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Abstract

An expression is derived for the analogue of the Laplace-Gauss or normal distribution in palaeomagnetism. For unit vector populations characterized by precision indexes greater than about 15, this distribution turns out to be identical to that previously postulated by Fisher (1953). An application of Pearson's χ^2 test is described as a means of verifying whether a sampled population conforms to the proposed distribution. A simple example is used to illustrate that three equations previously suggested by Watson and Irving (1957) for the analysis of palaeomagnetic data are incompatible with the proposed distribution and it is suggested that the design of a palaeomagnetic sampling scheme on the basis of those equations may not be universally acceptable.

FURTHER CONSIDERATIONS ON CERTAIN STATISTICAL METHODS IN PALAEOMAGNETISM

In a previous paper (Larochelle, 1967), an expression was derived for calculating an estimate of the angular variance δ^2 of a unit vector population on the basis of a sample of N vectors drawn at random from that population. The derivation of this expression was based on the assumption that any one of the vectors in the sampled population deviated from the population mean by an angle θ_i equal to or smaller than θ_o and that

$$(1) \quad \frac{\theta_o^4}{4!} << 1 - \frac{\theta_o^2}{2}$$

so that the approximation

$$(2) \quad \cos \theta_i \simeq 1 - \frac{\theta_i^2}{2}$$

holds true for any one vector in the population. It was suggested that the palaeomagnetic vectors of a geological unit may often be assumed to satisfy this condition on the evidence that the maximum deviation of any vector in a palaeomagnetic sample from the sample mean is commonly one radian or less. In effect, the maximum value which may be assigned to θ_o in (1) is relative but it is clear that a theory based on the approximation given in (2) is less and less valid as θ_i increases beyond one radian. As over 99.0% of the vectors forming a palaeomagnetic sample should be included in a cone whose half angle is equal to 2.25 times the angular standard deviation of the sample, it seems reasonable to assign this value to θ_o in studying a particular case.

As the vectors of a palaeomagnetic sample are generally distributed somewhat symmetrically about their mean, it also seems reasonable to assume this condition for a normally distributed palaeomagnetic population. Thus, by analogy with the Laplace-Gauss or normal distribution, we may write

$$(3) \quad 2\pi \int_o^{\pi} \underline{K e^{-\theta^2/\delta^2} \sin \theta \, d\theta} = 1$$

where K is a constant to be determined. As $0 \leqslant \theta \leqslant \theta_o$ and $\cos \theta = 1-\theta^2/2$, (3) may be rewritten as

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$$(4) \quad .99 < 2\pi \int_0^{\theta_o} \frac{K\, e^{(2/\delta^2)\cos\theta}}{e^{2/\delta^2}} \sin\theta \, d\theta < 1$$

and then

$$(5) \quad K \simeq \frac{1}{\pi\,\delta^2 \left(1 - e^{2/\delta^2 (\cos\theta_o - 1)}\right)}$$

The probability that vectors drawn at random from a normal population diverge from the population mean by an angle between θ and $\theta + d\theta$ is then given by

$$(6) \quad \mathcal{P}_A \, dA \simeq \frac{2\, e^{2/\delta^2 \cos\theta} \sin\theta \, d\theta}{\delta^2\, e^{2/\delta^2} \left(1 - e^{(2/\delta^2)(\cos\theta_o - 1)}\right)}$$

Now, from the definition of the estimate of δ^2 (Larochelle, op.cit.) given by

$$(7) \quad \hat{\delta}^2 = \frac{2\,(N - R)}{(N - 1)}$$

where R is the resultant length of the N vectors in the sample, it will be recognized that

$$(8) \quad \hat{\delta}^2 = 2/k$$

where k is an estimate of the precision index κ defined by Fisher (1953) under the assumption that the vectors of a palaeomagnetic population are distributed according to the law

$$(9) \quad \mathcal{P}_A \, dA = \frac{\kappa\, e^{\kappa \cos\theta} \sin\theta \, d\theta}{2 \sinh \kappa}$$

It is interesting to note that equations (6) and (9) are identical if δ^2 and θ_o in the first one are replaced by $2/\kappa$ and π respectively. It is pointed out however that the angle π is far from satisfying the inequality (1) and this suggests that Fisher's density distribution may not be considered as the analogue of the normal distribution when $2.25\ \hat{\delta} = \pi$. The angular standard deviation of a sample being equal to $(2/k)^{\frac{1}{2}}$ radians, it is further suggested that Fisher's density distribution does not hold rigorously for a palaeomagnetic population represented by a sample characterized by a $k \leqslant 16$ i.e. when $2.25\ (2/k)^{\frac{1}{2}} \geqslant .8$. A similar point of view was already suggested by Cox (1964) on a different reasoning basis. For these reasons it appears that the application of the variance ratio tests (Watson, 1956; Larochelle, 1967) would hardly be meaningful when dealing with populations having precision index estimates of the order of 3 or 4.

Pearson's χ^2 criterion may be used to test the null hypothesis that a given palaeomagnetic sample is not drawn from a normally distributed population at a predetermined probability level.

The probability that a vector drawn at random deviates from the population mean by an angle smaller than θ_1 may be derived from (6) and expressed as

$$(10) \quad P(\theta < \theta_1) = \frac{(1 - e^{2/\delta^2 (\cos \theta_1 - 1)})}{(1 - e^{2/\delta^2 (\cos \theta_o - 1)})}$$

where the value of δ^2 may be estimated with (7) for a sample drawn at random from the population. Multiplying $P(\theta < \theta_1)$ by N gives, for a sample of size N, the expected number E_1 of vectors which would be enclosed by a cone of half-angle θ_1 and centred on the population mean. Similarly the expected numbers E_2, E_3 . . . E_n may be computed for the intervals $\theta_1 < \theta \leqslant \theta_2$, $\theta_2 < \theta \leqslant \theta_3$ . . . $\theta_n < \theta \leqslant \theta_o$. In these computations, it is not necessary that all intervals be equal but it is recommended that all E_i's be equal to or greater than 5.

The number O_1 of vectors in the sample deviating from the population mean by an angle equal to or smaller than θ_1 may be estimated by assuming that the sample mean corresponds to the population mean. This implies that two of the three direction cosines of the population mean are estimated. Similarly, the numbers O_2, O_3 . . . O_n for the intervals $\theta_1 < \theta \leqslant \theta_2$, $\theta_2 < \theta \leqslant \theta_3$ . . . $\theta_n < \theta \leqslant \theta_o$ may be obtained.

The goodness of fit test consists in comparing the summation

$$(11) \quad \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = S$$

with the statistic $\chi^2_{d,\alpha}$, which has been tabulated for various numbers of degrees of freedom (d) and at a number of probability levels (α). The number of degrees of freedom is given by (n - 1 - c) where c is the number of parameters estimated in calculating the summation. In the present problem, three such parameters are estimated, as explained above, and thus, the number of degrees of freedom is (n - 4). If

$$(12) \quad S < \chi^2_{d,\alpha}$$

the probability of obtaining a summation equal to S for a sample drawn at random from a normally distributed palaeomagnetic population of angular variance δ^2 is greater than α. The conventional value adopted for α is .05, i.e. when relation (12) is satisfied at this probability level, it is generally accepted that there is not sufficient evidence to reject the hypothesis that the sampled population is normally distributed.

The fulfilment of the second condition (i.e. that the vectors of the sampled population are distributed symmetrically about their mean) may also be verified with the same test. In this case, however, the number of degrees of freedom is (n - 3) as the computing of $(O_i - E_i)$ only requires the estimation of two of the direction cosines of the population mean.

On the basis of the above discussion, it is suggested that a palaeomagnetic sample having a precision index estimate of 16 or greater (which would imply that $\theta_o \leqslant 0.8$) and satisfying (12) at the .05 significance level in both tests, may be realistically analyzed in the light of the variance ratio tests mentioned earlier.

Assuming that this sample may be broken down into B groups composed of N_i vectors (i = 1, 2, 3, ...B) having resultant lengths R_i respectively, the within-group and between-group variance estimates are respectively defined by:

$$(13) \quad \hat{\delta}_w^2 = \frac{2\,(N - \Sigma R_i)}{(N - B)}$$

and

$$(14) \quad \hat{\delta}_b^2 = \frac{2\,(\Sigma R_i - R)}{(B - 1)}$$

Combining (7), (13) and (14) yields

$$(15) \quad \hat{\delta}^2 = \frac{1}{(N - 1)} \left((B - 1)\, \hat{\delta}_b^2 + (N - B)\, \hat{\delta}_w^2 \right)$$

or

$$(16) \quad \hat{\delta}_b^2 = \hat{\delta}_w^2 + \frac{(N - 1)}{(B - 1)} \left(\hat{\delta}^2 - \hat{\delta}_w^2 \right)$$

which, as mentioned earlier, is not easily reconcilable with an equation originally proposed by Watson and Irving (1957). This equation which was given as

$$(17) \quad \frac{\Sigma R_i - R}{2\,(B - 1)} = 1/2 \left(\frac{1}{\omega} + \frac{\bar{N}}{\beta} \right)$$

where

$$(18) \quad \omega = \frac{N - B}{N - \Sigma R_i} = \frac{2}{\hat{\delta}_w^2}$$

and

$$(19) \quad \overline{N} = \frac{1}{(B - 1)} \left(N - \frac{\Sigma N_i^2}{N} \right)$$

may be rewritten as

$$(20) \quad \frac{2 (\Sigma R_i - R)}{(B - 1)} = \hat{\delta}_w^2 + \overline{N} \hat{\gamma}^2$$

where $\hat{\gamma}^2$ obviously stands for $2/\beta$.

As the statistic β is treated by Watson and Irving (op.cit.) as a precision index, it is clear that $\hat{\gamma}^2$ can only have a meaning when it is greater than or equal to zero. In fact, Watson and Irving point out that when

$$\frac{\delta_b^2}{\hat{\delta}_w^2} < F_{2(B - 1),\ 2(N - B),\ \alpha},$$

β may be considered very large (i.e. $\hat{\gamma}^2 \to 0$) and the between-site variation may be ignored. If on the other hand $\hat{\delta}_b^2$ is significantly larger than $\hat{\delta}_w^2$ they suggest that ω and β may be estimated from (17) and (18) respectively and that these estimates ($\hat{\omega}$ and $\hat{\beta}$) may be introduced into

$$(21) \quad 1/k_o' = 1/(N\hat{\omega}) + 1/(B\hat{\beta})$$

where they define k_o' as the precision estimate of the resultant of the N vectors. On this basis, they essentially suggest another method to calculate a radius of confidence (α_{95}') about the mean of the N vectors with the equation

$$(22) \quad 1 - \cos \alpha_{95}' = (\log_e 20)/k_o'$$

The universal validity of their method is questionable however because it is found that α_{95}' does not necessarily correspond with the radius of confidence (α_{95}) defined either by Fisher's (1953) elaborate equation

$$(23) \quad \cos \alpha_{95} = 1 - \frac{N - R}{R} \left(20^{\frac{1}{(N - 1)}} - 1 \right)$$

or by the simple relation

$$(24) \quad \alpha_{95} = \hat{\delta}\,(R/3)^{-1/2} \text{ radians}$$

To illustrate this incompatibility, four simple hypothetical cases were analyzed. In the four cases it was assumed that $N = 20$, $B = 5$ and $R = 19$, which implies that the statistic $F_{2(B-1),\ 2(N-B),\ \alpha}$ is equal to 2.2662. The value of ΣR_i assumed in each case is given in the second column of Table I and for each of these values two values of $\hat{\delta}^2$ ($\hat{\delta}_1^2$ and $\hat{\delta}_2^2$) are listed to represent the two eventualities where the N vectors are evenly and non evenly distributed among the B groups.

TABLE I : STATISTICS OF 3 HYPOTHETICAL CASES

CASE	ΣR_i	$\hat{\delta}_w^2$	$\hat{\delta}_b^2$	$\hat{\delta}^2$	$\hat{\gamma}_1^2$	$\hat{\gamma}_2^2$	$\frac{\hat{\delta}_b^2}{\hat{\delta}_w^2}$	Significance
I	19.21	.105	.105	.105	.000	.000	1.0	No
II	19.53	.062	.265	.105	.051	.053	4.2	Yes
III	19.00	.133	.000	.105	-.033	-.035	0.0	No
IV	20.00	.000	.500	.105	.125	.132	∞	Yes

The first line of Table I shows that when $\hat{\delta}^2 = \hat{\delta}_b^2 = \hat{\delta}_w^2$ the statistics $\hat{\gamma}_1^2$ and $\hat{\gamma}_2^2$ are identically null and that the restriction imposed on $\hat{\gamma}^2$ is fulfilled. As the between-site variation is not significant in this case, the value of k_o' is given by $(N \hat{\omega})$ and is equal to 381. The value of α_{95}' calculated on the basis of equation (22) is 7.2°, which is very close to the values of α_{95} calculated on the basis of (23) or (24) (7.66 and 7.44 respectively). In this case the value of α_{95}' is independent of whether the vectors are evenly distributed among the B groups or not.

In the second case, where the between-site variation is significant, if the calculated estimates of $\hat{\omega}$ and $\hat{\beta}$ are introduced into (21) the value of 150 (or 146) is obtained for k_o' and this value, introduced into (24), yields the corresponding value of 11.5° (or 11.7°) for α_{95}'.

In the third case k_o' is again defined by $(N\hat{\omega})$ as the between-site variation is not significant. Its calculated value is 300 and the corresponding value of α_{95}' is 8.1°. However, in this case the value of $\hat{\gamma}^2$ is negative, which is incompatible with the restriction imposed on $\hat{\gamma}^2$.

Finally in the fourth case k_o' is given by $(B \hat{\beta})$ as $\hat{\omega}$ is equal to infinity. Its calculated value is then 80 (or 76) and the corresponding value of α_{95}' is 15.7° (or 16.2°).

It thus appears that, except in the limited cases where $\hat{\delta}^2 \simeq \hat{\delta}_w^2$, the values of α_{95}' are significantly distinct from the fixed value of α_{95}, despite the fact that the two statistics have essentially the same definition. This clearly illustrates that Watson and Irving's method yields an approximate value of α_{95}' only in a restricted number of cases and therefore that the validity of equation (17) is questionable. Furthermore it appears that the index β has generally no useful statistical connotation and that equation (19) does not really fit in the present context. Equation (19) appears to have been borrowed directly from the theory of scalar statistics without due consideration that it stems from the familiar relation

$$(25) \quad \Sigma (x_i - \overline{x})^2 = \Sigma (x_i - \overline{x}_i)^2 + \Sigma N_i (\overline{x}_i - \overline{x})^2$$

which is irrelevant in vectorial statistics. Finally, it is implied that the analyses currently described in the palaeomagnetic literature and based on equations (17), (19) and (21) are relatively meaningless or at least unnecessarily approximative.

The questions of how many independently oriented specimens should be collected at a site and from how many sites should a geological unit be sampled may now be examined. Watson and Irving (op. cit.) proposed a solution to these problems but as their reasoning was based on the validity of equations (17), (19) and (21) in this context, it is suggested that their procedure is probably not universally applicable.

In the analysis of this aspect of palaeomagnetic research, it is important to define first the meaning of the word site. When dealing with a geological unit of igneous origin, it may be assumed that all parts of an outcrop within say the limits of a 100 foot square, were magnetized simultaneously i.e. within a period of a few hundred years. Furthermore if no geological discontinuities (e.g. the contact of two lava flows) and no differential rotation or tilting can be observed within the limits of an outcrop of this size, the independently oriented specimens collected within these limits may be said to come from the same site. In sedimentary terrain, the stratigraphic level from which samples are collected must be taken into account. As a simple example, let us assume that a flat lying sequence of sedimentary rocks is exposed on the face of a vertical cliff. If two specimens be taken from the cliff face at significantly different altitudes, it is reasonable to assume that they may have acquired their stable remanent magnetization at widely distinct geological times. Although the lateral extent of a site in such a case compares to that of a site in igneous terrain, the vertical extension of the site may not exceed a few inches or so and the two specimens mentioned in the above example would be said to come from different sites.

The dispersion in the palaeomagnetic directions obtained for a number of independently oriented specimens at a site may be attributed almost entirely to orientation and measurement errors and to lateral inhomogeneity in the sampled rock. Local effects due to lightning will generally be detected by unusually large intensities of magnetization in the specimen once subjected to this phenomenon. Such specimens should obviously not be considered in the statistical analysis of the data unless the lightning effect may be removed from the specimens.

A measure of the angular dispersion at a site is given by the angular standard deviation estimate ($\hat{\delta}_i$) of the palaeomagnetic directions obtained at that site. The standard error of the mean of the ith site is given by $\hat{\delta}_{\bar{R}_i}$ but it will generally be found that this statistic varies from site to site. A joint estimate of $\hat{\delta}_i$ for the B sites is given by $\hat{\delta}_w$ and thus a joint estimate of $\hat{\delta}_{\bar{R}_i}$ may be expressed as

$$(26) \quad \hat{\delta}_w/(\Sigma R_i/B)^{1/2} \simeq \hat{\delta}_w/(N/B)^{1/2}$$

If a reliable estimate of $\hat{\delta}_\omega$ is available for a given geological unit, it is possible to determine the average number (N/B) of specimens that should be collected at the individual site to obtain a desired average site mean standard error. For example if $\hat{\delta}_w = .14$ radian and the average standard error of the site means aimed at is 5°, the average number of specimens to be collected at each site is given by

$$\frac{N}{B} = \left(\frac{.14 \times 57.2}{5}\right)^2 = 2.6$$ i.e. 3 specimens per site. As a general rule, it is suggested that because most of the money, time and effort spent in a collecting scheme are spent in travelling from site to site, a minimum of three or four independently oriented specimens should be collected at each site visited.

To decide on the number of sites a geological unit should be sampled from, it is again necessary to assume a value for $\hat{\delta}_\omega$ and to know whether the site mean directions are significantly distinct. Normally, however, this information is available only after the palaeomagnetic measurements are completed and after the data have been analyzed.

If the within-site dispersion is high, it may be considered that the experimental error and/or the rock inhomogeneity render the data rather unreliable and not worthy of further analysis. On the other hand, if the within-site dispersion is reasonably low, an F-ratio test will indicate whether the site mean directions are significantly distinct. If the sites have been chosen several miles apart and if the site mean directions are not significantly distinct, it may be concluded that the sampled geological unit acquired its magnetization within a relatively short period and that it has not been disturbed appreciably since the time of its formation. Obviously only a virtual geomagnetic pole position may be derived for that period from such a rock unit.

If the result of the F-ratio test indicates that the site mean directions are significantly distinct, this inconsistency may be explained either by between-site differential tilting or rotation of the unit since its formation, by the effects of secular variation of the earth's field at the time the geological unit was formed or by both. When good geological control is available for the geological unit, such as is the case where well bedded sedimentary rocks are adjacent to the sampled unit, the between-site variation may be attributed solely to the secular variation of the earth's field at the time the geological unit acquired its magnetization. Assigning unit length to the vector resultant at each site, the standard deviation of the B unit vectors is given by

$$\hat{\delta}' = \left[\frac{2(B - R')}{(B - 1)}\right]^{1/2}$$

where R' is the length of the resultant. If the value obtained for $\hat{\delta}'$ compares with the expected standard deviation of the field direction at the palaeomagnetic latitude of the geological unit, the direction of $\overline{R}'$ may be assumed to be a reliable basis for estimating the geographic pole position at the time the rock acquired its magnetization. The standard angular error of $\overline{R}'$ would then be given by $\hat{\delta}_{\overline{R}'}$ and would naturally be approximately inversely proportional to the number of sites at which the geological unit was sampled.

From the above discussion it is clear that to give an equation for the design of a collecting scheme would be rather difficult but it may be stated that in general the sites should be as widely separated as possible within the geological unit and a minimum of three and preferably four independently oriented specimens should be collected at each site visited.

As a general conclusion to the above considerations, it is pointed out that the fundamental statistical methods available in palaeomagnetic research may all be derived from first principles and that there is in fact no need to refer, for instance, to hypergeometric dimensions to have a practical understanding of the system. The first principles approach has the further advantage of illustrating simply the hazards in transposing into vectorial statistics certain tests of significance based on the Laplace-Gauss distribution.

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