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Abstract

An evaluation is made of the systematic error involved in determining the magnetization of a uniformly magnetized cylinder from the field measured along three of its mutually perpendicular axes of symmetry and using the assumption that the specimen's moment may be considered as a small dipole at its centre of symmetry. In addition, the optimum height/diameter ratio is established for cylindrical specimens to be examined with a biastatic magnetometer. It is shown that within a group of magnetization directions obtained for cylindrical specimens the component of angular dispersion related to the dipole approximation and to ordinary departures of the specimen's dimensions from their optimum values is negligible compared to that related to the in situ orientation errors which can be reasonably assumed to be of the order of 2 or 3 degrees.

A POSSIBLE SOURCE OF ERROR IN DETERMINING THE REMANENT MAGNETIZATION OF CYLINDRICAL ROCK SPECIMENS WITH A BIASTATIC MAGNETOMETER

Introduction

The astatic magnetometer, either in the two-magnet or in the three-magnet, biastatic form, is a basic instrument in many paleomagnetic and archeomagnetic laboratories around the world. Essentially, these instruments are used to measure the field produced by the remanent magnetic moment of rock specimens of simple geometric forms. These measurements are then used to determine the intensity and the direction of the remanent magnetization in the specimens.

Because the field produced by a magnetized body of any shape other than the sphere is a complicated function of position, the magnetic moment of the body is generally determined by approximating it to a dipole of equivalent strength at the geometric centre of the body. The approximation introduces a negligible error when the field is measured at a point at least three times as far from the centre of symmetry of the specimen as its longest dimension.

The object of this paper is to evaluate, under general conditions, the systematic error involved in making the approximation mentioned above when using a biastatic magnetometer for the measurements of the field. As a corollary, the optimum height/diameter ratio is established for cylindrical specimens to be studied with the biastatic magnetometer currently in use at the Geological Survey of Canada. Finally, the importance of the statistical error related to departures of the specimens' dimensions from their optimum values is evaluated.

Field Due to a Uniformly Magnetized Cylinder

Expressions for the field due to a uniformly magnetized cylinder along its principal axes of symmetry may be found in a number of Physics

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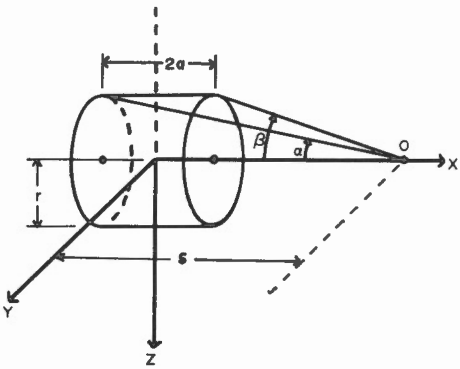


Figure 1.

Schematic representation of cylindrical specimen for the calculation of the axial component of magnetization $H_{x,o}$ from the field measured at point O.

textbooks. Using the notation of Figure 1, we may write the elementary result

$$(1)\quad H_{x,o} = 2\pi J_L (\cos\alpha - \cos\beta)$$

where J_L is the longitudinal component of magnetization and $H_{x,o}$ is the corresponding magnetic field at point O, along the X axis. In terms of the dimensions of the system, this expression may be rewritten (see Appendix) as

$$(2)\quad H_{x,o} = \frac{4\pi J_L r^2 a}{s^3}\left[1+(2\gamma^2-1.5)\delta^2+(3\gamma^4-7.5\gamma^2+1.875)\delta^4 + (4\gamma^6-21\gamma^4+17.5\gamma^2-2.19)\delta^6+ \dots\right]$$

where $\gamma = a/r$ and $\delta = r/s$. Replacing the terms inside the brackets by $(1+\epsilon_L)$ and neglecting the terms in δ whose order is greater than the sixth yields

$$(3)\quad H_{x,o} = \frac{4\pi r^2 J_L a}{s^3}(1+\epsilon_L) = 4\pi r^2 J_L' / s^3$$

where J_L' is the apparent value of J_L obtained under the dipole approximation. The systematic error related to the dipole approximation is then

$$(4)\quad \Delta J_L = J_L' - J_L = \epsilon_L J_L$$

Reference is made to Figure 2 for the derivation of $H_{x',o}$, the X component of the field at point O due to the transverse component in the X direction, J_x. The basic expression in this case is

$$(5)\quad V_o = \int_V \vec{J}_x \cdot \vec{r}\, dV/R^3 = \int_A \vec{J}_x \cdot \vec{r}\ dA/R$$

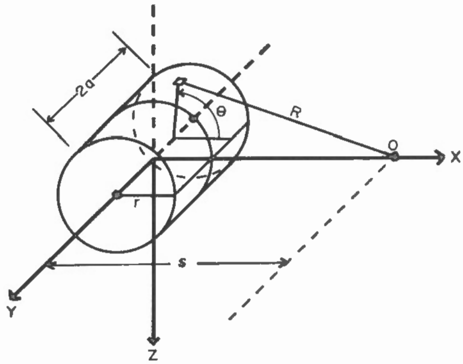


Figure 2.

Schematic representation of cylindrical specimen for the calculation of the transverse component of magnetization $H'_{x,o}$ from the field measured at point O.

Where V and A are respectively the volume and the surface of the specimen and V_o is the magnetic potential at point O. It can be shown (Appendix) that

$$(6)\quad H'_{x,o} = \frac{-\partial V_o}{\partial x} = \frac{4\pi r^2 a J_x}{s^3}\Big[1+(.75-\gamma^2)\delta^2+(.702-2.814\gamma^2 + 1.125\gamma^4)\delta^4+(.684-5.476\gamma^2+6.56\gamma^4-1.25\gamma^6)\delta^6+\ldots\Big]$$

or, with the expression inside the brackets written as $(1+\epsilon_T)$, that

$$(7)\quad H'_{x\,o} = \frac{4\pi r^2 a(1+\epsilon_T)J_x}{s^3} = \frac{J_x\, 4\pi r^2 a}{s^3}$$

where J'_x is the apparent value of J_x when the dipole approximation is used. The systematic error in determining J_x is then

$$(8)\quad \Delta J_x = J'_x - J_x = J_x\epsilon_T$$

It is obvious that if the cylinder is rotated through 90 degrees about its axis, the approximation error in determining the transverse component perpendicular to J_x is given by

$$(9)\quad \Delta J_z = J'_z - J_z = J_z\epsilon_T$$

Application to Magnetization Determination

The magnetization of the cylindrical specimen may be represented by a vector $\vec{J}$ whose modulus is defined by

$$(10)\quad J = \sqrt{J_x^2 + J_z^2 + J_L^2} = \sqrt{J_T^2 + J_L^2}$$

and whose orientation, reckoned to the J_xJ_z plane and to the direction of J_x, may be expressed in terms of the declination D, defined by

$$(11)\quad \tan D = J_z/J_x$$

and of the inclination I, defined by

$$(12)\quad \tan I = J_L/J_T$$

The systematic error in determining the modulus of $\vec{J}$ with the dipole approximation is then given by

$$(13)\quad dJ = \frac{J_T\Delta J_T + J_L\Delta J_L}{(J_T^2 + J_L^2)^{1/2}}$$

and, if $\epsilon_L = \epsilon_T$, by $dJ = J\epsilon_L = J\epsilon_T$

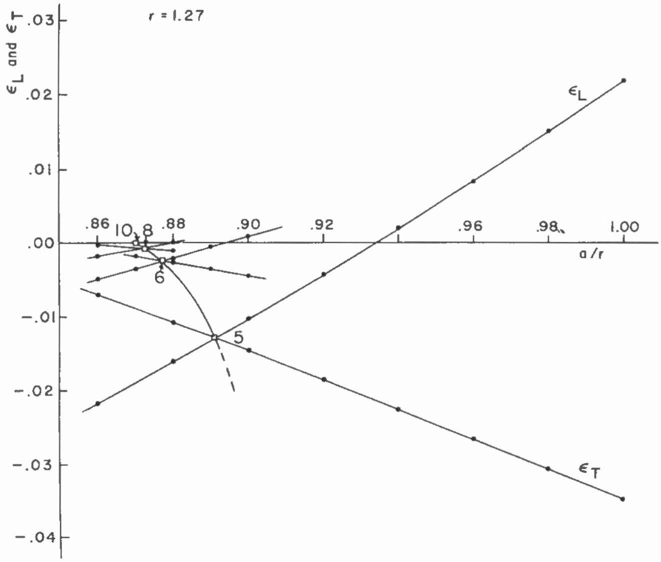
The systematic error in determining the inclination would be given by

$$(14)\quad dI = \frac{\partial \arctan (J_L/J_T)}{\partial J_L}\Delta J_L + \frac{\partial \arctan (J_L/J_T)}{\partial J_T}\Delta J_T$$

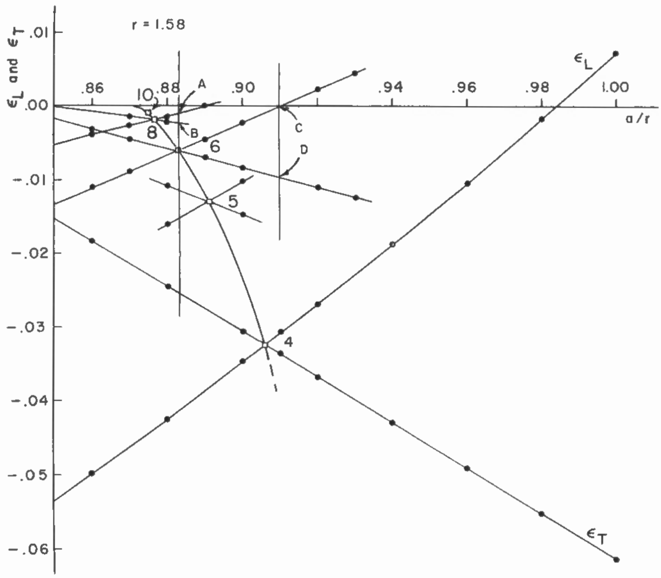
$$= \frac{J_LJ_T}{(J_L^2+J_T^2)}(\epsilon_L - \epsilon_T) \quad \text{radians}$$

In determining D, there is no systematic error inherent in the dipole approximation because $J_z\Delta J_x$ and $J_x\Delta J_z$ are theoretically equal.

The variation of ϵ_L and ϵ_T with a/r were calculated for discrete values of s and for r=1.27 cm (Fig. 3a) and r=1.58 cm (Fig. 3b). In both diagrams the absolute values of $\epsilon_T=\epsilon_L$ decrease assymptotically toward zero as s increases. If we denote the shortest distance s between the centre of symmetry of the specimen and that of the middle magnet of a given biastatic magnetometer by σ, the optimum value of r/a for a cylindrical specimen to be studied with this instrument is given by the abcissa of the point $\epsilon_L=\epsilon_T$ calculated for $s=\sigma$. For example, in determining the magnetization of cylindrical specimens with the biastatic magnetometer currently in use at the Geological Survey of Canada (Larochelle and Christie, 1967), the characteristic value of σ for this instrument is 6 cm and the specimen holder accepts cylinders either 1.27 or 1.58 cm in radius. In Figures 3a and 3b it may be verified that, for specimens of these radii, the optimum r/a ratios are .877 and .883 respectively. With specimens of these optimum dimensions there is



3a



3b

Figure 3a, b. Variations of ϵ_t and ϵ_L versus a/r for various distances s (numbers near intersections of ϵ_t and ϵ_L curves).

a – calculated for r=1.27 cm and

b – calculated for r=1.58 cm

no systematic error in determining D and I with the use of the dipole approximation when $s=\sigma$. The systematic errors in determining J are proportional to the ordinates of the points $\epsilon_L=\epsilon_T$ for s equal to 6 cm in both cases. Referring back to the equation 13, it turn out that dJ/J is equal to 0.25 and 0.60 per cent respectively in the two cases considered.

The vertical lines through the intersections of the ϵ_L and ϵ_T curves calculated for s=6 cm intersect the pairs of curves calculated for other values of s. In Figure 3b, for instance, this vertical line intersects the pair of curves calculated for s=8 cm, at points A and B. Assuming a specimen of optimum dimensions and centred 8 cm away from the magnet system, the difference between the ordinates of A and B is equal to twice the maximum value of dI (equation 14). It may be verified that the latter is equal to 0.04 degree. If, on the other hand, either J_T or J_L be null, the value of dI is also zero and values between zero and 0.04 degree are obtained for intermediate values of I. In a series of 28 repeat measurements performed (Larochelle and Christie, op.cit.) on a 3.16-cm cylinder with the biastatic magnetometer mentioned above, the angular standard deviation of the magnetization directions obtained was 0.2 degree i.e. 5 times the maximum value of dI derived above. It seems clear therefore that the angular error inherent in the dipole approximation is negligible compared with the measurement error. Inasmuch as "I" can vary widely within a group of specimens oriented independently in situ, dI may be considered as a statistical error although it must be considered as a systematic error in a set of repeat measurements of the same specimen. The same remark obviously applies to dJ, as illustrated in the following example where

$$dJ/J = (\epsilon_L+\epsilon_T)100/2 = 0.17\% \text{ when } J_L = J_T$$

$$dJ/J = 100\,\epsilon_T = 0.24\% \text{ when } J_L = 0$$

and

$$dJ/J = 100\,\epsilon_L = 0.10\% \text{ when } J_T = 0$$

It is noted in both Figures 3a and 3b that the angle between the ϵ_L and the ϵ_T curves of a given pair and the ordinates of their intersection decrease rapidly as the value of s increases. This indicates that the maximum values of both dJ and dI also decrease rapidly as the specimens are centred farther and farther away from the magnet system.

Figures 3a and 3b may also be used to determine the systematic errors involved when the specimen's dimensions depart from their optimum values. Here again, what is regarded as a systematic error with repeat measurements of one specimen becomes a statistical error when several independently oriented specimens are considered.

Suppose for example that the height of a 3.16-cm cylinder is 2.88 cm instead of 2.79 cm, its optimum height. The appropriate values of ϵ_L and ϵ_T for such a specimen (r/a=.91) centred 6 cm away from the magnet system are given by the ordinates of points C and D (Fig. 3b). The maximum values of dI and dJ would be 0.3 degree and 1 per cent respectively but the equally possible zero value of dJ and dI would be obtained if J_T were zero.

The dimensions of 75 3.16-cm cylindrical specimens (25 from each of 3 different collections) were measured accurately to determine the standard deviation of their r/a ratios. The value of 0.016 which was obtained indicates, in the light of the preceding paragraph, that the statistical error introduced in the cutting of the specimens can also be considered negligible. In the present example it can be shown that this error is well below 0.3 degree, which is considerably smaller than the 2- or 3-degree error generally assumed to be introduced during the orientation of the specimen in situ.

Conclusion

The current use of cylindrical specimens in paleomagnetic work bears many advantages but the relatively low degree of symmetry of cylinders compared to that of spheres or cubes can render cumbersome the theoretical calculation of their components of magnetization. Fortunately, as suggested at the beginning of this paper, the calculations are considerably simplified when the geometry of the measuring system is such that the distance between the sensing device and the centre of symmetry of the cylinder is at least thrice the long dimension of the cylinder. If the field is measured closer to the cylinder, the latter may only be equated to a dimensionless dipole at its centre of symmetry to a first approximation. It was shown, however, that the approximation is more and more valid as the height/diameter ratio of the cylinder reaches an optimum value which may be derived mathematically for a given biastatic measuring system. The systematic error inherent in the approximation when the specimen departs from these optimum dimensions may also be easily determined. On the basis of actual measurements of these departures within a large collection of cylindrical specimens, it was shown that the statistical error inherent in both the dipole approximation and the departure of the specimens' dimensions from their optimum values were small compared with that introduced during the specimens' orientation in situ. Finally, it is suggested that the above analysis could profitably be adapted to determine the optimum dimensions of cylindrical specimens to be studied with a particular 2-magnet astatic magnetometer.

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1967: An automatic 3-magnet or biastatic magnetometer; Geol. Surv. Can., Paper 67-28.

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APPENDIX

Appendix

The transition between equations (1) and (2) and between equations (5) and (6) involve a certain amount of laborious mathematical manipulation which was deliberately omitted from the text for obvious reasons. The sequence of these operations was outlined by Westphal (1967) in his unpublished doctoral thesis and they will be described in detail below for the sake of completeness.

To determine the longitudinal component of magnetization J_L, equation (1) may be rewritten as

$$\text{(a)}\quad H_{x,o} = 2\pi J_L \left[\left(1 + \frac{r^2}{(s+a)^2}\right)^{-1/2} - \left(1 + \frac{r^2}{(s-a)^2}\right)^{-1/2} \right]$$

Expressing the second term inside the brackets as a Taylor series yields

$$\text{(b)}\quad \cos \beta = 1 + \sum_{n=1}^{\infty} (-1)^n C_n (r/s)^{2n} \quad \frac{1}{(1-a/s)^{2n}}$$

where

$$C_n = \frac{1.3.5.\ldots.(2n-1)}{2.4.6.\ldots.2n}$$

A further expansion of the last factor in (b) leads to

$$\text{(c)}\quad \cos \beta = 1 + \sum_{n=1}^{\infty} (-1)^n C_n (r/s)^{2n} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k (2n+k-1)!}{(2n-1)!\,k!} \left(\frac{a}{s}\right)^k \right]$$

Similarly

$$\text{(d)}\quad \cos \alpha = 1 + \sum_{n=1}^{\infty} (-1)^n C_n (r/s)^{2n} \left[1 + \sum_{k=1}^{\infty} \frac{(2n+k-1)!}{(2n-1)!\,k!} \left(\frac{a}{s}\right)^k \right]$$

By subtraction of (c) from (d), the terms in even k's cancel and those in odd k's are doubled, and then

$$\text{(e)}\quad H_{x,o} = \frac{4\pi J_L r^2 a}{s^3} \left[\sum_{n=1}^{\infty} (-1)^{n+1} C_n (r/s)^{2(n-1)} \times \sum_{k'=o}^{\infty} \frac{2(n+k')!\,(a/s)^{2k'}}{(2n-1)!\,(2k'+1)!} \right]$$

where $k' = (k-1)/2$. Finally, the terms inside the brackets satisfying the relation $0 \leqslant 2(n+k'-1) \leqslant 6$ may be grouped to yield equation (2).

To determine either one of the transverse components of magnetization J_x or J_z, equation (5) may be rewritten as

$$\text{(f)}\quad V_o = \int_{-a}^{a} dy \int_{o}^{\pi} \frac{J_x \cos\theta\, d\theta}{(s^2+r^2+y^2-2rs\cos\theta)^{1/2}}$$

or

$$\text{(g)}\quad V_o = 2\int_{-a}^{a} \frac{rJ_x dy}{(s^2+r^2+y^2)^{1/2}} \int_{o}^{\pi} \frac{\cos\theta\, d\theta}{(1-B\cos\theta)^{1/2}}$$

where $B = 2rs/(y^2+r^2+s^2)$. By Taylor's theorem we may write

$$\text{(h)}\quad \frac{\cos\theta}{(1-B\cos\theta)^{1/2}} = \sum_{n=o}^{\infty} B^n C_n \cos^{(n+1)}\theta$$

and since

$$\text{(i)}\quad C_n \int_{o}^{\pi} B^n \cos^{(n+1)}\theta\, d\theta = \sum_{n'=o}^{\infty} \pi B^{2n'+1} C_{2n'+1} C_{n'+1}$$

where $2n'+1=n$ and $C_{2n'+1}$ and $C_{n'+1}$ have the same form as C_n, the expression for V_o may now be rewritten as

$$\text{(j)}\quad V_o = 2\pi r J_x \sum_{n'=o}^{\infty} C_{2n'+1} C_{n'+1} \int_{-a}^{a} \frac{(2rs)^{2n'+1} dy}{(s^2+r^2+y^2)^{2n'+3/2}}$$

or in terms of $(r^2+y^2)/s^2$,

$$\text{(k)}\quad V_o = \frac{F}{s^{2(n'+1)}} \int_{-a}^{a} \frac{dy}{\left(1+(r^2+y^2)/s^2\right)^{2n'+3/2}}$$

where F

$$\text{(l)}\quad F = 4\pi r^2 J_x \sum_{n'=o}^{\infty} C_{2n'+1} C_{n'+1} (2r)^{2n'}$$

A further Taylor expansion inside the integral sign of (k) yields

$$\text{(m)}\quad V_o = \frac{F}{s^{2(n'+1)}} \int_{-a}^{a} \sum_{k=o}^{\infty} \frac{(-1)^k (2n'+k+1/2)!\,(r^2+y^2)^k dy}{(2n'+1/2)!\,k!\,s^{2k}}$$

where $(2n'+1/2)! = \Gamma(2n'+3/2)$

After evaluation of the integral for general $(r^2+y^2)^k$,

$$\text{(n)}\quad V_o = \frac{FG}{s^{2(n'+1+k)}} \sum_{j=o}^{k} \frac{r^{2(k-j)} a^{2j}}{(k-j)!\,(2j+1)}$$

where

$$\text{(o)}\quad G = \sum_{k=o}^{\infty} \frac{(-1)^k (2n'+k+1/2)!\,2a}{(2n'+1/2)!}$$

Finally, differentiating V_o with respect to x yields

$$\text{(p)}\quad H'_{x,o} = \left(\frac{-\partial V_o}{\partial x}\right)_{x=s} = \frac{2FG(n'+k+1)}{s^3 s^{2(n'+k)}} \sum_{j=o}^{k} \frac{r^{2(k-j)} a^{2j}}{(k-j)!\,j!\,(2j+1)}$$

where $H'_{x,o}$ is the x component of the field at point O due to J_x. Replacing F and G by their values and grouping the terms satisfying the relation $O \leq 2(n'+k) \leq 6$ leads to equation (6).