36	-1.09915	-.74400	-1.85753

The purpose of the statistical analysis is to determine the periodicity of the system and the phase differences between the three variates.

The transition matrix as computed from R_0 and R_1 (cyclical scheme) is:

$$U = \begin{bmatrix} .41 & .39 & -.20 \\ .21 & .67 & -.05 \\ -.04 & .49 & .39 \end{bmatrix}$$

The first two eigenvalues of U appear to form a complex pair with

$$\lambda_{1,2} = .665 \pm .084i$$

It follows that r = .670, and θ = .126 (radians).
From $T = 2\pi/\theta$, it follows that the periodicity T = 49.9.
This value is higher than the periodicity of 36 sampling intervals assigned to the series before the random residuals were added.

The Quenouille's estimates of $\underline{t}_R$, $\underline{t}_I$, $\underline{v}_R$, and $\underline{v}_I$ are

$$\underline{t}_R = \begin{bmatrix} .221 \\ .337 \\ -.150 \end{bmatrix}; \underline{t}_I = \begin{bmatrix} -.233 \\ -.152 \\ .242 \end{bmatrix}; \underline{v}_R = \begin{bmatrix} 1.000 \\ 1.735 \\ 2.048 \end{bmatrix}; \underline{v}_I = \begin{bmatrix} .000 \\ -1.266 \\ -.286 \end{bmatrix}$$

From $\underline{v}_R \pm i\underline{v}_I$, the vectors $\underline{q}$ and $\underline{\varphi}$ can be estimated

$$\underline{q} = \begin{bmatrix} 1.000 \\ 2.147 \\ 3.514 \end{bmatrix} \text{ and } \underline{\varphi} = \begin{bmatrix} .000 \\ -.630 \\ -.949 \end{bmatrix}$$

where φ_j is reported in radians.

For the original series without random residuals the shift in phase angle of x_2 and x_3 with respect to x_1 are 20° and 50°, respectively, or

$$\underline{\varphi}_0 = \begin{bmatrix} .000 \\ -.349 \\ -.873 \end{bmatrix}$$

When U is estimated from R_1 and R_2, the computed periodicity is T = 44.3 and

$$\underline{\varphi}_2 = \begin{bmatrix} .000 \\ -.471 \\ -.895 \end{bmatrix}$$

In evaluating the results obtained by the method discussed here, in general, two types of error will occur:

(1) estimation errors due to the fact that the series is not infinitely large;

(2) errors inherent to the model that assumes a first order Markov scheme for the series, which, in most cases, is an approximation only.

Comparison to factor analysis

The linear model that is generally used in factor analysis (Harman, 1964, p. 16) is

$$
\begin{aligned}
z_1 &= a_{11}F_1 + a_{12}F_2 + \dots a_{1m}F_m + a_1U_1 \\
z_2 &= a_{21}F_1 + a_{22}F_2 + \dots a_{2m}F_m + a_2U_2 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
z_p &= a_{p1}F_1 + a_{p2}F_2 + \dots a_{pm}F_m + a_pU_p
\end{aligned}
\qquad (25)
$$

where $m \leq p$. Each variate z_j is expressed in terms of m common factors F_i and its unique factor U_j. The common factors are linear combinations of the p variates and have unity variance. Hence, they can be compared to the trend factors and the coefficients a_{ji} to the coefficients of the eigenvectors. In the method of Markov schemes there is no equivalent for the unique factors.

The coefficients a_{ji} and a_i in the factor model are computed from the correlation matrix R_0 as follows. The unit values along the leading diagonal of R_0 are replaced by the communalities $h_j^2 = 1 - a_j^2$, which are estimated after making an assumption on the number of relevant factors present in the system (m). The resulting modified correlation matrix R is divided into separate components similar to the components extracted from U in this paper. This leads to an initial factor matrix solution consisting of p x m elements, which is generally subjected to some rotation in order to make it possible to assign direct physical meaning to the m common factors and the coefficients a_{ji}. Methods of factor analysis have been refined extensively. Similar refinements are not available for the method of Markov schemes.

The usefulness of the present method, which applies to series only, is twofold:

(1) The first one or more trend factors may provide estimates of the linear combinations of the variates that show a relatively smooth variation pattern. The first trend factor represents the linear combination that has the largest first serial correlation coefficient that is possible.

(2) When oscillatory constituents are present in the system, they may be computed and described in terms of the complex components. When more than one of the variates are subject to an oscillatory constituent but when there are shifts in phase angle between the variates, these properties of the system can be described by the present method as it was illustrated by the preceding analysis of an artificial example.

APPLICATION TO QUENOUILLE'S PRACTICAL EXAMPLE (U.S. HOG SERIES)

Appendix I shows the input and part of the output obtained by the program for the practical examples discussed by Quenouille (1957, pp. 88-101). In the input, the five variates of Quenouille's Table 8. 1a are listed by year. In the graphical plots of the output, the 82 observations are coded 1-82. The fine columns of the input consist of transformed values for number of hogs ($x_{1,k}$), price of hogs ($x_{2,k}$), price of corn ($x_{3,k}$), supply of corn ($x_{4,k}$), and farm wage rate ($x_{5,k}$), respectively.

The computed values for the oscillatory constituent are close to those obtained by Quenouille who stated: "... this approach involves considerable computation with as many as five variates. While it may be possible to do this rapidly on an electronic computer, no programme was available for this purpose...".

The oscillatory constituent corresponds to the third and fourth eigenvalues. The oscillations can be seen in the graphical plots for the real and imaginary trend factor scores. A shift in phase angle of about T/4 or 90° as predicted by Eq. (23) between the two patterns seems to be present. In the graphical plots, the trend factor scores are shown as xes. Their five-point moving average is indicated by asterisks.

The listings for the example are followed by a listing of the computer program (Appendix II).

OPERATIONAL INSTRUCTIONS

The logical flow of the program is determined by a set of 10 indexes which are read from the header card of each data set. The indexes are summarized in the listing of the source program and in the flow chart (Fig. 1). The indexes 3, 5, 9, and 10 are "stopping" indexes, inasmuch as if any of them is zero, a problem is terminated at this point.

Input Formats

Each problem is defined by one header card which must be brought in from the card reader whether or not the data cards are on magnetic tape. Its format is as follows:

cols. 1-4 (R4) alpha-numeric identification;

cols. 5-6 (I2) NVAR = number of variates per observation
(maximum number allowed in present compilation is 8);

cols. 7-26 (10I2) the 10 indexes in 10 2-col. fields;

cols. 27-30 not used;

cols. 31-40 (F10.0) value of tolerance for convergence in case of dominant real roots. If this field is left blank, a value of .000005 is assumed.

cols. 41-50 (F10.0) value of tolerance for convergence in case of dominant pair of imaginary roots. If this field is left blank, a value of .0000005 is assumed.

Columns 27-30 and 51-80 are not read and may be punched with any alpha-numeric information.

The data cards are formated as follows: (If magnetic tape is used, the records on tape must be in this card image format)

cols. 1-4 (R4) any non-blank alpha-numeric identification, may differ from card to card;

cols. 5-14, 15-24, 25-34, 35-44, 45-54, 55-64, 65-74 (7F10.1) values of the variates, in order, up to 7. In case 8 variates are used, a second card follows with F10.1 in cols. 1-10.

For each observation, there are thus 2 cards (or card images on tape) if 8 variates are used, and 1 card (or card image) if 7 or less variates are used.

Concluding remarks

The program will accept up to 100 observations per problem, and up to 8 variates per observation. This restriction is imposed by the amount of memory core on the computer on which the program was compiled. For other computers, this may be changed by altering the dimension statements in the main program and all subroutines. It is noted that most doubly dimensioned variates are 9x9. This is because the matrix inversion subroutine used requires an extra row and extra column for scratch storage, and this must be allowed for in the dimension statements.

There is no limit on the number of problems that may be run simultaneously. Each problem is independent of all others and its flow is controlled by the information of the header card. When the input is from cards only, each individual job is followed by the same number of blank cards as there are cards per observation in that job. Therefore, if there are 8 variates per observation, so that each observation requires two cards, then two blank cards must follow that problem before the header card of the next problem. If there are 7 or less variates per observation, a problem is followed by a single blank card only. When the input data are from magnetic tape, each problem is separated by an end of file check.

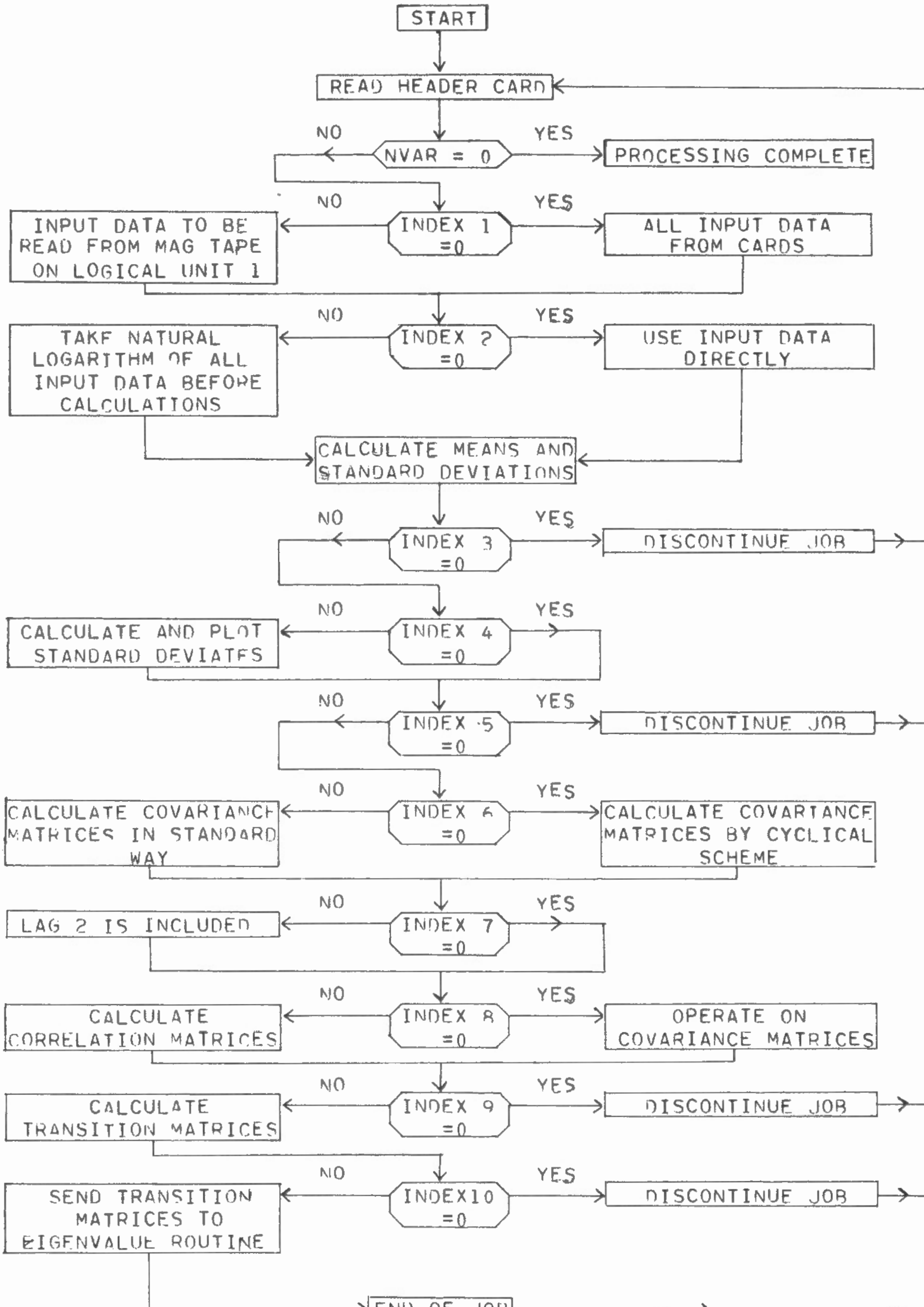


Figure 1. Flow chart for usage of the indexes.

In order to terminate processing at the end of a complete run, one final dummy header card is inserted which is actually a blank card. Therefore, when there are 8 variates per observation, a run is terminated when there are 3 blank cards following the last problem. When there are 7 or less variates, there are 2 blank cards at the end.

The input formats are controlled by format statement No. 1 for the header card, and No. 2 for the data cards. The input formats can be altered by changing these format cards in the main program.

In subroutine ROOT, 6 statements beyond statement 1540, the exponent of the first element of U^s is compared to 150. This number was chosen because all decimal numbers in the matrices U^s-U^{s+3} must have a resultant between 10^{-308} and 10^{308} (see theoretical part). For other computers, a number different from 150 may be used.

In subroutine ROOT, 1 statement beyond statement 1590, the power s of U^s is compared to 10,000. When $s > 10{,}000$, a job is discontinued. The purpose of this test is to safeguard the computer from unlimited powering when the structure of the matrix U would be such that U^s will not converge.

It may be assumed that the results obtained by subroutine ROOT are numerically precise when the elements of the check sum matrix printed at the end of a problem solution are nearly equal to those of the original transition matrix (see theoretical part).

In the graphical plots of the output, the x-es represent the standard deviates, and the asterisks their 5-point moving average values. Values equal to or larger than 4 and equal to or less than -4 are plotted as 4 and -4, respectively.

REFERENCES

Agterberg, F.P.
1966: The use of multivariate Markov schemes in petrology; J. Geol., vol. 74, pp. 764-786.

Francis, J.G.
1961: The QR transformation; Computer J., 4, pp. 265-271, and 332-340.

Frazer, R.A., Duncan, W.J., and Collar, A.R.
1960: Elementary matrices; Cambridge University Press.

Harman, H.H.
1960: Modern factor analysis; The University of Chicago Press.

Quenouille, M.H.
1957: The analysis of multiple time-series; London, Charles Griffin and Co.

APPENDIX I

LISTING OF INPUT AND OUTPUT

LISTING OF INPUT

TEST 500000101010101010101 .000005 .0000005

1867	.538	.597	.944	.900	.722	1908	.766	.777	.813	1.409	.971
1868	.522	.509	.841	.964	.719	1909	.720	.810	.790	1.417	.982
1869	.513	.663	.911	.893	.716	1910	.682	.957	.712	1.455	.987
1870	.529	.751	.768	1.051	.724	1911	.743	.970	.831	1.394	.991
1871	.565	.739	.718	1.057	.732	1912	.743	.903	.742	1.469	1.004
1872	.594	.598	.634	1.107	.740	1913	.730	.995	.847	1.357	1.013
1873	.600	.556	.735	1.003	.748	1914	.723	1.022	.850	1.402	1.004
1874	.584	.594	.858	1.025	.756	1915	.753	.998	.830	1.452	1.013
1875	.554	.667	.673	1.161	.748	1916	.782	.928	1.056	1.385	1.053
1876	.553	.776	.609	1.170	.740	1917	.760	1.073	1.163	1.464	1.149
1877	.595	.754	.604	1.181	.732	1918	.799	1.294	1.182	1.388	1.248
1878	.637	.689	.457	1.194	.744	1919	.808	1.346	1.180	1.428	1.316
1879	.641	.498	.612	1.244	.756	1920	.779	1.301	.805	1.487	1.384
1880	.647	.643	.642	1.232	.778	1921	.770	1.134	.714	1.467	1.190
1881	.634	.681	.849	1.095	.799	1922	.777	1.024	.865	1.432	1.179
1882	.629	.778	.733	1.244	.799	1923	.841	1.090	.911	1.459	1.228
1883	.638	.829	.672	1.218	.799	1924	.823	1.013	1.027	1.347	1.238
1884	.662	.751	.594	1.289	.799	1925	.746	1.119	.846	1.447	1.246
1885	.675	.704	.559	1.313	.801	1926	.717	1.195	.869	1.406	1.253
1886	.658	.633	.604	1.251	.803	1927	.744	1.235	.928	1.418	1.253
1887	.629	.663	.678	1.205	.806	1928	.791	1.120	.924	1.426	1.253
1888	.625	.709	.571	1.352	.806	1929	.771	1.112	.903	1.401	1.255
1889	.648	.763	.490	1.361	.806	1930	.746	1.129	.777	1.318	1.223
1890	.682	.681	.747	1.218	.810	1931	.739	1.055	.507	1.411	1.114
1891	.676	.627	.651	1.368	.813	1932	.773	.787	.500	1.467	.982
1892	.655	.667	.645	1.278	.810	1933	.793	.624	.716	1.380	.929
1893	.640	.804	.609	1.279	.806	1934	.768	.612	.911	1.161	.978
1894	.668	.782	.705	1.208	.771	1935	.592	.800	.816	1.362	1.013
1895	.678	.707	.453	1.404	.771	1936	.633	1.104	1.019	1.178	1.045
1896	.692	.653	.382	1.427	.780	1937	.634	1.075	.714	1.422	1.100
1897	.710	.639	.466	1.359	.789	1938	.649	1.052	.687	1.406	1.097
1898	.727	.672	.506	1.371	.799	1939	.699	1.048	.754	1.412	1.090
1899	.712	.669	.525	1.423	.820	1940	.786	.891	.791	1.390	1.100
1900	.708	.729	.595	1.425	.834	1941	.735	.921	.876	1.424	1.188
1901	.705	.784	.829	1.234	.848	1942	.782	1.193	.962	1.487	1.303
1902	.680	.842	.654	1.443	.863	1943	.869	1.352	1.050	1.472	1.422
1903	.682	.886	.673	1.401	.884	1944	.923	1.243	1.037	1.490	1.498
1904	.713	.784	.691	1.429	.906	1945	.774	1.314	1.104	1.458	1.544
1905	.726	.770	.660	1.470	.928	1946	.787	1.380	1.193	1.507	1.582
1906	.729	.783	.643	1.482	.949	1947	.754	1.556	1.334	1.372	1.607
1907	.752	.877	.754	1.417	.960	1948	.737	1.632	1.114	1.557	1.629

LISTING OF OUTPUT

STATISTICAL CALCULATIONS, PROBLEM TEST

MEANS OF VARIABLES ARE	.6993	.8913	.7755	1.3309	.9923
NUMBER OF OBSERVATIONS =	82				
STANDARD DEVIATIONS	.0828	.2510	.1957	.1485	.2411

CORRELATION MATRIX FOR PROBLEM NO. TEST LAG 0

1.000000	.624657	.413292	.784448	.743583
.624657	1.000000	.694763	.604086	.936911
.413292	.694763	1.000000	.123505	.726147
.784448	.604086	.123505	1.000000	.644287
.743583	.936911	.726147	.644287	1.000000

CORRELATION MATRIX FOR PROBLEM NO. TEST LAG 1

.882213	.704979	.314650	.857087	.729603
.608775	.911370	.791858	.554969	.922649
.481069	.584031	.789942	.299037	.686003
.756952	.648000	.266776	.797292	.644681
.770076	.906962	.756500	.626532	.983494

CORRELATION MATRIX FOR PROBLEM NO. TEST LAG 2

.752910	.733025	.271900	.831348	.721675
.676795	.764758	.806204	.529380	.874007
.535011	.506796	.608310	.380313	.644564
.704661	.650280	.252177	.806954	.633375
.775177	.854948	.712431	.649055	.948642

TRANSITION MATRICES FOR TEST

01

.640569	.564681	-0.109731	.262215	-0.365030
-0.218209	.207562	.396971	.279970	.421798
-0.166916	-0.678957	.838887	.337143	.619870
.432635	.617214	-0.031675	.408002	-0.495163
.107207	-0.070007	.124010	.004916	.876149

02

-1.448340	-0.392648	-0.325431	2.281432	.908104
-1.543117	-0.742281	.789587	1.713792	1.055650
-2.571517	-1.736446	.926737	2.714435	1.766346
-1.667863	-0.394263	.029634	2.713180	.452014
-1.354412	-0.765254	.211525	1.500618	1.556043

REAL ROOT NO. 1 AFTER POWER 64 ROOT = 9.77927194E-01

TREND VECTOR = T(J) / STF

2.44208E-01	2.08064E-01	3.42308E-01	4.17409E-01	4.04604E-02

EIGENVECTOR = V(I) * STF

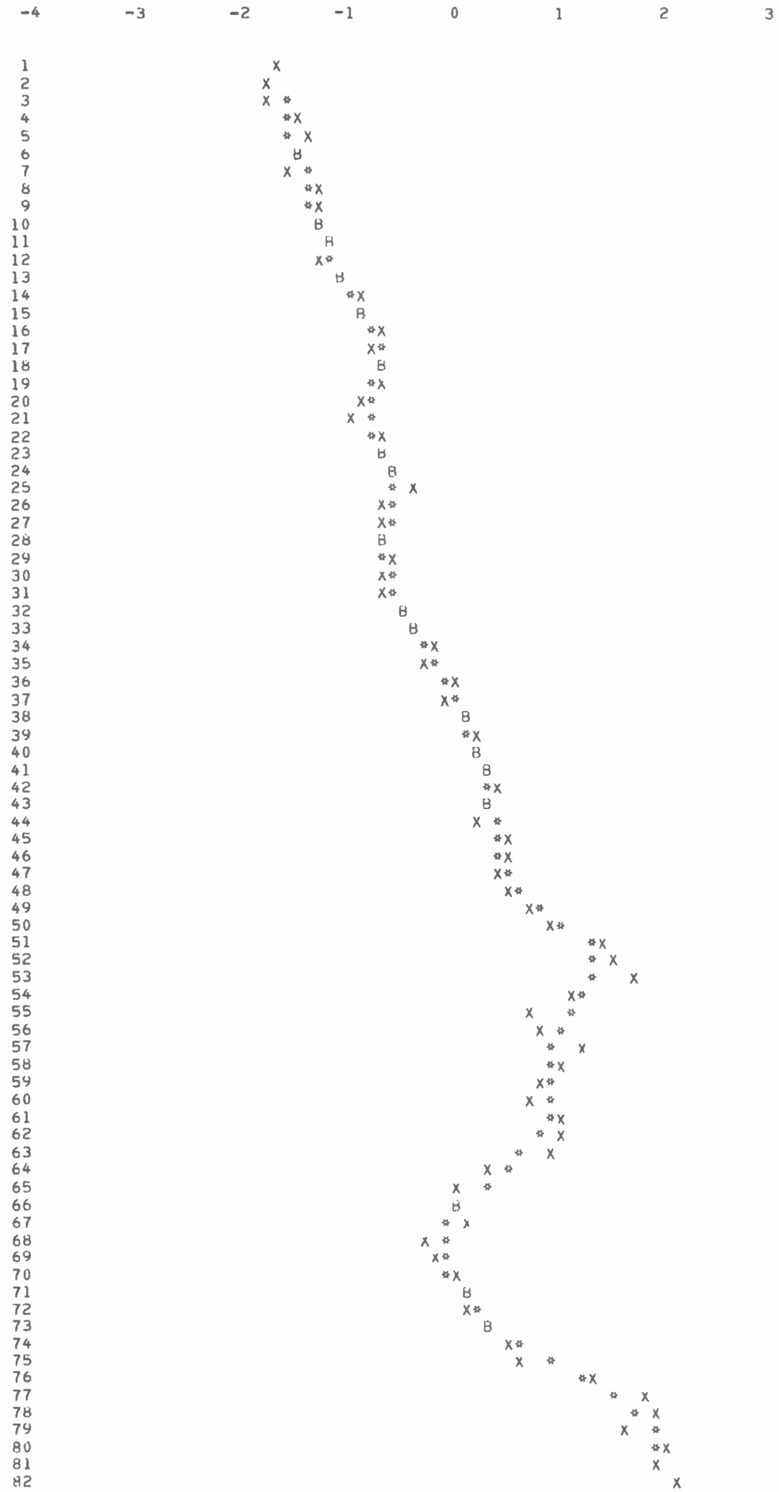
8.15445E-01	1.00248E 00	7.28816E-01	7.15428E-01	1.09198E 00

STANDARD DEVIATION OF RAW TREND FACTOR = STF 8.15444971E-01

COMPONENT = UC(I,J)

1.94742E-01	1.65920E-01	2.72972E-01	3.32861F-01	3.22650E-02
2.39409E-01	2.03974E-01	3.35580E-01	4.09204F-01	3.96686E-02
1.74054E-01	1.48293E-01	2.43972E-01	2.97499E-01	2.88388E-02
1.70857E-01	1.45569E-01	2.39491E-01	2.92035E-01	2.83073E-02
2.60783E-01	2.22185E-01	3.65540E-01	4.45738E-01	4.32105E-02

GRAPHICAL PLOT OF TREND FACTOR SCORES



REAL ROOT NO. 2 AFTER POWER 128 ROOT = 8.06366539E-01

TREND VECTOR = T(J) / STF

1.44650E-01	8.87700E-01	9.90707E-01	1.52564E 00	-2.58374E 00

EIGENVECTOR = V(I) * STF

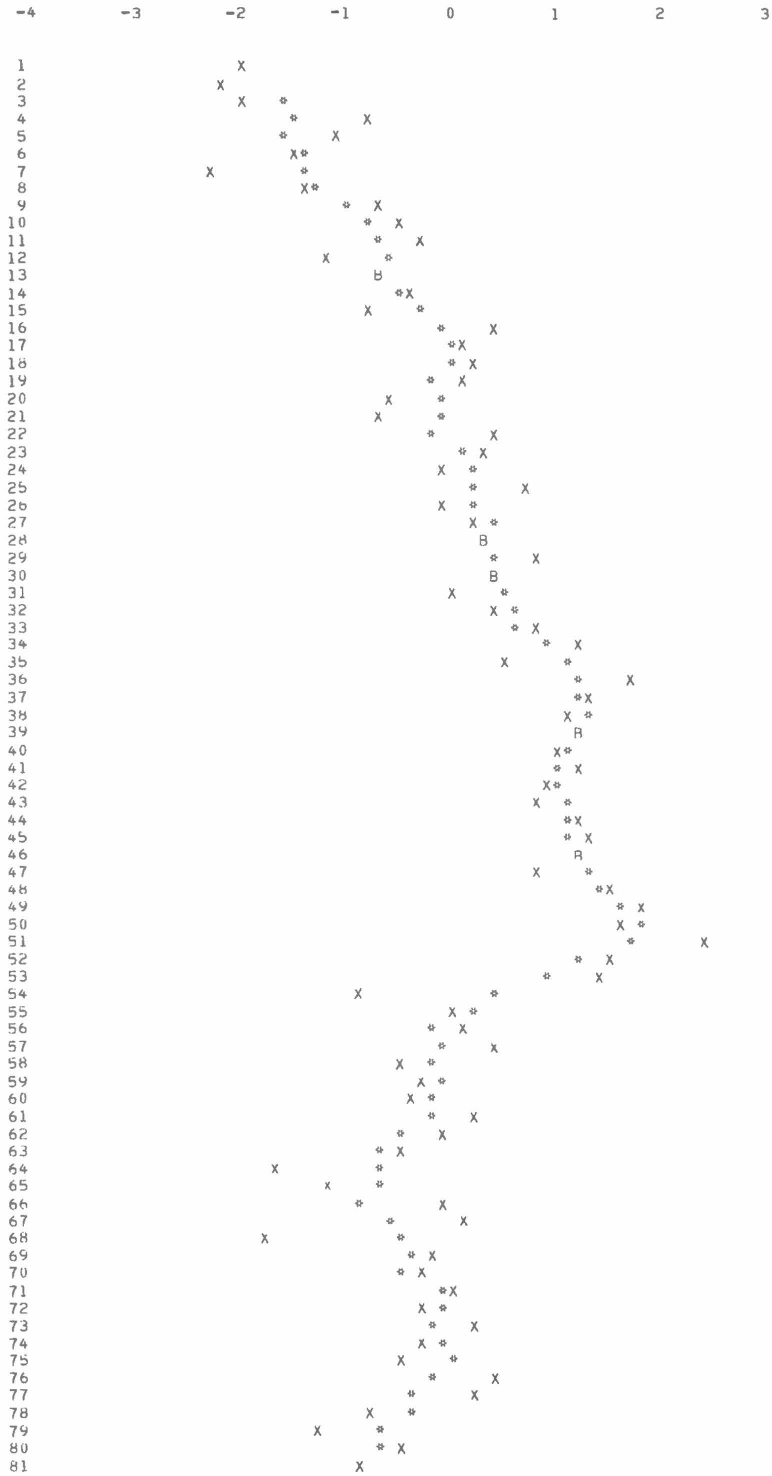
1.63361E-01	-3.30014E-01	-1.00727E-01	1.91911E-01	-4.16578E-01

STANDARD DEVIATION OF RAW TREND FACTOR = STF 1.63361025E-01

COMPONENT = UC(I,J)

1.90546E-02	1.16936E-01	1.30505E-01	2.00971E-01	-3.40353E-01
-3.84931E-02	-2.36228E-01	-2.63640E-01	-4.05993E-01	6.87565E-01
-1.17489E-02	-7.21018E-02	-8.04684E-02	-1.23918E-01	2.09860E-01
2.23846E-02	1.37372E-01	1.53312E-01	2.36094E-01	-3.99834E-01
-4.85900E-02	-2.98191E-01	-3.32793E-01	-5.12485E-01	8.67915E-01

GRAPHICAL PLOT OF TREND FACTOR SCORES



```
COMPLEX ROOT NO.    3 AND    4 AFTER POWER    8

MODULUS = 6.86571526E-01

REAL PART OF ROOTS            5.55784098E-01

IMAGINARY PART OF ROOTS       4.03093657E-01

PERIODICITY IN NO OF SAMPLING INTERVALS = 1.00132790E 01

QUENOUILLE*S ESTIMATE OF REAL PART OF TREND VECTORS
 3.04831E-01   -1.86157E-01   -2.18788E-01   -2.06683E-01    2.24701E-01

QUENOUILLE*S ESTIMATE OF IMAGINARY PART OF TREND VECTORS
-9.34187E-02   -6.00868E-01    3.34602E-01    3.54784E-02    3.74814E-01

QUENOUILLE*S ESTIMATE OF REAL PART OF EIGENVECTORS
 1.00000E 00   -8.53883E-01   -1.10160E 00    7.64188E-01   -2.08546E-01

QUENOUILLE*S ESTIMATE OF IMAGINARY PART OF EIGENVECTORS
           0    5.76009E-01   -5.68941E-01    1.71597E-01    8.10704E-02

STANDARD DEVIATION FOR REAL PART            2.24223518E-01

STANDARD DEVIATION FOR IMAGINARY PART       2.92508632E-01
```

REAL PART OF TREND VECTORS

1.35950E 00	-8.30229E-01	-9.75760E-01	-9.21772E-01	1.00213E 00

IMAGINARY PART OF TREND VECTORS

-3.19371E-01	-2.05419E 00	1.14391E 00	1.21290F-01	1.28138E 00

REAL PART OF FIGENVECTORS

4.48447E-01	-3.82921E-01	-4.94010E-01	3.42698E-01	-9.35217E-02

IMAGINARY PART OF EIGENVECTORS

0	-3.36975E-01	3.32840E-01	-1.00387E-01	-4.74276E-02

REAL PART OF COMPONENTS = ZR(I,J)

3.04831E-01	-1.86157E-01	-2.18788E-01	-2.06683E-01	2.24701E-01
-2.06480E-01	5.05062E-01	-5.91430E-03	1.56047E-01	-4.07764E-01
-3.88953E-01	-1.36788E-01	4.31387E-01	2.47868E-01	-3.42839E-02
2.48979E-01	-3.91519E-02	-2.24612E-01	-1.64033E-01	1.07397E-01
-5.59978E-02	8.75348E-02	1.85010E-02	4.02266F-02	-7.72467E-02

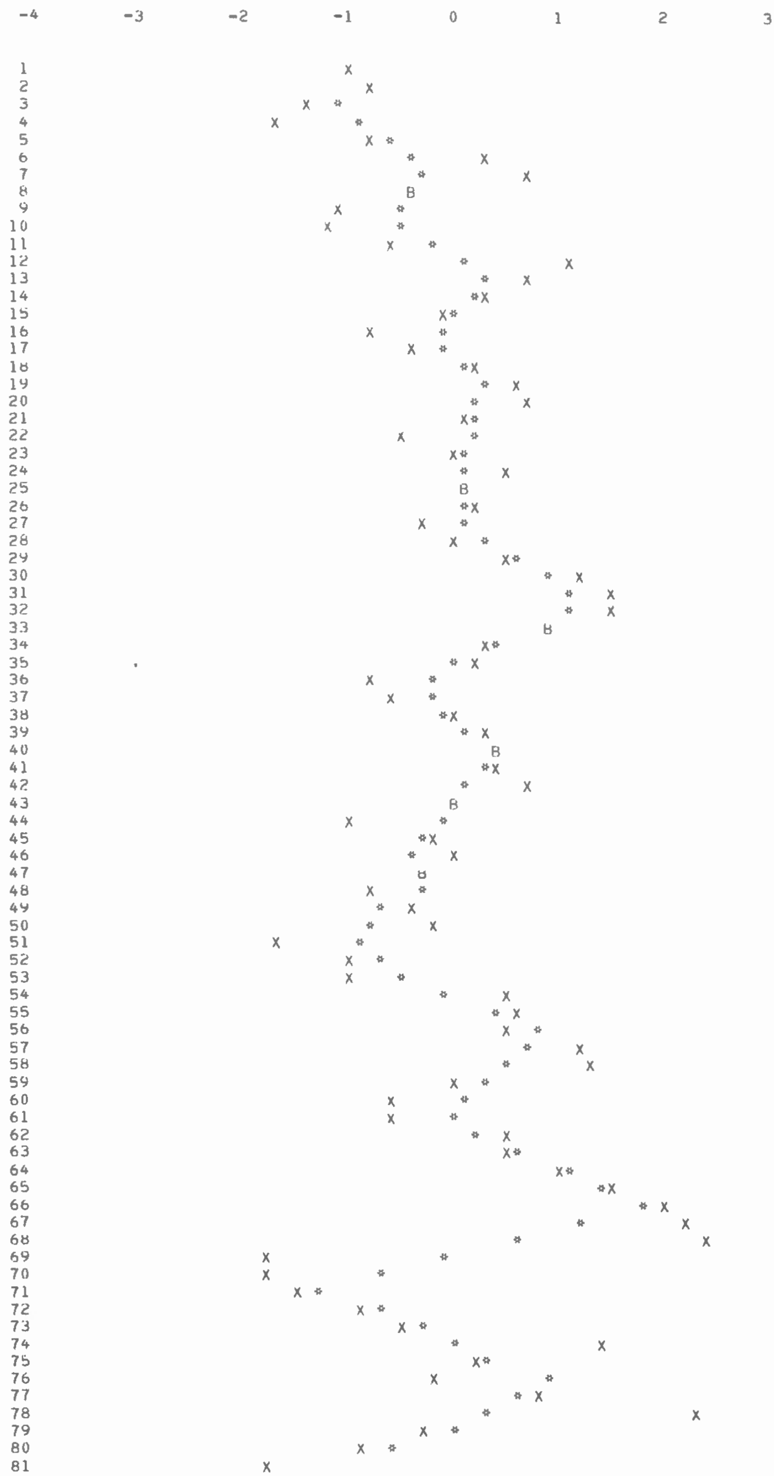
IMAGINARY PART OF COMPONENTS = ZI(I.J)

-9.34187E-02	-6.00868E-01	3.34602E-01	3.54784E-02	3.74814E-01
2.55354E-01	4.05843E-01	-4.11735E-01	-1.49346E-01	-1.90618E-01
-7.05208E-02	7.67830E-01	-2.44121E-01	7.85074E-02	-5.40737E-01
-1.90814E-02	-4.91120E-01	2.18156E-01	-8.35401E-03	3.24986E-01
4.41949E-02	1.10217E-01	-8.75171E-02	-2.41547E-02	-5.99492E-02

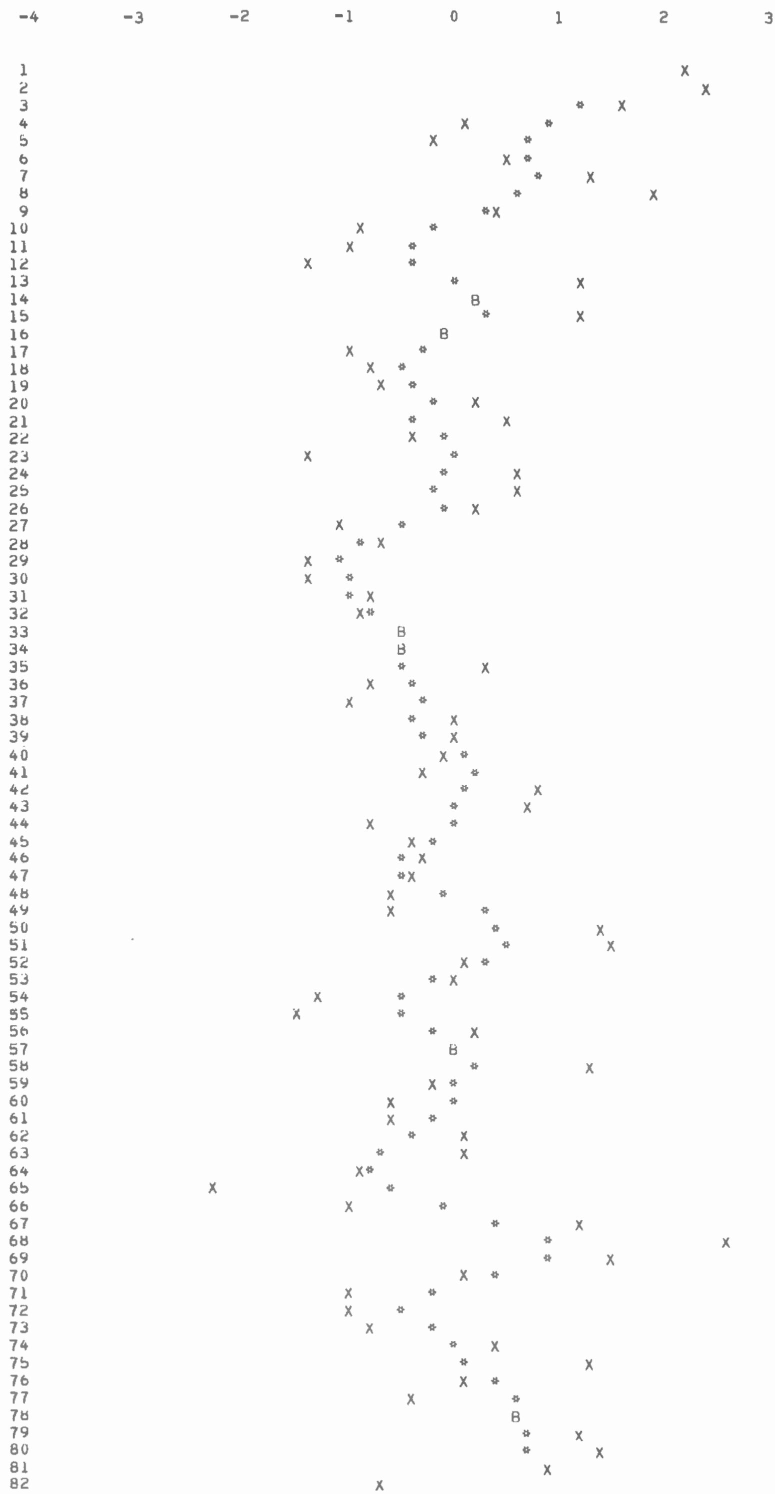
SUM COMPONENTS = UC(I.J)

4.14154E-01	2.77486E-01	-5.12950E-01	-2.58345F-01	-5.23999E-02
-4.35380E-01	2.34225E-01	3.25362E-01	2.93858E-01	-2.99585E-01
-3.75494E-01	-7.71063E-01	6.76323E-01	2.12230E-01	3.97827E-01
2.92140E-01	3.52415E-01	-4.25546E-01	-1.75599E-01	-1.42621E-01
-9.78747E-02	8.44574E-03	9.11203E-02	6.41879E-02	-3.75347E-02

GRAPHICAL PLOT FOR REAL TREND FACTOR SCORES



GRAPHICAL PLOT FOR IMAGINARY TREND FACTOR SCORES



REAL ROOT NO. 5 AFTER POWER 1 ROOT = 7.53043066E-02

TREND VECTOR = T(J) / STF

1.56759E 00	5.39163E-01	-3.17851E-02	-1.64866E 00	-5.64223E-01

EIGENVECTOR = V(I) * STF

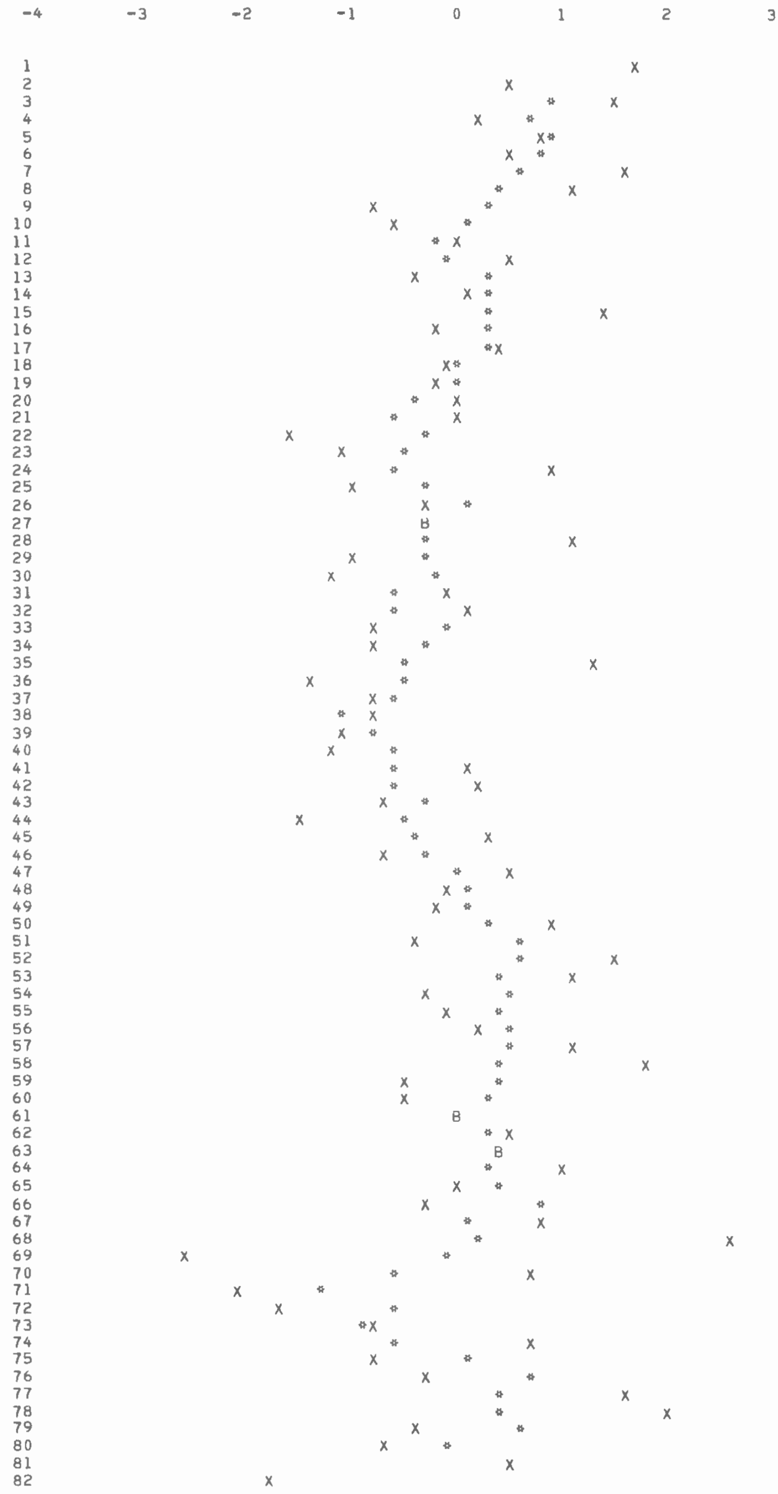
1.06897E-01	1.37718E-01	3.91999E-01	-4.46820E-01	-6.02291E-02

STANDARD DEVIATION OF RAW TREND FACTOR = STF 1.06896722E-01

COMPONENT = UC(I,J)

1.26188E-02	4.34015E-03	-2.55863E-04	-1.32714E-02	-4.54187E-03
1.62571E-02	5.59154E-03	-3.29636E-04	-1.70979E-02	-5.85142E-03
4.62740E-02	1.59157E-02	-9.38270E-04	-4.86672E-02	-1.66554E-02
-5.27454E-02	-1.81415E-02	1.06949E-03	5.54732E-02	1.89846E-02
-7.10982E-03	-2.44538E-03	1.44162E-04	7.47752E-03	2.55904E-03

GRAPHICAL PLOT OF TREND FACTOR SCORES



ORIGINAL TRANSITION MATRIX

6.40569E-01	5.64681E-01	-1.09731E-01	2.62215E-01	-3.65030E-01
-2.18209E-01	2.07562E-01	3.96971E-01	2.79970E-01	4.21798E-01
-1.66916E-01	-6.78957E-01	8.38887E-01	3.37143E-01	6.19870E-01
4.32635E-01	6.17214E-01	-3.16745E-02	4.08002E-01	-4.95163E-01
1.07207E-01	-7.00069E-02	1.24010E-01	4.91601E-03	8.76149E-01

CHECK SUM MATRIX

6.40569E-01	5.64681E-01	-1.09730E-01	2.62216E-01	-3.65030E-01
-2.18208E-01	2.07563E-01	3.96972E-01	2.79972E-01	4.21798E-01
-1.66915E-01	-6.78957E-01	8.38888E-01	3.37144E-01	6.19870E-01
4.32636E-01	6.17214E-01	-3.16735E-02	4.08003E-01	-4.95163E-01
1.07208E-01	-7.00061E-02	1.24012E-01	4.91770E-03	8.76150E-01

CHECK SUM MATRIX MINUS ORIGINAL TRANSITION MATRIX

7.93079E-07	7.02363E-07	1.13855E-06	1.39846E-06	4.89745E-08
9.62566E-07	7.68636E-07	1.29968E-06	1.56357E-06	3.25250E-07
7.05186E-07	5.83954E-07	9.68968E-07	1.17619E-06	1.67463E-07
6.97131E-07	6.23768E-07	1.00833E-06	1.24102E-06	1.95432E-08
1.04652E-06	8.30318E-07	1.40427E-06	1.68699E-06	3.80387E-07

APPENDIX II

LISTING OF PROGRAM

```
      PROGRAM C60431
C     PROGRAMMED BY
C     F.P.AGTERBERG, GEOLOGICAL SURVEY OF CANADA.
C     AND
C     G.D.CAMERON, COMPUTER SCIENCES DIVISION,
C     DEPARTMENT OF MINES AND TECHNICAL SURVEYS
C     ANALYSIS OF MULTIVARIATE SERIES AND EIGENVALUE ROUTINE FOR
C          ASYMMETRICAL MATRICES
C     INPUT MAY BE FROM CARDS OR MAG-TAPE
C     PUBLICATION OF OCTOBER 1966
C
C               INDEXES AS FOLLOW
C        INDEX 1 NOT ZERO. READ TAPE UNIT 1.    ZERO. READ CARDS
C        INDEX 2 NOT ZERO. TAKE LOG, BASE E, ALL INPUT (OPTIONAL)
C        INDEX 3 ZERO,DISCONTINUE THIS JOB AFTER CALCULATION OF MEAN AND
C                     STANDARD DEVIATION.      NOT ZERO, CONTINUE
C        INDEX 4 NOT ZERO. CALCULATE STANDARD DEVIATES, GRAPHICAL PLOT(OPTIONAL)
C        INDEX 5 ZERO. DISCONTINUE THIS JOB.  NOT ZERO, CONTINUE
C        INDEX 6 NOT ZERO. CALCULATE COVARIANCE MATRICES CO AND C1.
C                     ZERO, COVARIANCES BY CYCLICAL SCHEME
C        INDEX 7 NOT ZERO. CALCULATE C2, LAG 2 INCLUDED (OPTIONAL)
C        INDEX 8 NOT ZERO. CALCULATE CORRELATION MATRICES.  ZERO, USE
C                     COVARIANCES FOR FURTHER COMPUTATIONS
C        INDEX 9 ZERO,DISCONTINUE THIS JOB. NOT ZERO, CALCULATE INVERSE AND
C                     TRANSITION MATRICES
C        INDEX 10 ZERO,DISCONTINUE THIS JOB. NOT ZERO, SEND TRANSITION MATRICES
C                     TO EIGENVALUE ROUTINE
C
      COMMON R1, R2, X, SD, IDENT, N , CR, CI
      EQUIVALENCE(COVO,COV0)
      EQUIVALENCE(RO, R0)
      EQUIVALENCE (XBAR.PF)
      DIMENSION INDEX(10). X(100,8),XBAR(8),SD(8), SUM(8), SUM2(8),
     C COV0(9,9).COV1(9,9),COV2(9,9),R0(9,9),R1(9,9),R2(9,9),
     CSD1(8).SD2(8).SDN(8).SD2N(8),PF(8),XXXX(100)
      DIMENSION COVO(9,9). RO(9,9)
      REAL LOGF
  980 KZ=1
      IK=1
```

```
      NULL=4R
      DO 10 I = 1,8
      SUM(I)=0.0
   10 SUM2(I)=0.0
      DO 20 I = 1,9
      DO 20 J = 1,9
   20 COV0(I,J)=COV1(I,J)=COV2(I,J)=0.0
C
C          READ HEADER CARD
C
      READ(60,1) IDENT,NVAR,(INDEX(J),J = 1,10),  CR, CI
    1 FORMAT(R4, 11I2, 4X,  2F10.0)
      IF (NVAR) 1110, 1120, 1110
 1110 IF(INDEX(1))30,31,30
   30 IGO=1
   32 READ(01,2)NAM ,(X(IK,J),J=1,NVAR)
      K = EOFCK(01)
      IF(K - 1)132, 100,132
   31 IGO=2
   33 READ(60,2)NAM,(X(IK,J),J=1,NVAR)
      IF(NAM.EQ.NULL) 100, 132
  132 IF(INDEX(2))  450, 470, 450
  450 DO 580 J = 1,NVAR
  580 X(IK,J) = LOGF(X(IK,J))
  470 DO 50  J = 1,NVAR
         SUM(J)=SUM(J)+X(IK,J)
   50    SUM2(J)=SUM2(J)+X(IK,J)**2
      IK=IK+1
      GO TO(32,33),IGO
  100 IK=IK-1
      DO 102 I=1,NVAR
    2 FORMAT(R4, 7F10.1,  6X,  / F10.1)
  102 XBAR(I)=SUM(I)/IK
      PRINT 1540, IDENT
 1540 FORMAT(1H1, 35H STATISTICAL CALCULATIONS, PROBLEM , R4//)
      WRITE(61,103)(XBAR(I),I=1,NVAR)
      WRITE(61,104)IK
  104 FORMAT(1H0,25HNUMBER OF OBSERVATIONS = ,I4//)
      DO 202 I=1,NVAR
```

```
  202 SD(I)=SQRTF(( SUM2(I)- (SUM(I)**2/IK  ))/(IK-1))
      WRITE(61,203)(SD(I),I=1,NVAR)
  203 FORMAT(1H0,24H STANDARD DEVIATIONS     ,  8(F10.4, 2X))
      IF(INDEX(3))  312, 980, 312
  312 N=IK
      IF(INDEX(4)) 4070, 4080, 4070
 4070 DO 4040  I = 1,NVAR
      DO 4060 J = 1, N
 4060 XXXX(J) =(X(J, I)- XBAR(I))/SD(I)
      WRITE(61,4050) I
 4050 FORMAT(1H1 52H GRAPHICAL PLOT OF STANDARD DEVIATES FOR VARIABLE NO
     C  I3 //)
 4040 CALL GRPLOT(XXXX, N)
 4080 DO 302 I=1,NVAR
      X(N+1,I)=X(1,I)
  302 X(N+2,I)=X(2,I)
C
C         CALCULATE COVARIANCE MATRICES
C
      IF(INDEX(5)) 6000, 980, 6000
 6000 IF(INDEX(6)) 7000, 8000, 7000
 7000 DO 304 K = 1,NVAR
         DO 304 I=1,NVAR
      DO 303 J = 1,N
         J1=J+1
         J2=J+2
      COV0(K,I) = COV0(K,I) +X(J,I) * X(J,K)
      COV1(K,I) =  COV1(K,I) + X(J,I) * X(J1,K)
      IF(INDEX(7)) 5123, 303, 5123
 5123 COV2(K,I) =  COV2(K,I) + X(J,I) * X(J2,K)
  303 CONTINUE
      COV0(K,I) =(COV0(K,I) - (SUM(K) * SUM(I) / N)) / (N-1)
      COV1(K,I) =(COV1(K,I) - X(N,I)* X(1,K)-((SUM(K)-X(1,K)) * (SUM(I)
     C -X(N,I))) / (N-1)) / (N-2)
      IF(INDEX(7)) 7010, 304, 7010
 7010 COV2(K,I) = (COV2(K,I) -X(N-1,I) * X(1,K) - X(N,I) * X(2,K) -((
     C SUM(K) - X(1,K) - X(2,K)) * (SUM(I)-X(N-1,I)-X(N,I))) /
     C (N-2)) /(N-3)
  304 CONTINUE
```

```
      GO TO 7020
C
C          CALCULATE COVARIANCE MATRICES BY CYCLICAL SCHEME
C
 8000 DO 9304 K = 1,NVAR
      DO 9304 I = 1,NVAR
      DO 9303 J = 1,N
      J1 = J + 1
      J2 = J + 2
      COV0(K,I) = COV0(K,I) +X(J,I) * X(J,K)
      COV1(K,I) =  COV1(K,I) + X(J,I) * X(J1,K)
      IF(INDEX(7)) 5143, 9303, 5143
 5143 COV2(K,I) =  COV2(K,I) + X(J,I) * X(J2,K)
 9303 CONTINUE
      COV0(K,I) = (COV0(K,I)-(SUM(K)* SUM(I) / N)) /(N-1)
      COV1(K,I) = (COV1(K,I)-(SUM(K)* SUM(I) / N)) / (N-1)
      IF(INDEX(7)) 9390, 9304, 9390
 9390 COV2(K,I) = (COV2(K,I) - (SUM(K) * SUM(I) /N)) /(N-1)
 9304 CONTINUE
 7020 WRITE(61,305) IDENT
  305 FORMAT(1H1,35H  COVARIANCE MATRIX FOR PROBLEM NO.,R4)
      DO 306 I=1,NVAR
  306 WRITE(61,307)(COV0(I,J),J=1,NVAR)
  307 FORMAT(1H0, 10F12.6)
      WRITE(61,308)IDENT
  308 FORMAT(1H0,///,2X,41HCOVARIANCE (LAG 1) MATRIX FOR PROBLEM NO.,R4)
  310 FORMAT(1H0,///,2X,41HCOVARIANCE (LAG 2) MATRIX FOR PROBLEM NO.,R4)
  103 FORMAT(1H0,2X,22HMEANS OF VARIABLES ARE,8(F10.4,2X)//)
      DO 309 I=1,NVAR
  309 WRITE(61,307)(COV1(I,J),J=1,NVAR)
      IF(INDEX(7)) 9112, 9115, 9112
C
C          CALCULATE CORRELATION MATRICES
C
 9112 WRITE(61,310) IDENT
      DO 311 I=1,NVAR
  311 WRITE(61,307)(COV2(I,J),J=1,NVAR)
 9115 IF(INDEX(8)) 501, 7787, 501
  501 NOX=0
```

```
      WRITE(61,502)IDENT,NOX
  502 FORMAT(1H1,36H CORRELATION MATRIX FOR PROBLEM NO. ,R4,6H  LAG ,I3)
      IF(INDEX(6)) 5025,5026,5025
 5025 DO 5027 I=1,NVAR
      SD1(I)=SQRTF((SUM2(I)-X(N,I)**2-((SUM(I)-X(N,I))**2)/(N-1))/(N-2))
      SD2(I)=SQRTF((SUM2(I)-X(N,I)**2-X(N-1,I)**2-((SUM(I)-X(N,I)-X(N-1,
     CI))**2)/(N-2))/(N-3))
      SDN(I)=SQRTF((SUM2(I)-X(1,I)**2-((SUM(I)-X(1,I))**2)/(N-1))/(N-2))
      SD2N(I)=SQRTF((SUM2(I)-X(1,I)**2-X(2,I)**2-((SUM(I)-X(1,I)-X(2,I))
     C**2)/(N-2))/(N-3))
 5027 CONTINUE
      DO 504 I=1,NVAR
      DO 503 K=1,NVAR
      DEN=SD(I)*SD(K)
      DEN1=SD1(K)*SDN(I)
      DEN2=SD2(K)*SD2N(I)
      R0(I,K)=COV0(I,K)/DEN
      R1(I,K)=COV1(I,K)/DEN1
 7787 IF(INDEX(7)) 6678, 503, 6678
 6678 R2(I,K) = COV2(I,K) /DEN2
  503 CONTINUE
  504 WRITE(61,307)(R0(I,J),J=1,NVAR)
      GO TO 5055
 5026 DO 5045 I=1,NVAR
      DO 5035 K=1,NVAR
      DEN = SD(I)*SD(K)
      R0(I,K)=COV0(I,K)/DEN
      R1(I,K)=COV1(I,K)/DEN
      IF(INDEX(7)) 5036,5035,5036
 5036 R2(I,K)=COV2(I,K)/DEN
 5035 CONTINUE
 5045 WRITE(61,307)(R0(I,J),J=1,NVAR)
 5055 NOX=1
      PRINT 1400, IDENT, NOX
 1400 FORMAT(1H0,///,36H CORRELATION MATRIX FOR PROBLEM NO. ,   R4,
     C 6H  LAG ,  I3)
      DO 505 I=1,NVAR
  505 WRITE(61,307)(R1(I,J),J=1,NVAR)
      IF(INDEX(7)) 6687, 600, 6687
```

```
 6687 NOX = 2
      PRINT 1400, IDENT, NOX
      DO 506 I=1,NVAR
  506 WRITE(61,307) (R2(I,J), J = 1,NVAR)
  600 IF(INDEX(9)) 4545, 980, 4545
 4545 IF(INDEX(8)) 2323, 1199, 2323
 1199 DO 7123 I = 1,NVAR
      DO 7123 J = 1,NVAR
      R0(I,J) = COV0(I,J)
      R1(I,J) = COV1(I,J)
 2323 IF(INDEX(7)) 7132, 7123, 7132
 7132 R2(I,J) = COV2(I,J)
 7123 CONTINUE
C
C          CALCULATE TRANSITION MATRICES
C
      CALL XINVR(R0,NVAR,NVAR,DET,KZ,KZ)
         DO 602 I=1,NVAR
         DO 602 J=1,NVAR
  602   COV0(I,J)=R1(I,J)
      IF(INDEX(7))  2384, 2387, 2384
 2384    CALL XINVR(R1,NVAR,NVAR,DET,KZ,KZ)
 2387 WRITE(61,603)
  603 FORMAT(1H1,21H INVERSE OF R0 MATRIX)
         DO 604 I=1,NVAR
  604   WRITE(61,307)(R0(I,J),J=1,NVAR)
      IF(INDEX(7)) 7384, 7387, 7384
 7384 WRITE(61,605)
  605 FORMAT(///,21H INVERSE OF R1 MATRIX)
         DO 606 I=1,NVAR
  606 WRITE(61,307) (R1(I,J), J = 1,NVAR)
 7387 IF(INDEX(10)) 2183, 980, 2183
 2183 IF(INDEX(8)) 3183, 3184, 3183
 3183 DO 4185 I = 1,N
      DO 4185 J = 1,NVAR
 4185 X(I,J) = X(I,J) - XBAR(J)
      GO TO 701
 3184 DO 4186 I = 1,N
      DO 4186 J = 1,NVAR
```

```
4186 X(I,J) = (X(I,J) - XBAR(J)) * SD(J)
 701 WRITE(61,702)IDENT
 702 FORMAT(1H1,25H TRANSITION MATRICES FOR ,R4,/3H U1)
     CALL XMULT(COV0, R0, COV1, NVAR, NVAR, NVAR)
     CALL XMULT( R2,  R1, COV2, NVAR, NVAR, NVAR)
        DO 703 I=1,NVAR
 703   WRITE(61,307)(COV1(I,J),J=1,NVAR)
     WRITE(61,704)
 704 FORMAT(1H0,//,3H U2)
     DO 705 I=1,NVAR
 705 WRITE(61,307)(COV2(I,J),J=1,NVAR)
     WRITE(61, 3114)
3114 FORMAT(1H1, 26H EIGENVALUE ROUTINE FOR U1//)
     CALL ROOT(COV1,NVAR)
     IF(INDEX(7)) 9915, 980, 9915
9915 WRITE(61,3116)
3116 FORMAT(1H1, 26H EIGENVALUE ROUTINE FOR U2//)
     CALL ROOT(COV2,NVAR)
     GO TO 980
1120 WRITE(59, 801)
 801 FORMAT(20H PROCESSING COMPLETE)
     STOP
     END



     SUBROUTINE XINVR(X,N,NN,DET,N1,N2)
     DIMENSION X(9,9),Y(9)
     NP=(N-N1)+1
     NQ = (NN - N2) + 1
     DET=1.0
     ICOUNT=1
     NK=NN+1
  10 X(N1,NK)=1.0
     NS=N1+1
     DO 4 I=NS,N
```

```
    4 X(I,NK)=0.0
      T=X(N1,N2)
      DET=DET*T
      DO 5 J=N2,NN
      K=J+1
      Y(J)=X(N1,K)/T
      TT = ABSF(Y(J))
      IF(TT .LE. 0.0009) 12,5
   12 WRITE(61,13)
   13 FORMAT(37H ERROR IN XINV,MATRIX NEARLY SINGULAR)
    5 CONTINUE
      DO 6 I=NS,N
      TT=X(I,N1)
      DO 6 J=N2,NN
      JC=J+1
    6 X(I,J)=X(I,JC)-TT*Y(J)
      NZ=N-1
      DO 7 I=N1,NZ
      JC=I+1
      DO 7 J=N2,NK
    7 X(I,J)=X(JC,J)
      DO 8 J=N1,NK
    8 X(N,J)=Y(J)
      ICOUNT=ICOUNT+1
      IF(ICOUNT-NP)10,10,11
   11 RETURN
      END


      SUBROUTINE GRPLOT(TF,NN)
      DIMENSION TF(100),LINE(80)
      NNM2 = NN - 2
      WRITE(61,1830)
 1830 FORMAT(4X , 82H-4         -3         -2         -1         0
     C1         2         3         4//)
      DO 1840 I = 1,NN
      DO 1850 J = 1,80
```

```
1850 LINE(J) = 1R
     TFF = TF(I) * 10.
     INX = TFF + 41.5
     IF(INX - 81) 41,41,42
  42 INX = 81
     GO TO 1860
  41 IF(INX) 51, 51,1860
  51 INX = 1
1860 LINE(INX) = 1RX
     IF(I .GE. 3  .AND.  I  .LE. NNM2) 1900, 1910
1900 TFG = (TF(I - 2) + TF(I - 1) + TF(I) + TF(I+1) +  TF(I+2)) / 5.
     INZ = TFG*10. + 41.5
     IF(INZ - 81) 141,141,142
 142 INZ = 81
     GO TO 1960
 141 IF(INZ) 151, 151, 1960
 151 INZ = 1
1960 IF(INX .EQ. INZ) 1970, 1980
1970 LINE(INX) = 1RB
     GO TO 1910
1980 LINE(INZ) = 1R*
1910 WRITE(61, 1990) I, LINE
1990 FORMAT(1H ,I4,81R1)
1840 CONTINUE
     RETURN
     END

     SUBROUTINE XMULT(X, Y, Z, M, N, K)
     DIMENSION X(9,9),Y(9,9), Z(9,9)
     DO 10  I = 1,M
     DO 10  J = 1,K
  10 Z(I,J) = 0.0
     DO 20  L = 1,K
     DO 20  I = 1,M
     DO 20  J = 1,N
  20 Z(I,L) = Z(I,L) + X(I,J) * Y(J,L)
     RETURN
     END
```

```
      SUBROUTINE ROOT(U,N)
C     PROGRAMMED BY
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      EQUIVALENCE(TF,VP,TFI)
      COMMON B, C, X, SD, IDENT, NN, CR, CI
      DIMENSION B(9,9), C(9,9),D(9,9),U(9,9),A(9,9)
      DIMENSION X(100,8),SD(8),T(8),V(8),TF(100),UC(9,9)
      DIMENSION YZR(9,9),ZI(9,9),ZR(9,9)
      DIMENSION TFI(100)  ,                VR(8),VI(8),TFR(100)
      DIMENSION PVR( 8), PVI( 8), PTVR( 8), PTVI( 8)
      DIMENSION UORIG( 9, 9), CSUM( 9, 9)
      REAL LOGF
      IF(CR .EQ. .0  .AND.  CI .EQ. .0) 1700, 1710
 1700 CR = .000005
      CI = .0000005
      GO TO 1710
 1710 WRITE(61,1715) CR, CI
 1715 FORMAT(1H0, 5H CR =,E11.4, 5H CI =,E11.4)
      L = 1
      DO 1760 I = 1,N
      DO 1760 J = 1,N
      CSUM(I,J) = 0.
 1760 UORIG(I,J) = U(I,J)
      M = 1
      CO = 1.
      DO 1280 I = 1,N
      DO 1280 J = 1,N
 1280 A(I,J) = U(I,J)
 1540 CALL XMULT(U, A, B, N, N, N)
      CALL XMULT(B, A, C, N, N, N)
      CALL XMULT(C, A, D, N, N, N)
      U11 =  .4342944819 * LOGF(ABSF(U(1,1)))
      U11BS = ABSF(U(1,1))
      IF(ABSF(U11)  .GE. 150.) 1730, 1750
 1730 WRITE(61,1740)  M, U(1,1)
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1740 FORMAT(1H0, 12H AFTER POWER, I5,37H ALL ELEMENTS ARE DIVIDED BY A(
    C1,1) =, E15.8)
     CO = CO * EXPF(U11 / (.4342944819 * M))
     COSQ = CO * CO
     COCU = COSQ * CO
     DO 7128 I = 1,N
     DO 7128 J = 1,N
     U(I,J) =  U(I,J) / U11BS
     B(I,J) = B(I,J) / (U11BS * CO)
     C(I,J) = C(I,J) / (U11BS * COSQ)
7128 D(I,J) = D(I,J) / (U11BS * COCU)
1750 R1 = B(1,1) / U(1,1)
     R2 = D(1,1) / C(1,1)
     R2M2  = R2 ** (M+2)
     R2M3  = R2 ** (M+3)
     QR = ABSF(R1 - R2)
     IF(QR - CR) 1090, 1050, 1050
C
C          CALCULATE REAL COMPONENT
C
1090 R2 = R2 * CO
     WRITE(61,580) L, M, R2
 580 FORMAT(1H1,14H REAL ROOT NO., I5, 12H AFTER POWER, I5, 10X
    C 6HROOT =, E15.8)
     DO 540 J = 1,N
 540  T(J) =D(1,J) / R2M3
     DO 550 I = 1,N
     V(I) = D(I,1) /D(1,1)
 550 CONTINUE
     DO 560 I = 1,N
     DO 560 J = 1,N
 560 UC(I,J) =D(I,J) / R2M2 * CO
     DO 1770 I = 1,N
     DO 1770 J = 1,N
1770 CSUM(I,J) = CSUM(I,J) + UC(I,J)
     DO 700 K = 1,NN
 700 TF(K) = 0.
     DO 800 K = 1,NN
     DO 800 J = 1,N
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 800 TF(K) = TF(K) + (T(J) * X(K,J) / SD(J))
     SUM = 0.
     DO 630 K = 1,NN
 630 SUM = SUM + TF(K) ** 2
     STF = SQRTF(SUM / (NN - 1))
     DO 640 J = 1,N
     T(J) = T(J) / STF
 640 V(J) = V(J) * STF
     IF(T(1)) 645,655,645
 655 GO TO 1590
 645 PRINT 650
 650 FORMAT(1H0, 26H TREND VECTOR = T(J) / STF)
     PRINT 660.(T(I). I = 1,N)
 660 FORMAT(1H . 8(E13.5. 2X))
     PRINT 670
 670 FORMAT(// 1H0, 25H EIGENVECTOR = V(I) * STF)
     PRINT 660.(V(I), I = 1,N)
     PRINT 690. STF
 690 FORMAT(// 1H0,  45H STANDARD DEVIATION OF RAW TREND FACTOR = STF,
    C  F15.8)
     PRINT 695
 695 FORMAT(//,1H0,20H COMPONENT = UC(I,J))
     DO 770 I = 1,N
 770 PRINT 666.(UC(I,J) . J = 1,N)
 666 FORMAT(1H0, 8(E13.5. 2X))
     DO 710 K = 1,NN
710  TF(K)= TF(K) / STF
     PRINT 720
 720 FORMAT(// 1H0, 34H TREND FACTOR SCORES = TF(K)./ STF)
     PRINT 660,(TF(I). I = 1,NN)
     WRITE (61. 1820)
1820 FORMAT(1H1,38H GRAPHICAL PLOT OF TREND FACTOR SCORES////)
     CALL GRPLOT(TF.NN)
     L = L + 1
1270 IF (L - N - 1) 1400. 1490. 1490
1490 WRITE(61,2100)
2100 FORMAT(1H1,27H ORIGINAL TRANSITION MATRIX//)
     DO 2110  I = 1.N
2110 WRITE(61,660) (UORIG(I.J). J = 1,N)
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      WRITE(61,2120)
 2120 FORMAT(////, 1H0, 17H CHECK SUM MATRIX//)
      DO 2130  I = 1,N
 2130 WRITE(61,660) (CSUM(I,J), J = 1,N)
      DO 2140  I = 1,N
      DO 2140  J = 1,N
 2140 CSUM(I,J) = CSUM(I,J) - UORIG(I,J)
      WRITE(61,2150)
 2150 FORMAT(////,1H0,50H CHECK SUM MATRIX MINUS ORIGINAL TRANSITION MAT
     CRIX)
      DO 4130  I = 1,N
 4130 WRITE(61,660) (CSUM(I,J), J = 1,N)
      RETURN
 1400 DO 810 I = 1,N
      DO 810 J = 1,N
      U(I,J)=A(I,J) - UC(I,J)
  810 A(I,J) = U(I,J)
      M = 1
      CO =  1.
      GO TO 1540
 1050  P1N = B(1,1) * C(1,2) - B(1,2) * C(1,1)
      P1D = U(1,1) * B(1,2) - U(1,2) * B(1,1)
      P1 =  ABSF(P1N/P1D)
      P1 = SQRTF(P1)
      P2N = C(1,1) * D(1,2) - C(1,2) * D(1,1)
      P2 =  ABSF(P2N/P1N)
      P2 = SQRTF(P2)
      QI = ABSF(P1 - P2)
      IF(QI - CI) 1570, 1590, 1590
 1590 M = 2 * M
      IF(M - 10000) 910, 920, 920
  920 PRINT 930, L
  930 FORMAT(1H1,  36H NO CONVERGENCE AFTER POWER OF 10000,
     C 5H ROOT, I5/////////)
      PRINT 940
  940 FORMAT(1H0,25H CONTINUE NOW TO NEXT JOB)
      RETURN
  910 CALL XMULT( U, U, B,N, N, N)
      DO 1600 I = 1,N
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      DO 1600 J = 1,N
 1600 U(I,J) = B(I,J)
      GO TO 1540
C
C           CALCULATE PAIR OF COMPLEX COMPONENTS
C
 1570 LP1 = L + 1
      PRINT 1000, L, LP1, M
 1000 FORMAT(1H1,17H COMPLEX ROOT NO.,I5, 4H AND, I5,
     C 12H AFTER POWER, I5//)
      P2 = CO * P2
      PRINT 1010, P2
 1010 FORMAT(1H0,10H MODULUS =, E15.8//)
      P22 = (P2 * P2) / (CO * CO)
      AN = B(1,1) * P22 + D(1,1)
      AD = 2. * C(1,1)
      AA = AN/AD
      P22AA = P22 -(AA*AA)
      IF(P22AA) 1590,1590,1591
 1591 BB = SQRTF(P22AA)
      AA = AA * CO
      BB = BB * CO
      PRINT 1020, AA
 1020 FORMAT(1H0,19H REAL PART OF ROOTS, 10X, E15.8)
      PRINT 1030, BB
 1030 FORMAT(1H0,24H IMAGINARY PART OF ROOTS, 5X, E15.8//)
      AA = AA / CO
      BB = BB / CO
      DD = ATANF(BB/AA)
      EE = 6.28318531/DD
      WRITE(61,1035) EE
 1035 FORMAT(1H0,42H PERIODICITY IN NO OF SAMPLING INTERVALS =,E15.8)
      CC =AA/(COSF(DD))
      CM2 = CC ** (M+2)
      P2D = (M+2) * DD
      AHP = CM2 * COSF(P2D)
      BHP = CM2 * SINF(P2D)
      AD = 2. * BHP * BB
      BD = (-2.) * AHP * BB
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      AD2 = AD * AD
      BD2 = BD * BD
      ADBD2 = AD2 + BD2
      DO 1040 I = 1,N
      DO 1040 J = 1,N
      YZR(I,J) = AA * C(I,J) - D(I,J)
      ZI(I,J) =-BB * C(I,J)
      ZR(I,J) = (AD * YZR(I,J) + BD * ZI(I,J)) / ADBD2
 1040 ZI(I,J) = (AD * ZI(I,J) - BD * YZR(I,J)) / ADBD2
      PRINT 1060
 1060 FORMAT(1H0,52H QUENOUILLE*S ESTIMATE OF RFAL PART OF TREND VECTORS
     C)
      PRINT 660. (ZR(1,J), J = 1,N)
      PRINT 1070
 1070 FORMAT(//1H0,57H QUFNOUILLE*S ESTIMATE OF IMAGINARY PART OF TREND
     CVECTORS)
      PRINT 660. (ZI(1,J) , J = 1,N)
      ZR112 = ZR(1,1) ** 2
      ZI112 = ZI(1,1) ** 2
      ZRIRI = ZR112 + ZI112
      DO 1080 I = 1,N
      VR(I) = (ZR(1,1) * ZR(I,1) + ZI(1,1) * ZI(I,1)) / ZRIRI
 1080 VI(I) = (ZR(1,1) * ZI(I,1) - ZI(1,1) * ZR(I,1)) / ZRIRI
      PRINT 1100
 1100 FORMAT(//, 1H0, 51H QUENOUILLE*S ESTIMATE OF REAL PART OF EIGENVEC
     CTORS)
      PRINT 660.(VR(I), I = 1,N)
      PRINT 1110
 1110 FORMAT(//, 1H0, 56H QUENOUILLE*S ESTIMATE OF IMAGINARY PART OF EIG
     CENVECTORS)
      PRINT 660.(VI(I), I = 1,N)
      DO 1120 K = 1,NN
 1120 TFR(K) = TFI(K) = 0.
      DO 1130 K = 1,NN
      DO 1130 J = 1,N
      TFR(K) = TFR(K) + ZR(1,J) * X(K,J) / SD(J)
 1130 TFI(K) = TFI(K) + ZI(1,J) * X(K,J) / SD(J)
      SUMR = SUMI = 0.
      DO 1140 K = 1,NN
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      SUMR = SUMR + TFR(K) ** 2
 1140 SUMI = SUMI + TFI(K) ** 2
      STFR = SQRTF(SUMR / (NN - 1))
      STFI = SQRTF(SUMI / (NN - 1))
      PRINT 1150, STFR
 1150 FORMAT(//.1H0,33H STANDARD DEVIATION FOR REAL PART, 10X, E15.8)
      PRINT 1160, STFI
 1160 FORMAT(//.1H0,38H STANDARD DEVIATION FOR IMAGINARY PART,5X,E15.8)
      DO 5001 I=1,N
      PTVR(I)=ZR(1,I)/STFR
 5001 PTVI(I)=ZI(1,I)/STFI
      WRITE(61,5002)
 5002 FORMAT(//. 1H0. 27H REAL PART OF TREND VECTORS)
      WRITE(61,660)(PTVR(I).I=1,N)
      WRITE(61,5003)
 5003 FORMAT(//. 1H0. 32H IMAGINARY PART OF TREND VECTORS)
      WRITE(61,660)(PTVI(I).I=1.N)
      DO 5007  I = 1.N
      PVR(I)= 2.*VR(I)*STFR
 5007 PVI(I)=-2.*VI(I)*STFI
      WRITE(61,5008)
 5008 FORMAT(//. 1H0. 26H REAL PART OF EIGENVECTORS)
      WRITE(61,660) (PVR(I). I = 1,N)
      WRITE(61.5009)
 5009 FORMAT(//. 1H0. 31H IMAGINARY PART OF EIGENVECTORS)
      WRITE(61,660) (PVI(I). I = 1,N)
      PRINT 1170
 1170 FORMAT(//,1H0,34H REAL PART OF COMPONENTS = ZR(I,J))
      DO 1180 I = 1,N
 1180 PRINT 660. (ZR(I,J). J = 1,N)
      PRINT 1190
 1190 FORMAT(//.1H0,39H IMAGINARY PART OF COMPONENTS = ZI(I,J))
      DO 1200 I = 1.N
 1200 PRINT 660. (ZI(I.J). J = 1,N)
      DO 1210 I = 1,N
      DO 1210 J = 1,N
      AA = AA * CO
      BB = BB * CO
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1210 UC(I,J) = 2. * AA * ZR(I,J) - 2. * BB * ZI(I,J)
     DO 1780 I = 1,N
     DO 1780 J = 1,N
1780 CSUM(I,J) = CSUM(I,J) + UC(I,J)
     PRINT 1220
1220 FORMAT(//,1H0,25H SUM COMPONENTS = UC(I,J))
     DO 1230 I = 1,N
1230 PRINT 660,(UC(I,J) , J = 1,N)
     DO 5006  I = 1,NN
     TFR(I) = TFR(I) / STFR
5006 TFI(I) = TFI(I) / STFI
     PRINT 1240,L,  LP1
1240 FORMAT(////35H REAL TREND FACTOR SCORES FOR ROOTS, I5, 4H AND I5)
     PRINT 660,(TFR(I), I = 1,NN)
     PRINT 1250
1250 FORMAT(//, 1H0,  30H IMAGINARY TREND FACTOR SCORES)
     PRINT 660,(TFI(I), I = 1,NN)
     WRITE(61,5020)
5020 FORMAT(1H1,44H GRAPHICAL PLOT FOR REAL TREND FACTOR SCORES////)
     CALL GRPLOT(TFR,NN)
     WRITE(61,5030)
5030 FORMAT(1H1,49H GRAPHICAL PLOT FOR IMAGINARY TREND FACTOR SCORES///
    C/)
     CALL GRPLOT(TFI,NN)
     L = L + 2
     GO TO 1270
     END
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